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α-IDEALS IN DE MORGAN RESIDUATED LATTICES

Supeng Wu, Xiaogang An, Hui Liu and Jiang Yang

The main purpose of this paper is to study the lattice structure of α-ideals and investigate the hull-kernel topology on prime α-ideals in a De Morgan residuated lattice. First, the lattice structure of all α-ideals in a De Morgan residuated lattice is studied and it is proved that the set of all α-ideals forms a complete Heyting algebra. Furthermore, by means of annihilator operators, the concept of quasi-complemented De Morgan residuated lattices is proposed and some equivalent conditions are derived for them. Finally, the space of all prime α-ideals is studied and it is stated that the space is a spectral space. In addition, some conditions are given for the space to be a Hausdorff space. Some topological characterizations of quasi-complemented De Morgan residuated lattices are demonstrated.

Keywords: De Morgan residuated lattice, α-ideal, prime α-ideal, quasi-complemented De Morgan residuated lattice, hull-kernel topology

Classification: 06D35, 06F05, 03G10, 03G25

1. INTRODUCTION

Residuated lattices have been playing a significant role in studying fuzzy logic algebras. The theory of ideals in residuated lattices or EQ-algebras has been extensively developed by some scholars [2, 3, 7, 11, 12, 13, 16, 18, 23, 26, 29, 35, 36, 40] since Lele and Nganou [17] introduced and studied ideals in BL-algebras. In order to further study ideals in residuated lattices, Holdon [11] presented an important subclass of residuated lattices called De Morgan residuated lattices. The notion of α-ideals was originally proposed by Cornish [5] in distributive lattices with 0. The concept of α-ideals was generalized by Jayaram [14] to 0-distributive lattices and some more properties concerning α-ideals were further studied by Pawar et al. [22]. By means of coannihilators introduced by Turunen [33] in BL-algebras, Haveshki et al. [8] presented the concept of α-filters. Subsequently, the notion of α-filters was extended by Dong and Xin [6] to residuated lattices and the lattice structure of all α-filters was studied by Dong and Xin [6] and by Rasouli[27], respectively.

Varlet [34] introduced the notion of quasi-complemented lattices, which is a natural generalization of that of distributive pseudo-complemented lattices. Speed [31] introduced distributive $*$-lattices with zero as a subclass of distributive lattices, which was inspired by the work of Henriksen and Jerison [9]. Speed ([31], Proposition 3.4) proved

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that quasi-complemented lattices are equivalent to distributive $*$-lattices. Moreover, Speed [30] studied some significant algebraic and topological characterizations of the quasi-complemented properties in commutative semigroups and subsequently investigated a series of topological characterizations for distributive $*$-lattices with zero [31, 32]. Rao [25] demonstrated some equivalent conditions for $*$-almost distributive lattices by means of α-ideals and prime ideals. Mokbel [19] investigated 0-ideals in 0-distributive posets and quasi-complemented 0-distributive posets.

The spectrum theory on many kinds of algebraic structures has been extensively investigated by some scholars [9, 12, 15, 20, 21, 22, 23, 28, 32, 36, 40]. α-ideals (or α-filters) as an important kind of ideals (or filters) have attracted the attention of several scholars and the study of α-ideals (or α-filters) in several lattice structures has achieved some interesting results. For example, the hull-kernel topology on the set of all prime α-ideals in a 0-distributive lattice was introduced by Jayaram [14]. The hull-kernel topology on the set of all prime α-ideals in an almost distributive lattice has been studied by Pawar et al. [21]. The hull-kernel topology on prime α-filters in a residuated lattice was intensively investigated by Dong [6] and by Rasouli [27, 28], respectively.

The main goal of this work is to investigate the hull-kernel topology on prime α-ideals in De Morgan residuated lattices. Our findings show that many important results obtained from the literatures mentioned above can also be reproduced in the context of De Morgan residuated lattices.

2. PRELIMINARY

In this section, we will summarize some basic concepts and results concerning residuated lattices, which will be used in the following sections.

Definition 2.1. (Buşneag et al. [2], Buşneag et al. [3], Holdon [11]) A residuated lattice is an algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \odot, 1)$ is a commutative monoid;
(3) $x \odot y \leq z$ if and only if $x \leq y \to z$, for all $x, y, z \in L$.

In the sequel, by L we denote the universe of a residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$. For any $x \in L$, we define: $x^* = x \to 0$.

Definition 2.2. (Holdon [11]) A residuated lattice L is called De Morgan provided that $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$.

It was shown by Holdon [11] that the class of De Morgan residuated lattices consists of some significant subclasses of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, Stonean and involutive residuated lattices.

Let L be a residuated lattice. For $x, y \in L$, we define two binary operations $\oplus$ and $\boxplus$ on L: $x \oplus y = x^* \to y$ and $x \boxplus y = (x^* \odot y^*)^* = x^* \to y^{**}$. Clearly, $x \boxplus y = x \oplus y^{**}$. $0x = 0$ and $nx = x \boxplus (n-1)x$, for some $n \in \mathbb{N}$ and $n \geq 1$. Concerning many properties of two operations $\oplus$ and $\boxplus$ in a BL-algebra or a residuated lattice, we refer the reader to the references [2, 3, 7, 11, 13, 17, 18, 35].

Definition 2.3. (Buşneag et al. [2], Buşneag et al. [3], Piciu [23]) Let L be a residuated lattice. A nonempty subset I of L is called an ideal if the following conditions are satisfied for all $x, y \in L$:

(I_1) $x \leq y \in I$ implies $x \in I$;

(I_2) $x, y \in I$ implies $x \boxplus y \in I$.

Note that for an ideal I of L, $x \in I$ iff $x^{**} \in I$ in any residuated lattice L. Condition (I_2) is equivalent to the following condition (I_2'): $x, y \in I$ implies $x \oplus y \in I$.

We denote by $\mathcal{I}(L)$ the set of all ideals of L . An ideal I of L is proper if $I \neq L$. A proper ideal of L is called a maximal ideal if it is not strictly contained in any proper ideal of L. Let A be a subset of L. The least ideal of L containing A is called the ideal generated by A and denoted by $\langle A \rangle$. In particular, for any $x \in L$, $\langle \{x\} \rangle$ is simply denoted by $\langle x \rangle$, and it is called the principal ideal of L generated by x. Let I, J be ideals of L and $a \notin I$. We denote $\langle I \cup J \rangle$ and $\langle I \cup \{a\} \rangle$ by $I \vee J$ and $\langle I, a \rangle$, respectively.

The ideal generated by a subset of a residuated lattice is explicitly described in the following. Moreover, our proofs depend on a concrete description of a generated ideal. So we shall enumerate the following properties of the generated ideals.

Proposition 2.4. (Dong and Wang [7], Holdon [11], Piciu [23]) Let L be a residuated lattice and I, J ideals of L. Then the following hold for $a, b \in L$:

(1) $\langle A \rangle = \{x \in L \mid x \leq a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n, \text{ for some } a_i \in A \text{ and } 1 \leq i \leq n, n \geq 1\}$;

(2) $I \vee J = \{x \in L \mid x \leq a \boxplus b, \text{ for some } a \in I \text{ and } b \in J\}$;

(3) $\langle I, a \rangle = \{x \in L \mid x \leq b \boxplus na, \text{ for some } b \in I, n \geq 1\}$;

(4) $\langle a \rangle = \{x \in L \mid x \leq na, \text{ for some } n \geq 1\}$ and so $\langle a \rangle = \langle na \rangle$, for all $n \geq 1$;

(5) $\langle a \rangle = \{0\}$ iff $a = 0$, $\langle a \rangle = L$ iff $na = 1$, for some $n \in \mathbb{N}$;

(6) $a \leq b$ implies $\langle a \rangle \subseteq \langle b \rangle$ and so $\langle I, a \rangle \subseteq \langle I, b \rangle$;

(7) $\langle a \rangle \cap \langle b \rangle = \langle a^{**} \wedge b^{**} \rangle$ and $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a \oplus b \rangle = \langle a \boxplus b \rangle$;

(8) $\langle I, a \rangle \cap \langle I, b \rangle = \langle I, a^{**} \wedge b^{**} \rangle$ and $\langle I, a \rangle \vee \langle I, b \rangle = \langle I, a \vee b \rangle = \langle I, a \oplus b \rangle = \langle I, a \boxplus b \rangle$;

(9) if $L = \langle X \rangle$, then $L = \langle A \rangle$, for some finite subset A of X;

(10) if L is finite, then any ideal of L is principal;

(11) if $\{I_i \mid i \in \Lambda\}$ is a family of ideals of L, then

$$\bigvee_{i \in \Lambda} I_i = \{x \in L \mid x \leq a_{i_1} \boxplus \cdots \boxplus a_{i_n}, \text{ for some } a_{i_j} \in I_{i_j}, 1 \leq j \leq n\}.$$

Furthermore, if L is a De Morgan residuated lattice, then:

(12) $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$;

(13) $\langle I, a \rangle \cap \langle I, b \rangle = \langle I, a \wedge b \rangle$.

Theorem 2.5. (Dong and Wang [7], Holdon [11], Piciu [23]) Let L be a residuated lattice. Then:

(1) $(\mathcal{I}(L), \cap, \vee, \to, \{0\}, L)$ is a complete Heyting algebra (or a frame), where for all ideals $I, J \in \mathcal{I}(L)$, $J \to I = \{x \in L \mid x \wedge y \in I, \text{for all } y \in J\}$ is the pseudocomplement of J with respect to I;

(2) $(\mathcal{I}(L), \cap, \vee, \{0\}, L)$ is an algebraic lattice, where compact elements of $\mathcal{I}(L)$ are exactly principal ideals of L.

The rest of this paper is based on some properties of annihilators in De Morgan residuated lattices. We next illustrate annihilators in De Morgan residuated lattices.

Definition 2.6. (Dong and Wang [7], Holdon [11]) Let L be a De Morgan residuated lattice and $A \subseteq L$. The set

$$A^{\perp} = \{x \in L \mid x \wedge a = 0, \text{for all } a \in A\}$$

is called the annihilator of A.

In general, such an annihilator in Definition 2.6 is not always an ideal in a residuated lattice, but an annihilator is an ideal in a De Morgan residuated lattice. It is clear that $A^{\perp}$ is precisely the pseudocomplement of A with respect to the ideal $\{0\}$, for a subset A of a De Morgan residuated lattice L. Particularly, $\{x\}^{\perp}$ is simply denoted by $x^{\perp}$, for all $x \in L$. It is easily proved that $L^{\perp} = 1^{\perp} = \{0\}$ and $0^{\perp} = \emptyset^{\perp} = L$. The set of all annihilators of L is denoted by $\mathcal{A}(L)$. Set $\mathcal{A}_0(L) = \{x^{\perp} \mid x \in L\}$ and elements of $\mathcal{A}_0(L)$ are called annulets of L.

The following proposition provides some useful properties of annihilators in De Morgan residuated lattices.

Proposition 2.7. (Dong and Wang [7], Holdon [11]) Let L be a De Morgan residuated lattice. Then the following hold for all $A, B \subseteq L$:

(1) $A^{\perp}$ is an ideal of L;
(2) $A \subseteq B$ implies $B^{\perp} \subseteq A^{\perp}$ and so $A^{\perp\perp} \subseteq B^{\perp\perp}$;
(3) $A \subseteq A^{\perp\perp}$ and $A^{\perp} = A^{\perp\perp\perp}$;
(4) $B^{\perp} \subseteq A^{\perp}$ if and only if $A^{\perp\perp} \subseteq B^{\perp\perp}$;
(5) $A^{\perp} = (\langle A \rangle)^{\perp}$;
(6) $\bigcap_{i\in\Lambda}(A_i)^{\perp} = (\bigcup_{i\in\Lambda} A_i)^{\perp}$, for a family $\{A_i \mid i \in \Lambda\}$ of subsets of L;
(7) $(A \cap B)^{\perp\perp} = A^{\perp\perp} \cap B^{\perp\perp}$;
(8) $\langle A \rangle \cap A^{\perp} = \{0\}$ and so $A^{\perp} \cap A^{\perp\perp} = \{0\}$;
(9) $A^{\perp} = L$ iff $\langle A \rangle = \{0\}$.

Note that from Proposition 2.7(2) and (3) it follows that the map $^{\perp\perp}$ is a closure operator on the lattice $\mathcal{I}(L)$. It is obvious that $\mathcal{A}(L) = \{X^{\perp} \mid X \subseteq L\} = \{\langle X \rangle \in \mathcal{I}(L) \mid X^{\perp\perp} = \langle X \rangle\}$. We define a join operation $\vee_{\mathcal{A}}$ on $\mathcal{A}(L)$ as follows: $\vee_{\mathcal{A}}\{I_i \mid i \in \Lambda\} = (\bigcap_{i\in\Lambda} I_i^{\perp})^{\perp} = (\bigcup_{i\in\Lambda} I_i)^{\perp\perp}$, where $\{I_i \mid i \in \Lambda\} \subseteq \mathcal{A}(L)$. According to Theorem 7.2 ([1], p. 105), we have $(\mathcal{A}(L), \cap, \vee_{\mathcal{A}}, \{0\}, L)$ is a complete Boolean algebra.

Lemma 2.8. (Dong and Wang [7], Holdon [11], Istrata [13]) Let L be a De Morgan residuated lattice. Then the following hold for all $a, b \in L$:

(1) $a^{\perp} = \langle a \rangle^{\perp}$ and $a^{\perp} = (na)^{\perp}$, for all $n \geq 1, n \in \mathbb{N}$;
(2) $a \leq b$ implies $b^{\perp} \subseteq a^{\perp}$ and $a^{\perp\perp} \subseteq b^{\perp\perp}$;
(3) $a^{\perp} \cap b^{\perp} = (a \vee b)^{\perp} = (a \oplus b)^{\perp} = (a \boxplus b)^{\perp}$;
(4) $a \in a^{\perp\perp}$ and $a^{\perp} = a^{\perp\perp\perp}$;
(5) $b^{\perp} \subseteq a^{\perp}$ iff $a^{\perp\perp} \subseteq b^{\perp\perp}$;
(6) $\langle a \rangle \cap a^{\perp} = \{0\}$ and $a^{\perp} \cap a^{\perp\perp} = \{0\}$;
(7) $a^{\perp} \cap b^{\perp} = \{0\}$ iff $a^{\perp} \subseteq b^{\perp\perp}$;
(8) $a^{\perp} = L$ iff $a = 0$.

Lemma 2.9. Let L be a De Morgan residuated lattice. Then the following hold for all $a, b \in L$:
(1) $a^{\perp\perp} \cap b^{\perp\perp} = (a \wedge b)^{\perp\perp}$;
(2) $a^{\perp} \vee_{\mathcal{A}} b^{\perp} = (a \wedge b)^{\perp}$;
(3) $a^{\perp\perp} \vee_{\mathcal{A}} b^{\perp\perp} = (a \vee b)^{\perp\perp} = (a \oplus b)^{\perp\perp} = (a \boxplus b)^{\perp\perp}$.

Proof. (1) Applying Proposition 2.7(7) and Proposition 2.4(12), $a^{\perp\perp} \cap b^{\perp\perp} = \langle a \rangle^{\perp\perp} \cap \langle b \rangle^{\perp\perp} = (\langle a \rangle \cap \langle b \rangle)^{\perp\perp} = (\langle a \wedge b \rangle)^{\perp\perp} = (a \wedge b)^{\perp\perp}$.
(2) By (1) and Lemma 2.8(4), $a^{\perp} \vee_{\mathcal{A}} b^{\perp} = (a^{\perp\perp} \cap b^{\perp\perp})^{\perp} = (a \wedge b)^{\perp\perp\perp} = (a \wedge b)^{\perp}$.
(3) By Lemma 2.8(3) and (4), we have $a^{\perp\perp} \vee_{\mathcal{A}} b^{\perp\perp} = (a^{\perp\perp\perp} \cap b^{\perp\perp\perp})^{\perp} = (a \vee b)^{\perp\perp} = (a \oplus b)^{\perp\perp} = (a \boxplus b)^{\perp\perp}$.
Note that from Lemmas 2.8(3) and 2.9(2) it follows that $(\mathcal{A}_0(L), \cap, \vee_{\mathcal{A}}, \{0\}, L)$ is a sublattice of $(\mathcal{A}(L), \cap, \vee_{\mathcal{A}}, \{0\}, L)$. □

Proposition 2.10. (Holdon [11], Piciu [23]) Let L be a De Morgan residuated lattice and P a proper ideal of L. Then the following are equivalent:
(1) If $I, J \in \mathcal{I}(L)$ such that $I \cap J = P$, then $I = P$ or $J = P$;
(2) if $I, J \in \mathcal{I}(L)$ such that $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$;
(3) if $x \wedge y \in P$ for all $x, y \in L$, then $x \in P$ or $y \in P$.

Definition 2.11. (Holdon [11], Piciu [23]) Let L be a De Morgan residuated lattice. A proper ideal P of L is called a prime ideal if it satisfies one of the equivalent conditions of Proposition 2.10. We denote the set of all prime ideals of L by $\mathcal{PI}(L)$.

Let L be a De Morgan residuated lattice. A nonempty subset S of L is a $\wedge$-closed subset if $x \wedge y \in S$, for all $x, y \in S$. The prime ideal theorems in De Morgan residuated lattices will be given in the following.

Theorem 2.12. (Holdon [11], Piciu [23]) Let L be a De Morgan residuated lattice and I an ideal of L. If S is a $\wedge$-closed subset of L such that $I \cap S = \emptyset$, then there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$.

Proposition 2.13. (Holdon [11, 12]) Let L be a De Morgan residuated lattice and I an proper ideal of L. Then for all $X \subseteq L$ and $a \in L$:
(1) If $a \notin I$, then there is $P \in \mathcal{PI}(L)$ such that $I \subseteq P$ and $a \notin P$;
(2) if $X \nsubseteq I$, then there is $P \in \mathcal{PI}(L)$ such that $I \subseteq P$ and $X \nsubseteq P$;
(3) $I = \bigcap\{P \in \mathcal{PI}(L) \mid I \subseteq P\}$ and hence $\bigcap \mathcal{PI}(L) = \{0\}$.

In a De Morgan residuated lattice L, a prime ideal P of L is minimal if any prime ideal Q of L is such that $Q \subseteq P$, then $Q = P$. The set of all minimal prime ideals of L is denoted by $\mathcal{MI}(L)$.

Lemma 2.14. (Piciu [23]) Let L be a De Morgan residuated lattice and P a prime ideal of L. Then there is a minimal prime ideal M of L such that $M \subseteq P$.

Proposition 2.15. (Piciu [23]) Let L be a De Morgan lattice and P a minimal prime ideal of L. Then for all $x \in P$ there is $a \notin P$ such that $x \wedge a = 0$.

Let L be a De Morgan lattice and P a prime ideal of L. Set $0(P) = \{x \in L \mid x \wedge a = 0, \text{ for some } a \notin P\}$. One can easily prove that $0(P) = \{x \in L \mid x^{\perp} \nsubseteq P\} = \bigcup_{x \notin P} x^{\perp}$ and $0(P) \subseteq P$. We next characterize minimal prime ideals by using the set $0(P)$. The following results are analogous to the ones obtained for minimal prime ideals in a distributive lattice with zero by Speed [32] and the ones obtained for minimal prime filters in a residuated lattice by Rasouli [28].

Theorem 2.16. Let L be a De Morgan lattice. Then the following are equivalent:
(1) M is a minimal prime ideal of L;
(2) $M = 0(M)$;
(3) for all $x \in L$, M contains precisely one of x and $x^{\perp}$;
(4) $x^{\perp} \setminus M \neq \emptyset$, for any $x \in M$.

Proof. (1) $\Rightarrow$ (2) This follows from Proposition 2.15.
(2) $\Rightarrow$ (3) If $x \in M$, then $x \in 0(M)$, so $x^{\perp} \nsubseteq M$. Let $x^{\perp} \subseteq M$. Then $x \notin 0(M)$. By $0(M) = M$, we get $x \notin M$.
(3) $\Rightarrow$ (4) Assume that $x \in M$. Then, by (3), $x^{\perp} \nsubseteq M$. This shows that $x^{\perp} \setminus M \neq \emptyset$.
(4) $\Rightarrow$ (1) Let Q be a prime ideal such that $Q \subseteq M$. By hypothesis, there is $y \in x^{\perp}$ such that $y \notin M$, for any $x \in M$. From $Q \subseteq M$, we have $y \notin Q$. Since Q is prime and $x \wedge y = 0 \in Q$, we deduce that $x \in Q$. Thus $M \subseteq Q$. We conclude that $Q = M$. □

Corollary 2.17. Let L be a De Morgan lattice and I a proper ideal of L. Then:
(1) $I = \bigcap\{P \in \mathcal{P}(L) \mid I \subseteq P\} = \bigcap\{M \in \mathcal{MI}(L) \mid I \subseteq M\}$.
(2) $x^{\perp} = \bigcap\{M \in \mathcal{MI}(L) \mid x \notin M\}$, for any $x \in L$.

Proof. (1) This follows from Lemma 2.14.
(2) It is a direct consequence of Theorem 2.16(3). □

3. THE LATTICE OF ALL α-IDEALS

In this section, some further properties of α-ideals in De Morgan residuated lattices will be discussed and we will characterize α-ideals by means of annihilators and prime α-ideals. The lattice structure of all α-ideals will be mainly investigated and a prime α-ideal theorem for De Morgan residuated lattices will be shown.

Definition 3.1. Let L be a De Morgan residuated lattice. An ideal I of L is called an α-ideal if $x^{\perp\perp} \subseteq I$, for all $x \in I$.

Notice that an α-ideal in a De Morgan residuated lattice L is also called a regular ideal by Holdon [11]. One can easily check that $0^{\perp} = L$ and $L^{\perp} = \{0\}$ and so $L^{\perp\perp} = L$ and $0^{\perp\perp} = \{0\}$. Hence both $\{0\}$ and L are α-ideals of L. The set of all α-ideals of L is denoted by $\mathcal{I}_{\alpha}(L)$.

Let I be an ideal of a De Morgan residuated lattice L. We denote by $\alpha(I)$ the α-ideal of L generated by an ideal I and so $\mathcal{I}_{\alpha}(L) = \{\alpha(I) \mid I \in \mathcal{I}(L)\}$. The following proposition provides a concrete description of the generated α-ideal $\alpha(I)$.

Proposition 3.2. (Holdon [11]) Let L be a De Morgan residuated lattice and I, J be two ideals of L. Then:

(1) $\alpha(I)$ is the smallest α-ideal of L containing I;
(2) $\alpha(I) = \{x \in L \mid a^{\perp} \subseteq x^{\perp}, \text{for some } a \in I\}$;
(3) I is an α-ideal of L iff $\alpha(I) = I$;
(4) $I \subseteq J$ implies $\alpha(I) \subseteq \alpha(J)$;
(5) $X^{\perp}$ is an α-ideal of L, for all $X \subseteq L$.

We mean by $\alpha(A)$ the α-ideal generated by a subset A of a De Morgan residuated lattice L. It is obvious that $\alpha(A) = \alpha(\langle A\rangle)$. The following proposition provides some additional basic properties of the generated α-ideals.

Proposition 3.3. Let L be a De Morgan residuated lattice and I, J two ideals of L. Then the following hold for $A \subseteq L$ and $a, b \in L$:

(1) $\alpha(I) = \{x \in L \mid x^{\perp\perp} \subseteq a^{\perp\perp}, \text{ for some } a \in I\} = \bigcup_{x\in I} x^{\perp\perp}$;
(2) $\alpha(A) = \{x \in L \mid x \in (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}, \text{ for some } a_1, \cdots, a_n \in A, n \geq 1\}$;
(3) $\alpha(I, a) = \bigcup_{b\in I, n\geq 1}(b \boxplus (na))^{\perp\perp}$, for some $n \in \mathbb{N}$;
(4) $\alpha(I) \cap \alpha(J) = \alpha(I \cap J)$ and $\alpha(I) \vee \alpha(J) \subseteq \alpha(I \vee J)$;
(5) $\alpha(I, a) \cap \alpha(I, b) = \alpha(I, a \wedge b)$;
(6) $\alpha(\langle I, a\rangle \vee \langle I, b\rangle) = \alpha(I, a \vee b) = \alpha(I, a \oplus b) = \alpha(I, a \boxplus b)$;
(7) if $\{I_i \mid i \in \Lambda\}$ is a family of α-ideals of L, then $\bigcap_{i\in\Lambda} I_i$ is an α-idea of L;
(8) if $\{I_i \mid i \in \Lambda\}$ be a family of α-ideals of L, then

$$\alpha(\vee_{i\in\Lambda} I_i) = \{x \in L \mid x \in (a_{i_1} \boxplus a_{i_2} \boxplus \cdots \boxplus a_{i_n})^{\perp\perp}, \text{ for some } a_{i_j} \in I_{i_j}, 1 \leq j \leq n, n \geq 1\}.$$

Proof. (1) This follows from Proposition 3.2(2) and Lemma 2.8(5).

(2) Set $I_A = \{x \in L \mid x \in (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}, \ for\ a_1, \cdots, a_n \in A\}$. Suppose that $x \in \alpha(A) = \alpha(\langle A\rangle)$. By (1), $x^{\perp\perp} \subseteq b^{\perp\perp}$, for some $b \in \langle X\rangle$ and so there are $a_1, \cdots, a_n \in A$ such that $b \leq a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n$, for some $n \geq 1$. By Lemma 2.8(2), we have $b^{\perp\perp} \subseteq (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}$. Thus $x^{\perp\perp} \subseteq (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}$ and so $x \in x^{\perp\perp} \subseteq (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}$. This shows that $x \in I_A$. Conversely, let $x \in I_A$. Then there are $a_1, \cdots, a_n \in A$ such that $x \in (a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n)^{\perp\perp}$. This means that $x \in \alpha(\langle A\rangle)$ by (1).

(3) By (2) and Propositions 2.4(3), $\alpha(I, a) = \bigcup_{b\in I, n\geq 1}(b \boxplus (na))^{\perp\perp}$.

(4) Since $I \cap J \subseteq I, J$, by Proposition 3.2(4) we have $\alpha(I \cap J) \subseteq \alpha(I) \cap \alpha(J)$. Let $x \in \alpha(I) \cap \alpha(J)$. Then $x^{\perp\perp} \subseteq a^{\perp\perp}$ and $x^{\perp\perp} \subseteq b^{\perp\perp}$, for some $a \in I$ and $b \in J$. By Lemma 2.9(1), $x^{\perp\perp} \subseteq a^{\perp\perp} \cap b^{\perp\perp} = (a \wedge b)^{\perp\perp}$, for $a \wedge b \in I \cap J$, which means that $x \in \alpha(I \cap J)$. We conclude that $\alpha(I) \cap \alpha(J) = \alpha(I \cap J)$.

Furthermore, from $I, J \subseteq I \vee J$, by Proposition 3.2(4), we obtain that $\alpha(I), \alpha(J) \subseteq \alpha(I \vee J)$ and so $\alpha(I) \vee \alpha(J) \subseteq \alpha(I \vee J)$.

(5) Applying (4) and Proposition 2.4(13), we have the following formulas: $\alpha(I, a) \cap \alpha(I, b) = \alpha(\langle I, a\rangle \cap \langle I, b\rangle) = \alpha(\langle I, a \wedge b\rangle) = \alpha(I, a \wedge b)$.

(6) Using Proposition 2.4(8), we have: $\alpha(\langle I, a\rangle \vee \langle I, b\rangle) = \alpha(I, a \vee b) = \alpha(I, a \oplus b) = \alpha(I, a \boxplus b)$.

(7) Let $x \in \bigcap_{i\in\Lambda} I_i$. Since I_i is an α-ideal for all $i \in \Lambda$, it follows that $x^{\perp\perp} \subseteq \bigcap_{i\in\Lambda} I_i$. This states that $\bigcap_{i\in\Lambda} I_i$ is an α-idea.

(8) This follows from (2) and Proposition 2.4(11). □

Proposition 3.4. Let L be a De Morgan residuated lattice and I an ideal of L. Then the following are equivalent for $x, y \in L$:

(1) I is an α-ideal of L;

(2) $x^{\perp} = y^{\perp}$ and $x \in I$ imply $y \in I$;

(3) $I = \bigcup_{x\in I} x^{\perp\perp} = \bigvee_{x\in I} x^{\perp\perp}$.

Proof. (1) $\Rightarrow$ (2) Suppose $x^{\perp} = y^{\perp}$ and $x \in I$. It follows from Proposition 3.2(2) and (3) that $y \in \alpha(I) = I$.

(2) $\Rightarrow$ (1) Let $x \in \alpha(I)$. Then, by Proposition 3.3(1), we get $x^{\perp\perp} \subseteq a^{\perp\perp}$, for some $a \in I$. By Lemma 2.9(1), we have $x^{\perp\perp} = x^{\perp\perp} \cap a^{\perp\perp} = (x \wedge a)^{\perp\perp}$, so $(x \wedge a)^{\perp} = x^{\perp}$. From $a \in I$, it follows that $x \wedge a \in I$. By hypothesis, $x \in I$ as $a \in I$ implies that $a \wedge x \in I$. Hence $\alpha(I) \subseteq I$. Obviously, $I \subseteq \alpha(I)$. We conclude that $\alpha(I) = I$. From Proposition 3.2(3) it follows that I is an α-ideal of L.

(1) $\Leftrightarrow$ (3) Suppose that I is an α-ideal of L. Then $I = \bigcup_{x\in I}\{x\} \subseteq \bigcup_{x\in I}\{x\}^{\perp\perp} \subseteq \bigvee_{x\in I}\{x\}^{\perp\perp} \subseteq I$. Hence $I = \bigcup_{x\in I} x^{\perp\perp} = \bigvee_{x\in I} x^{\perp\perp}$. The converse is evident. □

Theorem 3.5. Let L be a De Morgan residuated lattice. The map $\Phi : \mathcal{I}(L) \to \mathcal{F}(\mathcal{A}_0(L))$, defined by, $I \mapsto \{x^{\perp} \mid x \in I\}$ and the map $\Psi : \mathcal{F}(\mathcal{A}_0(L)) \to \mathcal{I}(L)$, defined by, $\mathcal{F} \mapsto \{x \mid x^{\perp} \in \mathcal{F}\}$, where $\mathcal{F}(\mathcal{A}_0(L))$ represents the set of all filters of $\mathcal{A}_0(L)$ as a lattice. Then:

(1) The pair (Φ, Ψ) is an adjunction and so $\Psi \circ \Phi$ is a closure operator on $\mathcal{I}(L)$;

(2) $\Psi(\Phi(I)) = \alpha(I)$, for all $I \in \mathcal{I}(L)$;

(3) I is an α-ideal of L if and only if $\Psi(\Phi(I)) = I$, that is, an α-ideal I of L is a closed element with respect to the closure operator $\Psi \circ \Phi$ on $\mathcal{I}(L)$.

Proof. (1) Assume that $\Phi(I) \subseteq \mathcal{F}$. Consider $x \in I$. Then $x^{\perp} \in \Phi(I)$ and so $x^{\perp} \in \mathcal{F}$, which means that $I \subseteq \Psi(\mathcal{F})$. Conversely, suppose that $I \subseteq \Psi(\mathcal{F})$. Consider $x \in \Phi(I)$. Then $x = a^{\perp}$, for some $a \in I$ and so $a \in \Psi(\mathcal{F})$. Thus $a^{\perp} \in \mathcal{F}$ and so $x \in \mathcal{F}$. This shows that $\Phi(I) \subseteq \mathcal{F}$. Hence the pair (Φ, Ψ) is an adjunction and so $\Psi \circ \Phi$ is a closure operator on $\mathcal{I}(L)$.

(2) Let I be an ideal of L. Then $\Psi(\Phi(I)) = \{x \in L \mid x^{\perp} \in \Phi(I)\} = \{x \in L \mid x^{\perp} = a^{\perp}, \text{ for some } a \in I\} = \{x \in L \mid x^{\perp\perp} = a^{\perp\perp}, \text{ for some } a \in I\} = \{x \in L \mid x \in a^{\perp\perp}, \text{ for some } a \in I\} = \bigcup_{x\in I} x^{\perp\perp} = \alpha(I)$.

(3) This follows from (2) and Proposition 3.2(3). □

Notice that condition (3) of Theorem 3.5 is similar to the concept of α-ideals of a distributive lattice with zero, which was introduced by Cornish [5].

In a De Morgan residuated lattice L, an ideal I of L is called dense if $I^{\perp} = \{0\}$. Otherwise, I is said to be non-dense. An element $x \in L$ is called dense if $x^{\perp} = \{0\}$.

Proposition 3.6. Let L be a De Morgan residuated lattice and I an ideal of L. Then:

(1) $\alpha(I) = L$ if and only if I contains a dense element;

(2) $\alpha(I)$ is proper if and only if I contains no dense element;

(3) if I is maxmal and $\alpha(I)$ is proper, then I is an α-ideal.

Proof. (1) If $\alpha(I) = L$, then $1 \in \alpha(I)$. Using Proposition 3.2(2), $x^\perp \subseteq 1^\perp = \{0\}$ for some $x \in I$. Conversely, suppose that there exists $x \in I$ such that $x^\perp = \{0\}$. Then, by $1^\perp = \{0\}$, we have $x^\perp = 1^\perp$ and so $1 \in \alpha(I)$. Hence $\alpha(I) = L$.

(2) It is obvious by (1).

(3) Since I is maxmal and $\alpha(I)$ is proper, we obtain $\alpha(I) = I$. It follows from Proposition 3.2(3) that I is an α-ideal. □

In what follows, we study the lattice structure of $\mathcal{I}_\alpha(L)$. We define a join operation $\vee_\alpha$ on $\mathcal{I}_\alpha(L)$: $\vee_\alpha\{\alpha(I_i) \mid i \in \Lambda\} = \alpha(\vee_{i\in\Lambda} I_i)$, for $\{I_i \mid i \in \Lambda\} \subseteq \mathcal{I}(L)$. The following theorem shows that the structure of the lattice $\mathcal{I}_\alpha(L)$ under the two binary operations $\cap$ and $\vee_\alpha$ is the same as that of the lattice $\mathcal{I}(L)$ and so $\mathcal{I}_\alpha(L)$ forms a frame.

Theorem 3.7. Let L be a De Morgan residuated lattice. Then:

(1) $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, \{0\}, L)$ is a frame;

(2) $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, \{0\}, L)$ is an algebraic lattice, where a compact element of $\mathcal{I}_\alpha(L)$ is exactly of the form $\alpha(x)$, for $x \in L$.

Proof. (1) Suppose that I is an ideal of L and $\{J_i \mid i \in \Lambda\}$ is a family of ideals of L. Then $\alpha(I) \cap (\vee_\alpha\{\alpha(J_i) \mid i \in \Lambda\}) = \alpha(I) \cap \alpha(\vee_{i\in\Lambda} J_i) = \alpha(I \cap (\vee_{i\in\Lambda} J_i)) = \alpha(\vee_{i\in\Lambda}(I \cap J_i)) = \vee_\alpha\{\alpha(I \cap J_i) \mid i \in \Lambda\} = \vee_\alpha\{(\alpha(I) \cap \alpha(J_i)) \mid i \in \Lambda\}$. Therefore, $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha)$ is a frame.

(2) From Theorem 2.5(2) it follows that for any $I \in \mathcal{I}(L)$, $I = \vee_{i=1}^n \langle x_i\rangle$. By the definition of the join operation $\vee_\alpha$ on $\mathcal{I}_\alpha(L)$, we have $\alpha(I) = \alpha(\vee_{i=1}^n \langle x_i\rangle) = \alpha(x_1) \vee_\alpha \alpha(x_2) \vee_\alpha \cdots \vee_\alpha \alpha(x_n)$. Thus, $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, \{0\}, L)$ is an algebraic lattice. Moreover, a compact element of $\mathcal{I}_\alpha(L)$ is exactly of the form $\alpha(x)$, for $x \in L$. □

It is well-known that a lattice is a frame if and only if it is a complete Heyting algebra. $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, \to_\alpha, \{0\}, L)$ is a complete Heyting algebra, where $I \to_\alpha J = \bigvee\{H \in \mathcal{I}_\alpha(L) \mid I \cap H \subseteq J\}$, for all $I, J \in \mathcal{I}_\alpha(L)$.

Theorem 3.8. Let L be a De Morgan residuated lattice. Then $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, ^*, \{1\}, L)$ is a Boolean algebra if and only if for all $I \in \mathcal{I}_\alpha(L)$, there exist $x \in I$ and $y \in I^\perp$ such that $x^\perp \cap y^\perp = \{0\}$, where $(\alpha(I))^* = I^\perp$.

Proof. Suppose that $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, ^*, \{0\}, L)$ is a Boolean algebra and $I \in \mathcal{I}_\alpha(L)$. Since $I^\perp$ is the complement of $\alpha(I)$, then $L = \alpha(I) \vee_\alpha I^\perp = \alpha(I) \vee_\alpha \alpha(I^\perp) = \alpha(I \vee I^\perp)$. Thus there exists $a \in I \vee I^\perp$ such that $a^\perp \subseteq 1^\perp = \{0\}$. From $a \in I \vee I^\perp$ it follows that there are $x \in I$ and $y \in I^\perp$ such that $a \leq x \boxplus y$. By Lemma 2.8(2) and (3), we have $x^\perp \cap y^\perp = (x \boxplus y)^\perp \subseteq a^\perp = \{0\}$. Conversely, assume that for all $I \in \mathcal{I}_\alpha(L)$, there are $x \in I$ and $y \in I^\perp$ such that $x^\perp \cap y^\perp = \{0\}$. So $x \boxplus y \in I \vee I^\perp$ and $(x \boxplus y)^\perp = x^\perp \cap y^\perp = \{0\}$. Since $1^\perp = \{0\}$, we get $(x \boxplus y)^\perp = 1^\perp$, which implies that $1 \in \alpha(I \vee I^\perp)$. Hence $\alpha(I) \vee_\alpha I^\perp = \alpha(I \vee I^\perp) = L$. Clearly, $\alpha(I) \cap I^\perp = \alpha(0) = \{0\}$. This means that $I^\perp$ is the complement of $\alpha(I)$. Therefore, $(\mathcal{I}_\alpha(L), \cap, \vee_\alpha, ^*, \{0\}, L)$ is a Boolean algebra. □

We will establish the following prime α-ideal theorem for De Morgan residuated lattices.

Theorem 3.9. (Prime α-ideal theorem) Let L be a De Morgan residuated lattice and I an α-ideal of L. If S is a $\wedge$-closed subset of L such that $I \cap S = \emptyset$, then there exists a prime α-ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$.

Proof. Denote $\mathcal{I}_I = \{J \in \mathcal{I}_\alpha(L) \mid I \subseteq J \text{ and } J \cap S = \emptyset\}$. Since $I \in \mathcal{I}_I$, then $\mathcal{I}_I \neq \emptyset$. Clearly, $(\mathcal{I}_I, \subseteq)$ is a partially ordered set. By Zorn's Lemma, there is a maximal element P in $\mathcal{I}_I$. We now prove that P is a prime ideal of L. Suppose that $x \wedge y \in P$ and consider $x \notin P$ and $y \notin P$. From the maximality of P, we have $\alpha(P, x) \notin \mathcal{I}_I$ and $\alpha(P, y) \notin \mathcal{I}_I$, which means that $\alpha(P, x) \cap S \neq \emptyset$ and $\alpha(P, y) \cap S \neq \emptyset$. Then there exist $a \in \alpha(P, x) \cap S$ and $b \in \alpha(P, y) \cap S$. By Proposition 3.3(5), we have $a \wedge b \in \alpha(P, x) \cap \alpha(P, y) = \alpha(P, x \wedge y)$. From $x \wedge y \in P$, we have $a \wedge b \in \alpha(P, x \wedge y) = \alpha(P) = P$. Hence $a \wedge b \in P \cap S$, a contradiction. This shows that P is a prime α-ideal of L. □

We denote by $\mathcal{PI}_\alpha(L)$ the set of prime α-ideals of L. Some properties of prime α-ideals of L will be shown in the following proposition.

Proposition 3.10. Let L be a De Morgan residuated lattice and I an proper α-ideal of L. Then for all $X \subseteq L$ and $a \in L$:

(1) If $a \notin I$, then there is $P \in \mathcal{PI}_\alpha(L)$ such that $I \subseteq P$ and $a \notin P$;
(2) if $X \nsubseteq I$, then there is $P \in \mathcal{PI}_\alpha(L)$ such that $I \subseteq P$ and $X \nsubseteq P$;
(3) $I = \bigcap\{P \in \mathcal{PI}_\alpha(L) \mid I \subseteq P\}$;
(4) $\bigcap \mathcal{PI}_\alpha(L) = \{0\}$.

Proof. (1) Consider the set $S_a = \{a\}$ for $a \in L$. It is evident that S_a is a closed $\wedge$-subset of L. From Theorem 3.9 it follows that there is a prime α-ideal P of L such that $I \subseteq P$ and $a \notin P$.

(2) If $X \nsubseteq I$, then there is $a \in X$ and $a \notin I$. By (1), there is a prime α-ideal P of L such that $a \notin P$. Thus $X \nsubseteq P$.

(3) Obviously, $I \subseteq \bigcap\{P \in \mathcal{PI}_\alpha(L) \mid I \subseteq P\}$. Conversely, suppose that $a \notin I$. By (1), there is a prime α-ideals P of L such that $I \subseteq P$ and $a \notin P$. So $a \notin \bigcap\{P \in \mathcal{PI}_\alpha(L) \mid I \subseteq P\}$. Thus $\bigcap\{P \in \mathcal{PI}_\alpha(L) \mid I \subseteq P\} \subseteq I$.

(4) Take $I = \{0\}$. The result holds by (3). □

4. QUASI-COMPLEMENTED DE MORGAN RESIDUATED LATTICES

By means of annihilator operators, the notion of quasi-complemented De Morgan residuated lattices will be introduced. Some equivalent characterizations of quasi-complemented De Morgan residuated lattices will be shown in terms of prime ideals and α-ideals.

Definition 4.1. Let L be a De Morgan residuated lattice. L is called quasi-complemented if for all $x \in L$, $x^{\perp\perp} = y^\perp$, for some $y \in L$.

The following lemma provides a sufficient condition for a De Morgan residuated lattice to be quasi-complemented.

Lemma 4.2. Let L be a De Morgan residuated lattice. If any annulet of L is a principal ideal, then L is quasi-complemented.

Proof. For $x \in L$, by hypothesis we have $x^{\perp} = \langle y \rangle$, for some $y \in L$, so $x^{\perp\perp} = y^{\perp}$. This means that L is quasi-complemented. □

Notice that any ideal of a finite De Morgan residuated lattice is always principal, so any annulet in a finite De Morgan residuated lattice is also principal. The above lemma ensures that any finite De Morgan residuated lattice is quasi-complemented.

Proposition 4.3. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;

(2) for all $x \in L$, there is $y \in L$ such that $x \wedge y = 0$ and $x \boxplus y$ is dense;

(3) $\mathcal{A}_0(L)$ is a Boolean lattice.

Proof. (1) $\Rightarrow$ (2) Let L be quasi-complemented. For $x \in L$, there exists $y \in L$ such that $x^{\perp\perp} = y^{\perp}$. Applying Lemma 2.8(3) and (7), it follows that $(x \boxplus y)^{\perp} = x^{\perp} \cap y^{\perp} = x^{\perp} \cap x^{\perp\perp} = \{0\}$. From $x \in x^{\perp\perp} = y^{\perp}$, it follows that $x \wedge y = 0$.

(2) $\Rightarrow$ (3) Applying Lemmas 2.8(3) and 2.9(2), for all $x, y \in L$, $x^{\perp} \cap y^{\perp} = (x \boxplus y)^{\perp} = \{0\}$ and $x^{\perp} \vee_{\mathcal{A}} y^{\perp} = (x \wedge y)^{\perp} = 0^{\perp} = L$. Hence $\mathcal{A}_0(L)$ is a Boolean lattice.

(3) $\Rightarrow$ (1) For all $x \in L$, there is $y \in L$ such that $x^{\perp} \cap y^{\perp} = \{0\}$ and $x^{\perp} \vee_{\mathcal{A}} y^{\perp} = L$. By Lemma 2.8(7), we have $y^{\perp} \subseteq x^{\perp\perp}$ and so $x^{\perp\perp} = x^{\perp\perp} \cap L = x^{\perp\perp} \cap (x^{\perp} \vee_{\mathcal{A}} y^{\perp}) = (x^{\perp\perp} \cap x^{\perp}) \vee_{\mathcal{A}} (x^{\perp\perp} \cap y^{\perp}) = x^{\perp\perp} \cap y^{\perp} \subseteq y^{\perp}$. Hence $x^{\perp\perp} = y^{\perp}$. □

Proposition 4.4. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;

(2) any prime ideal having no dense element is a minimal prime ideal;

(3) any ideal having no dense element is contained in a minimal prime ideal.

Proof. (1) $\Rightarrow$ (2) Let P be a prime ideal containing no dense element. Suppose that $x \in P$. Since L is quasi-complemented, by Proposition 4.3(2) we have $x \wedge y = 0$ and $(x \boxplus y)^{\perp} = \{0\}$, for some $y \in L$. From $(x \boxplus y)^{\perp} = \{0\}$, it follows that $x \boxplus y \notin P$ as P has no dense element. Since P is an ideal and $x \in P$, we get $y \notin P$, which means that $x \in 0(P)$ due to $x \wedge y = 0$. By Theorem 2.16(2), P is minimal.

(2) $\Rightarrow$ (3) This follows from Proposition 2.13(3).

(3) $\Rightarrow$ (1) For all $x \in L$, from Theorem 2.16(3) it follows that $\langle x \rangle \vee x^{\perp}$ can not be contained in any minimal prime ideal and so $\langle x \rangle \vee x^{\perp}$ contains a dense element d. This implies that $d \leq a \boxplus b$, for $a \in \langle x \rangle$ and $b \in x^{\perp}$. By Lemma 2.8(2) and (3), we have $(a \boxplus b)^{\perp} \subseteq d^{\perp} = \{0\}$, so $(a \boxplus b)^{\perp} = \{0\}$, which means that $a^{\perp} \cap b^{\perp} = (a \boxplus b)^{\perp} = \{0\}$, so $b^{\perp} \subseteq a^{\perp\perp}$. From $a \in \langle x \rangle$, we have $a^{\perp\perp} \subseteq x^{\perp\perp}$, which implies that $b^{\perp} \subseteq x^{\perp\perp}$. Applying Proposition 2.7(2), $x^{\perp\perp} \subseteq b^{\perp}$ as $b \in x^{\perp}$. Hence $x^{\perp\perp} = b^{\perp}$ and so L is quasi-complemented. □

In the subsequent proposition, some necessary and sufficient conditions are given for an α-ideal to be an annulet.

Proposition 4.5. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) $I_\alpha(L)$ possesses the unique dense element L itself;
(2) any proper α-ideal is non-dense;
(3) any dense ideal contains a dense element;
(4) any α-ideal is an annulet;
(5) for any proper α-ideal $I \neq \{0\}$ of L, there exists a proper α-ideal $J \neq \{0\}$ of L such that $I \cap J = \{0\}$, that is, $\mathcal{I}_\alpha(L)$ is semi-complemented.

Moreover, one of the above conditions implies that a De Morgan residuated lattice L is quasi-complemented.

P r o o f. It is clear that (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) Let I be a dense ideal of L. Then $I^\perp = \{0\}$. Since $I \subseteq \alpha(I)$, by Proposition 2.7(2), we have $(\alpha(I))^\perp \subseteq I^\perp = \{0\}$. By hypothesis, we get $\alpha(I) = L$. It follows from Proposition 3.6(1) that $a^\perp = \{0\}$ for some $a \in I$.

(3) $\Rightarrow$ (4) Suppose that I is an α-ideal of L. From $I, I^\perp \subseteq I \vee I^\perp$, we have $(I \vee I^\perp)^\perp \subseteq I^\perp \cap I^{\perp\perp} = \{0\}$. This means that $I \vee I^\perp$ is a dense ideal and so it has a dense element. Assume that $x \in L$ is a dense element of $I \vee I^\perp$. Then $x \leq a \boxplus b$, for some $a \in I$ and $b \in I^\perp$. Applying Lemma 2.8(2) and (3), we have $a^\perp \cap b^\perp = (a \boxplus b)^\perp \subseteq x^\perp = \{0\}$. It follows that $b^\perp \subseteq a^{\perp\perp} \subseteq I$. Applying Proposition 2.7(2), from $b \in I^\perp$, it follows that $I^{\perp\perp} \subseteq b^\perp$. Thus $I = b^\perp$ and so I is an annulet of L.

(4) $\Rightarrow$ (5) It is obvious by Lemma 2.8(6).

(5) $\Rightarrow$ (1) Since $L^\perp = \{0\}$, we have L is a dense element of $\mathcal{I}_\alpha(L)$. Suppose that I is a proper dense α-ideal. From (5), it follows that there exists a proper α-ideal J such that $I \cap J = \{0\}$. Hence $J \subseteq I^\perp = \{0\}$ and so $J = \{0\}$, a contradiction.

We now prove that L is quasi-complemented. Let L satisfy condition (1). Set $I = x^\perp \vee x^{\perp\perp}$, for all $x \in L$. Then $I^\perp = (x^\perp \vee x^{\perp\perp})^\perp = x^{\perp\perp} \cap (x^{\perp\perp})^\perp = x^{\perp\perp} \cap x^\perp = \{0\}$. From $I \subseteq \alpha(I)$, it follows that $(\alpha(I))^\perp \subseteq I^\perp$ and so $(\alpha(I))^\perp = \{0\}$. It means that $\alpha(I)$ is a dense ideal. Hence $\alpha(I) = L$, which implies that $y^\perp = \{0\}$, for some $y \in I$. From $I = x^\perp \vee x^{\perp\perp}$, it follows that there exist $a \in x^\perp$ and $b \in x^{\perp\perp}$ such that $y \leq a \boxplus b$. By Lemma 2.8(2) and (3), we have $a^\perp \cap b^\perp = (a \boxplus b)^\perp \subseteq y^\perp = \{0\}$, which implies that $a^\perp \subseteq b^{\perp\perp}$. Since $b \in x^{\perp\perp}$, by Proposition 2.7(2) we get $b^{\perp\perp} \subseteq x^{\perp\perp}$. So $a^\perp \subseteq x^{\perp\perp}$. Since $a \in x^\perp$, by Proposition 2.7(2) we have $x^{\perp\perp} \subseteq a^\perp$. We conclude that $x^{\perp\perp} = a^\perp$, which means that L is quasi-complemented. □

Definition 4.6. Let L be a De Morgan residuated lattice. L is called weakly disjunctive provided that $x^\perp = y^\perp$ implies $\langle x \rangle = \langle y \rangle$, for all $x, y \in L$.

Weakly disjunctive De Morgan residuated lattices will be characterized by means of α-ideals in the following proposition.

Proposition 4.7. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is weakly disjunctive;
(2) any ideal of L is an α-ideal;
(3) any prime ideal of L is an α-ideal.

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of L. Suppose that $x^{\perp} = y^{\perp}$ and $x \in I$, for all $x, y \in L$. Since L is weakly disjunctive, we get $\langle x \rangle = \langle y \rangle$ and so $y \in I$ by $x \in I$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Assume that $x^{\perp} = y^{\perp}$ but $\langle x \rangle \neq \langle y \rangle$, for some $x, y \in L$. Without loss of generality, suppose that $\langle x \rangle \nsubseteq \langle y \rangle$. Let $\Sigma = \{I \in \mathcal{I}(L) \mid y \in I, x \notin I\}$. obviously, $\Sigma \neq \emptyset$ due to $\langle y \rangle \in \Sigma$ and Σ satisfies the conditions of Zorn's lemma. Hence, Σ contains a maximal element P. We now prove that P is prime. Suppose that $a \wedge b \in P$ and $a, b \notin P$. By the maximality of P, we have $\langle P, a \rangle \notin \Sigma$ and $\langle P, b \rangle \notin \Sigma$. This states that $x \in \langle P, a \rangle \cap \langle P, b \rangle$. It follows that $x \in \langle P, a \rangle \cap \langle P, b \rangle = \langle P, a \wedge b \rangle = P$, a contradiction. Thus P is a prime ideal. By hypothesis we have $x \in x^{\perp\perp} = y^{\perp\perp} \subseteq P$, which contradicts the fact $P \in \Sigma$. □

Proposition 4.8. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented and weakly disjunctive;

(2) The lattice of all principal ideals is a Boolean lattice.

Proof. (1) $\Rightarrow$ (2) Let L be quasi-complemented and weakly disjunctive. By Proposition 4.3(2), for all $x \in L$, $x \wedge y = 0$ and $(x \boxplus y)^{\perp} = \{0\}$, for some $y \in L$. Thus $\langle x \rangle \cap \langle y \rangle = \langle x \wedge y \rangle = \{0\}$. Since L is weakly disjunctive, by $1^{\perp} = \{0\}$, we obtain that $\langle x \boxplus y \rangle = \langle 1 \rangle = L$ and so $\langle x \rangle \vee \langle y \rangle = L$.

(2) $\Rightarrow$ (1) Suppose that for all $x \in L$, $\langle x \rangle^* = \langle y \rangle$, for some $y \in L$. Then $\langle x \rangle \cap \langle y \rangle = \{0\}$ and $\langle x \rangle \vee \langle y \rangle = L$. This implies that $\langle y \rangle \subseteq (\langle x \rangle)^{\perp} = x^{\perp}$ and $x^{\perp} = x^{\perp} \cap L = x^{\perp} \cap (\langle x \rangle \vee \langle y \rangle) = (x^{\perp} \cap \langle x \rangle) \vee (x^{\perp} \cap \langle y \rangle) = \{0\} \vee (x^{\perp} \cap \langle y \rangle) = x^{\perp} \cap \langle y \rangle \subseteq \langle y \rangle$. Thus $\langle y \rangle = x^{\perp}$ and so $\langle x \rangle^* = x^{\perp}$. Hence $(x^{\perp})^{\perp} = (\langle y \rangle)^{\perp} = y^{\perp}$, which means that L is quasi-complemented. Assume that $x^{\perp} = y^{\perp}$, that is, $\langle x \rangle^* = \langle y \rangle^*$. Then $\langle y \rangle = \langle x \rangle$. This shows that L is weakly disjunctive. □

For any subset A of a De Morgan residuated lattice L, we define $O(A) = \{x \in L \mid x \wedge a = 0, \text{ for some } a \in A\}$. Obviously, $O(A) = \{x \in L \mid x \in a^{\perp}, \text{ for some } a \in A\} = \bigcup_{a \in A} a^{\perp}$. Moreover, if A is a $\wedge$-closed subset of L, then $O(A)$ is an ideal of L. We shall give the notion of O-ideals in a De Morgan residuated lattice in the following.

Definition 4.9. Let L be a De Morgan residuated lattice. An ideal of L is called an O-ideal provided that $I = O(F)$, for some lattice filter F of L.

Lemma 4.10. Let L be a De Morgan residuated lattice. Then:

(1) Any O-ideal of L is an α-ideal;

(2) any minimal prime ideal of L is an α-ideal;

(3) any proper O-ideal of L contains no dense element.

Proof. (1) Let I be an O-ideal of L. Then $I = O(F)$, for some lattice filter F of L. If $x \in I$, then $x \in O(F)$, which implies that $x \in a^{\perp}$, for some $a \in F$. By Proposition 2.7(2), $x^{\perp\perp} \subseteq a^{\perp\perp\perp} = a^{\perp} \subseteq O(F) = I$. This means that I is an α-ideal of L.

(2) Let M be a minimal prime ideal. Applying Theorem 2.16(2), we have $M = 0(M) = O(L \setminus M)$. By (1), we have M is an α-ideal.

(3) Let I be a proper O-ideal of L. Then $I = O(F)$, for some lattice filter F of L. Suppose that I contains a dense element d. Hence $d \in a^{\perp}$, for some $a \in F$, which implies that $a \in d^{\perp} = \{0\}$ and so $0 \in F$. It follows that $I = L$, a contradiction. □

Some equivalent conditions will be given for an α-ideal to be an O-ideal in a De Morgan residuated lattice in the following proposition.

Proposition 4.11. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;
(2) any α-ideal is an O-ideal;
(3) any annihilator is an O-ideal;
(4) for all $x \in L$, $x^{\perp\perp}$ is an O-ideal.

Proof. (1) $\Rightarrow$ (2) Suppose that I is an α-ideal of L. Set $F_I = \{x \in L \mid x^{\perp} \subseteq a^{\perp\perp}$, for some $a \in I\}$. We now prove that F_I is a lattice filter. Since $1^{\perp} = 0^{\perp\perp} = \{0\}$ and $0 \in I$, then $1 \in F_I$. Let $x \leq y$ and $x \in F_I$. Then there is $a \in I$ such that $x^{\perp} \subseteq a^{\perp\perp}$. From $x \leq y$ and Lemma 2.8(2), it follows that $y^{\perp} \subseteq x^{\perp}$ and so $y^{\perp} \subseteq a^{\perp\perp}$. Thus $y \in F_I$. Let $x, y \in F_I$. Then there are $a, b \in I$ such that $x^{\perp} \subseteq a^{\perp\perp}$ and $y^{\perp} \subseteq b^{\perp\perp}$, so $a^{\perp} \subseteq x^{\perp\perp}$ and $b^{\perp} \subseteq y^{\perp\perp}$. Hence $a^{\perp} \cap b^{\perp} \subseteq x^{\perp\perp} \cap y^{\perp\perp}$. From Lemmas 2.8(3) and 2.9(1) it follows that $(a \boxplus b)^{\perp} \subseteq (x \wedge y)^{\perp\perp}$ and so $(x \wedge y)^{\perp} \subseteq (a \boxplus b)^{\perp\perp}$. This states that $x \wedge y \in F_I$ due to $a \boxplus b \in I$. We conclude that F_I is a lattice filter.

We next prove that $I = O(F_I)$. Let $x \in I$. Since L is quasi-complemented, we obtain that $x^{\perp\perp} = a^{\perp}$ for some $a \in L$. So $a \in F_I$ and $x \in a^{\perp} \subseteq O(F_I)$. Hence $I \subseteq O(F_I)$. Conversely, suppose that $x \in O(F_I)$. Then there is $a \in F_I$ such that $x \in a^{\perp}$. From $a \in F_I$ it follows that $a^{\perp} \subseteq b^{\perp\perp}$, for some $b \in I$. So $x \in b^{\perp\perp}$. Since I is an α-ideal, we have $x \in b^{\perp\perp} \subseteq I$. Thus $O(F_I) \subseteq I$. We conclude that $I = O(F_I)$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are evident.

(4) $\Rightarrow$ (1) For any $x \in L$, there is a lattice filter F such that $x^{\perp\perp} = O(F)$, so $x \in O(F)$. It follows that $x \in a^{\perp}$, for some $a \in F$. Applying Proposition 2.7(2) and (3), we get $x^{\perp\perp} \subseteq a^{\perp\perp\perp} = a^{\perp} \subseteq O(F) = x^{\perp\perp}$. This means that $x^{\perp\perp} = a^{\perp}$, for some $a \in F$. Therefore, L is quasi-complemented. □

Lemma 4.12. Let L be quasi-complemented and P a prime ideal of L. Then the following are equivalent:

(1) P is an α-ideal of L;
(2) P contains no dense element;
(3) P is a minimal prime ideal;
(4) P contains exactly one of $x, y \in L$ such that $x^{\perp\perp} = y^{\perp}$.

Proof. (1) $\Rightarrow$ (2) Since P is a prime α-ideal of L, we have $\alpha(P) = P \neq L$. By Proposition 3.6(2), we have that P contains no dense element.

(2) $\Rightarrow$ (3) It follows from Proposition 4.4(2).

(3) $\Rightarrow$ (4) Suppose that $x^{\perp\perp} = y^{\perp}$, for all $x, y \in L$. Then $x \in y^{\perp}$ and so $x \wedge y = 0$. Since P is prime and $0 \in P$, then $x \in P$ or $y \in P$. From $x^{\perp} \cap x^{\perp\perp} = \{0\}$, it follows

that $x^\perp \cap y^\perp = \{0\} \subseteq P$, so $x^\perp \subseteq P$ or $y^\perp \subseteq P$. It follows from Theorem 2.16(3) that P contains exactly one of x and y.

(4) $\Rightarrow$ (1) Let $x \in P$. By (4), we have $x^{\perp\perp} = y^\perp$, for some $y \notin P$. Hence $y^\perp \subseteq P$ and so $x^{\perp\perp} \subseteq P$. □

We shall characterize quasi-complemented residuated lattices in terms of α-ideals in De Morgan residuated lattices.

Corollary 4.13. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;

(2) any prime α-ideal is a minimal prime ideal, that is, $\mathcal{PI}_\alpha(L) = \mathcal{MI}(L)$;

(3) any proper α-ideal is the intersection of minimal prime ideals containing it;

(4) any proper α-ideal is contained in a minimal prime ideal.

Proof. (1) $\Rightarrow$ (2) This follows from Lemmas 4.10(2) and 4.12(3).

(2) $\Rightarrow$ (3) This follows from Proposition 3.10(3).

(3) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (1) Let I be an ideal of L containing no dense element. By Proposition 3.6(2), $\alpha(I)$ is proper and so, by hypothesis, it is contained in a minimal prime ideal M. From $I \subseteq \alpha(I)$, it follows that I is contained in the minimal prime ideal M. From Proposition 4.4(3) it follows that L is quasi-complemented. □

In a De Morgan residuated lattice L, for any $x \in L$, $\alpha(x)$ is called the principal α-ideal of L generated by x. We denote by $Prin_\alpha(L) = \{\alpha(x) \mid x \in L\}$ the set of all principal α-ideals. Several equivalent conditions will be shown for an α-ideal to be a principal α-ideal in the following.

Lemma 4.14. Let L be a De Morgan residuated lattice and I an α-ideal of L. If $I \subseteq \alpha(x)$ for some $x \in I$, then $I = \alpha(x)$.

Proof. Let $x \in I$. By Proposition 3.2(4), we get $\alpha(x) \subseteq \alpha(I) = I$. From $I \subseteq \alpha(x)$, we get $I = \alpha(x)$. □

Proposition 4.15. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) For any α-ideal I and $\mathcal{P} \subseteq \mathcal{PI}_\alpha(L)$, $I \subseteq \cup\mathcal{P}$ implies $I \subseteq P$, for some $P \in \mathcal{P}$;

(2) for any $P \in \mathcal{PI}_\alpha(L)$ and $\mathcal{P} \subseteq \mathcal{PI}_\alpha(L)$, $P \subseteq \cup\mathcal{P}$ implies $P \subseteq Q$, for some $Q \in \mathcal{P}$;

(3) any prime α-ideal is principal;

(4) any α-ideal is principal.

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (3) Let P be a prime α-ideal that is not a principal α-ideal. By Lemma 4.14 and Proposition 3.10(3), for any $x \in P$, there is a prime α-ideal P_x such that $P \nsubseteq P_x$, which means that $P \subseteq \cup\{P_x \mid x \in P\}$. By (2), $P \subseteq P_a$ for some $a \in P$, a contradiction.

(3) $\Rightarrow$ (4) Since $L = \alpha(1)$, it follows that L is a principal α-ideal. Set $\Gamma = \{L \neq I \in \mathcal{I}_\alpha(L) \mid I \neq \alpha(x), \text{ for all } x \in L\}$. We claim that Γ is empty. Suppose that $\Gamma \neq \emptyset$. If $\{I_i \mid i \in \Lambda\}$ is a chain in Γ, then $\cup\{I_i \mid i \in \Lambda\} \in \Gamma$ and it is an upper bound of $\{I_i \mid i \in \Lambda\}$. By Zorn's lemma, we get that Γ contains a maximal element M. Let $x \wedge y \in M$. Consider $x \notin M$ and $y \notin M$. By the maximality of M, we have $\alpha(M, x) \notin \Gamma$ and $\alpha(M, y) \notin \Gamma$. Hence $\alpha(M, x) = \alpha(a)$ and $\alpha(M, y) = \alpha(b)$, for some $a, b \in L$. Applying Proposition 3.3(5), $\alpha(M, x) \cap \alpha(M, y) = \alpha(M, x \wedge y) = M$. This means that $M = \alpha(a) \cap \alpha(b) = \alpha(\langle a \rangle \cap \langle b \rangle) = \alpha(a \wedge b)$, which contradicts the fact $M \in \Gamma$. Thus M is a prime α-ideal. By hypothesis, M is principal, a contradiction. Therefore, $\Gamma = \emptyset$.

(4) $\Rightarrow$ (1) Let I be an α-ideal such that $I \subseteq \cup\mathcal{P}$, for some $\mathcal{P} \subseteq \mathcal{PI}_\alpha(L)$. By hypothesis, $I = \alpha(a)$, for some $a \in L$, which implies that $a \in P$, for some $P \in \mathcal{P}$. Hence $I \subseteq P$. □

Lemma 4.16. Let L be a De Morgan residuated lattice and P a non-dense prime ideal of L. Then P is an annulet.

Proof. Let P be a non-dense prime ideal of L. Then $P^\perp \neq \{0\}$ and so there is $0 \neq a \in P^\perp$. Applying Proposition 2.7(2) and (3), $P \subseteq P^{\perp\perp} \subseteq a^\perp$. Let $x \in a^\perp$. Since P is prime and $x \wedge a = 0 \in P$, we deduce that $x \in P$ or $a \in P$. From $a \in P^\perp$ and Proposition 2.7(8), it follows that $a \notin P$ and so $x \in P$. This yields that $a^\perp \subseteq P$. We conclude that $P = a^\perp$. □

Proposition 4.17. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) Any α-ideal is principal;
(2) any O-ideal is principal;
(3) any annulet is principal and any minimal prime ideal is non-dense;
(4) any prime α-ideal is principal.

Proof. (1) $\Rightarrow$ (2) This follows from hypothesis and Lemma 4.10(1).

(2) $\Rightarrow$ (3) Since $x^\perp = O(\{x\})$ for all $x \in L$, by hypothesis we have $x^\perp$ is a principal ideal. Let M be a minimal prime ideal. Then M is an O-ideal and so, by hypothesis, $M = \alpha(a)$ for some $a \in L$. Assume that M is dense. Then $\{0\} = M^\perp = (\alpha(a))^\perp = a^\perp$, which contradicts Lemma 4.10(3).

(3) $\Rightarrow$ (4) Let P be a prime α-ideal. By Lemma 4.2 and Corollary 4.13, P is a minimal prime α-ideal, which implies that P is non-dense. By Lemma 4.16, P is an annulet. It follows that P is principal by hypothesis.

(4) $\Rightarrow$ (1) This follows from Proposition 3.10(3). □

5. THE HULL-KERNEL TOPOLOGY ON PRIME α-IDEALS

In this section, we investigate the hull-kernel topology (or the spectral topology) on the set of all prime α-ideals in a De Morgan residuated lattice. Let L be a De Morgan residuated lattice. By $\mathcal{PI}_\alpha(L)$ we mean the set of all prime α-ideals of L. For a subset X of L, we define: $H(X) = \{P \in \mathcal{PI}_\alpha(L) | X \subseteq P\}$ and $D(X) = \mathcal{PI}_\alpha(L) \setminus H(X) = \{P \in \mathcal{PI}_\alpha(L) | X \nsubseteq P\}$.

Proposition 5.1. Let L be a De Morgan residuated lattice. Then the following hold for $X, Y \subseteq L$:

(1) $X \subseteq Y$ implies $H(Y) \subseteq H(X)$;
(2) $H(X) = H(\langle X\rangle) = H(\alpha(X))$;
(3) $H(\emptyset) = H(\{0\}) = \mathcal{PI}_\alpha(L)$;
(4) $H(X) = \emptyset$ iff $\alpha(X) = L$ iff $\langle X\rangle$ contains a dense element. In particular, $H(L) = H(1) = \emptyset$;
(5) $H(X) = \mathcal{PI}_\alpha(L)$ iff $X = \emptyset$ or $X = \{0\}$;
(6) if $\{X_i\}_{i\in\Lambda}$ is any family of subsets of L, then $\bigcap_{i\in\Lambda} H(X_i) = H(\bigcup_{i\in\Lambda} X_i)$;
(7) $H(X) \cup H(Y) = H(\langle X\rangle \cap \langle Y\rangle)$. In particular, $H(X) \cup H(X^\perp) = \mathcal{PI}_\alpha(L)$;
(8) $H(X) \subseteq H(Y)$ iff $\alpha(Y) \subseteq \alpha(X)$;
(9) $H(X) = H(Y)$ iff $\alpha(X) = \alpha(Y)$.

Proof. (1) Suppose that $P \in H(Y)$. Then $Y \subseteq P$. From $X \subseteq Y$, it follows that $X \subseteq P$ and so $P \in H(X)$.

(2) From $X \subseteq \langle X\rangle \subseteq \alpha(X)$, by (1), we have $H(\alpha(X)) \subseteq H(\langle X\rangle) \subseteq H(X)$. Conversely, if $P \in H(X)$, then $X \subseteq P$ and so $\alpha(X) \subseteq P$. Thus $P \in H(\alpha(X))$. This yields that $H(X) \subseteq H(\alpha(X))$. We conclude that $H(X) = H(\langle X\rangle) = H(\alpha(X))$.

(3) Since $0 \in P$ and $\emptyset \subseteq P$, we have $H(\emptyset) = H(\{0\}) = \mathcal{PI}_\alpha(L)$.

(4) We first prove $H(X) = \emptyset$ iff $\alpha(X) = L$. Let $\alpha(X) = L$. Then, by (2), $H(X) = \emptyset$. Conversely, suppose that $H(X) = \emptyset$ and $\alpha(X) \neq L$. By Proposition 3.10(3), there is a prime α-ideal of L such that $\alpha(X) \subseteq P$. Hence $P \in H(X)$, a contradiction.

Next, it is trivial that $\alpha(X) = L$ iff $\langle X\rangle$ by Proposition 3.6(1).

(5) If $X = \emptyset$ or $X = \{0\}$, then, by (3), we have $H(X) = \mathcal{PI}_\alpha(L)$. Conversely, if $H(X) = \mathcal{PI}_\alpha(L)$, then $P \in H(X)$, for any $P \in \mathcal{PI}_\alpha(L)$ and so $X \subseteq P$, for any $P \in \mathcal{PI}_\alpha(L)$. By Proposition 3.10(4), we have $\alpha(X) = \{0\}$. From $X \subseteq \alpha(X)$, it follows that $X = \{0\}$ or $X = \emptyset$.

(6) Since $X_i \subseteq \bigcup_{i\in\Lambda} X_i$, by (1), we have $H(\bigcup_{i\in\Lambda} X_i) \subseteq \bigcap_{i\in\Lambda} H(X_i)$. For any $P \in \bigcap_{i\in\Lambda} H(X_i)$, we have $P \in H(X_i)$, for all $i \in \Lambda$ and so $\bigcup_{i\in\Lambda} X_i \subseteq P$. This means that $P \in H(\bigcup_{i\in\Lambda} X_i)$. Hence $\bigcap_{i\in\Lambda} H(X_i) \subseteq H(\bigcup_{i\in\Lambda} X_i)$. We conclude that $\bigcap_{i\in\Lambda} H(X_i) = H(\bigcup_{i\in\Lambda} X_i)$.

(7) Suppose that $P \in H(\langle X\rangle) \cup H(\langle Y\rangle)$, for $P \in \mathcal{I}_\alpha(L)$. Then $\langle X\rangle \subseteq P$ or $\langle Y\rangle \subseteq P$. This means that $\langle X\rangle \cap \langle Y\rangle \subseteq P$ and so $\alpha(\langle X\rangle \cap \langle Y\rangle) \subseteq P$. From (2), we have $P \in H(\alpha(\langle X\rangle \cap \langle Y\rangle)) = H(\langle X\rangle \cap \langle Y\rangle)$. Conversely, assume that $P \in H(\langle X\rangle \cap \langle Y\rangle)$. Then $\langle X\rangle \cap \langle Y\rangle \subseteq P$. So $\alpha(\langle X\rangle \cap \langle Y\rangle) \subseteq P$. From Proposition 2.10(2) and Proposition 3.3(4) it follows that $\alpha(X) \subseteq P$ or $\alpha(Y) \subseteq P$. By (2), $P \in H(\alpha(X)) = H(X)$ or $P \in H(\alpha(Y)) = H(Y)$. Therefore, $H(X) \cup H(Y) = H(\langle X\rangle \cap \langle Y\rangle)$.

(8) If $\alpha(Y) \subseteq \alpha(X)$, then, by (1), $H(\alpha(X)) \subseteq H(\alpha(Y))$. By (2), we have $H(X) \subseteq H(Y)$. Conversely, suppose that $H(X) \subseteq H(Y)$. Then $\bigcap H(Y) \subseteq \bigcap H(X)$. By Proposition 3.10(3), we have $\alpha(Y) = \bigcap H(Y) \subseteq \bigcap H(X) = \alpha(X)$.

(9) It follows from (8). □

From Proposition 5.1(3),(4),(6) and (7) it follows that $H(X)$ satisfies the axioms for closed sets in a topological space. If the set $\{H(x)|x \in L\}$ is taken as a closed base, the resulting topology is called the hull-kernel topology or the Zariski topology and denoted

by $\mathcal{T}_h$. $(\mathcal{PI}_\alpha(L), \mathcal{T}_h)$ is called the space of prime α-ideals. $D(X)$ is an open set and $D(x) = \{P \in \mathcal{PI}_\alpha(L) \mid x \notin P\}$ is called a basic open set of $\mathcal{PI}_\alpha(L)$.

The following corollary is a direct consequence of Proposition 5.1.

Corollary 5.2. Let L be a De Morgan residuated lattice. Then, for all $x, y \in L$:
(1) $H(x) = \emptyset$ iff $D(x) = \mathcal{PI}_\alpha(L)$ iff $\alpha(x) = L$ iff $x^\perp = \{0\}$;
(2) $H(x) = \mathcal{PI}_\alpha(L)$ iff $D(x) = \emptyset$ iff $x = 0$;
(3) if $x \leq y$, then $H(y) \subseteq H(x)$ and $D(x) \subseteq D(y)$;
(4) $H(y) \subseteq H(x)$ iff $D(x) \subseteq D(y)$ iff $\alpha(x) \subseteq \alpha(y)$;
(5) $D(x) \cap D(y) = D(x \wedge y)$ and $H(x) \cup H(y) = H(x \wedge y)$;
(6) $D(x) \cup D(y) = D(x \boxplus y)$ and $H(x) \cap H(y) = H(x \boxplus y)$;
(7) $H(x) \cup H(x^\perp) = \mathcal{PI}_\alpha(L)$ and $D(x) \cap D(x^\perp) = \emptyset$;
(8) $H(x) = H(x^{\perp\perp})$ and $D(x) = D(x^{\perp\perp})$;
(9) $D(x) \subseteq H(x^\perp)$ and $H(x) \supseteq D(x^\perp)$.

Proof. The proofs of conditions (1) – (7) are evident from Proposition 5.1. We only prove (8) and (9).

(8) If $P \in H(x)$, then $x \in P$. Since $P \in \mathcal{PI}_\alpha(L)$, we have $x^{\perp\perp} \subseteq P$ and so $P \in H(x^{\perp\perp})$. This implies that $H(x) \subseteq H(x^{\perp\perp})$. Conversely, if $P \in H(x^{\perp\perp})$, then $x^{\perp\perp} \subseteq P$. Since $x \in x^{\perp\perp}$, we have $P \in H(x)$. Thus $H(x^{\perp\perp}) \subseteq H(x)$. We conclude that $H(x) = H(x^{\perp\perp})$ and consequently $D(x) = D(x^{\perp\perp})$.

(9) If $P \in D(x)$, then $x \notin P$. Since P is prime, we have $x^\perp \subseteq P$, which means that $P \in H(x^\perp)$. Thus, $D(x) \subseteq H(x^\perp)$ and consequently $H(x) \supseteq D(x^\perp)$. □

Proposition 5.3. Let L be a De Morgan residuated lattice. Then the following are equivalent:
(1) C is an irreducible closed subset of $\mathcal{PI}_\alpha(L)$;
(2) $C = H(P)$, for some $P \in \mathcal{PI}_\alpha(L)$.

Proof. (1) ⇒ (2): Let C be an irreducible closed subset of $\mathcal{PI}_\alpha(L)$. Then $C = H(X)$, for some $X \subseteq L$. Let $a \wedge b \in \alpha(X)$, for $a, b \in L$. It follows from Proposition 5.1((1) and (2)) and Corollary 5.2(5) that $C = h(X) = h(\alpha(X)) \subseteq H(a \wedge b) = H(a) \cup H(b)$. By hypothesis, we have $C = H(X) \subseteq H(a)$ or $C = H(X) \subseteq H(b)$. Applying Proposition 5.1(2) and (8), we have $a \in \alpha(X)$ or $b \in \alpha(X)$. Hence, $\alpha(X)$ is a prime α-ideal.

(2) ⇒ (1): Let $C = H(P)$, for some $P \in \mathcal{PI}_\alpha(L)$ and $C = H(P) \subseteq H(X) \cup H(Y)$, for $X, Y \subseteq L$. By Proposition 5.1(7) and (8), we have $\alpha(X) \cap \alpha(Y) \subseteq P$. By Proposition 2.10(2), we have $\alpha(X) \subseteq P$ or $\alpha(Y) \subseteq P$. Using Proposition 5.1(8), $H(P) \subseteq H(X)$ or $H(P) \subseteq H(Y)$. Thus C is an irreducible closed subset of $\mathcal{PI}_\alpha(L)$. □

Proposition 5.4. Let L be a De Morgan residuated lattice. Then:
(1) The family $\mathcal{D}(L) = \{D(x) \mid x \in L\}$ is a base for open sets in $\mathcal{PI}_\alpha(L)$;
(2) $K^\circ(\mathcal{PI}_\alpha(L)) = \mathcal{D}(L)$, where $K^\circ(\mathcal{PI}_\alpha(L))$ is the set of all compact open sets in $\mathcal{PI}_\alpha(L)$.

Proof. (1) Since $D(X)$ is an open set of $\mathcal{PI}_\alpha(L)$, for some $X \subseteq L$, by Proposition 5.1(6) we have $D(X) = D(\bigcup_{x\in X}\{x\}) = \bigcup_{x\in X} D(x)$.

(2) Suppose that $D(a) \subseteq \bigcup_{x\in X} D(x) = D(X)$, for some $X \subseteq L$. It follows from Proposition 5.1(8) that $\alpha(a) \subseteq \alpha(X)$. Hence $a \in \alpha(X)$, which means that $a \in (x_1 \boxplus \cdots \boxplus x_n)^{\perp\perp}$, for some $x_1, \cdots, x_n \in X$. By Corollary 5.2(6) and (8),

$$\begin{aligned} D(a) &\subseteq D((x_1 \boxplus \cdots \boxplus x_n)^{\perp\perp}) \\ &= D(x_1 \boxplus \cdots \boxplus x_n) \\ &= D(x_1) \bigcup D(x_2) \bigcup \cdots \bigcup D(x_n). \end{aligned}$$

This shows that $\mathcal{D}(L) \subseteq K^\circ(\mathcal{PI}_\alpha(L))$.

Conversely, suppose that $D(X) \in K^\circ(\mathcal{PI}_\alpha(L))$. Then $D(X) = \bigcup_{i=1}^n D(a_i)$, for some $a_i \in X$, $1 \leq i \leq n$, $n \geq 1$. By Corollary 5.2(6), $D(X) = D(a_1 \boxplus a_2 \cdots \boxplus a_n)$, which means that $D(X) \in \mathcal{D}(L)$. □

Theorem 5.5. Let L be a De Morgan residuated lattice. Then $\mathcal{PI}_\alpha(L)$ is a compact T_0 topological space.

Proof. It follows from Proposition 5.1(4) that $\mathcal{PI}_\alpha(L) = D(1)$. Using Proposition 5.4(2), $\mathcal{PI}_\alpha(L)$ is a compact space. Let P and Q be two distinct elements of $\mathcal{PI}_\alpha(L)$. Then $P \nsubseteq Q$ or $Q \nsubseteq P$. If $P \nsubseteq Q$, then there exists $a \in P$ such that $a \notin Q$, which implies $P \notin D(a)$ and $Q \in D(a)$. This shows that $\mathcal{PI}_\alpha(L)$ is a compact T_0 topological space. □

A topological space X is called Sober provided that every irreducible closed subset C of X has a unique generic point, i. e., $C = cl(c)$, for some $c \in C$. It is well-known that a generic point of an irreducible closed subset in a T_0 topological space is unique.

Theorem 5.6. Let L be a De Morgan residuated lattice. Then $\mathcal{PI}_\alpha(L)$ is a Sober space.

Proof. It is a direct consequence of Proposition 5.3 and Theorem 5.5. □

Following Hochster [10], a topological space $(X, \mathcal{T})$ is called spectral provided that the following conditions are satisfied: (1) X is a compact T_0 space; (2) X is a Sober space; (3) the set of compact basic open sets of X is a base of open subsets of X; (4) the set of compact basic open sets of X is closed under finite intersections.

Theorem 5.7. Let L be a De Morgan residuated lattice. Then $\mathcal{PI}_\alpha(L)$ is a spectral space.

Proof. It follows from Corollary 5.2(5), Proposition 5.4 and Theorems 5.5 and 5.6. □

Theorem 5.8. Let L be a De Morgan residuated lattice. Then $\mathcal{PI}_\alpha(L)$ is a T_1 space if and only if it is an antichain.

Proof. Suppose that $\mathcal{PI}_\alpha(L)$ is a T_1 space and P, Q are two distinct prime α-ideals. Then there exist basic open sets $D(x)$ and $D(y)$ such that $P \in D(x) \setminus D(y)$ and $Q \in D(y) \setminus D(x)$, respectively. Thus $x \in Q \setminus P$ and $y \in P \setminus Q$, which means that $P \nsubseteq Q$ and $Q \nsubseteq P$. Hence, $\mathcal{PI}_\alpha(L)$ is an antichain.

Conversely, assume that $\mathcal{PI}_\alpha(L)$ is antichain and P, Q are two distinct elements in $\mathcal{PI}_\alpha(L)$. Then $P \nsubseteq Q$ and $Q \nsubseteq P$, so there exist $x \in P \setminus Q$ and $y \in Q \setminus P$. This states that $P \in D(y) \setminus D(x)$ and $Q \in D(x) \setminus D(y)$. We conclude that $\mathcal{PI}_\alpha(L)$ is a T_1 space. □

Theorem 5.9. Let L be a De Morgan residuated lattice. Then the following conditions are equivalent:

(1) $\mathcal{PI}_\alpha(L)$ is a Hausdorff space

(2) for two distinct prime α-ideals P and Q, there are $a \notin P$ and $b \notin Q$ such that $a \wedge b = 0$.

(3) For any prime α-ideal P, P is the unique element of $\mathcal{PI}_\alpha(L)$ containing $0(P)$.

Proof. (1) ⇒ (2) Let $\mathcal{PI}_\alpha(L)$ be a Hausdorff space and P, Q be two distinct elements of $\mathcal{PI}_\alpha(L)$. Then there are open subsets $D(a)$ of P and $D(b)$ of Q such that $D(a) \cap D(b) = \emptyset$. From Corollary 5.2(2) and (5) we deduce that $a \notin P$ and $b \notin Q$ is such that $a \wedge b = 0$.

(2) ⇒ (3) Let $P, Q \in \mathcal{PI}_\alpha(L)$ be such that $P \neq Q$ and $0(P) \subseteq Q$. By hypothesis, there are $a \notin P$ and $b \notin Q$ such that $a \wedge b = 0$, which implies that $b \in 0(P)$ and so $b \in Q$, a contradiction.

(3) ⇒ (1) let P and Q be two distinct elements of $\mathcal{PI}_\alpha(L)$. Then $0(P) \nsubseteq Q$, which means that there exist $a \notin P$ and $b \notin Q$ such that $a \wedge b = 0$. From Corollary 5.2(2) and (5) it follows that $D(a) \cap D(b) = D(a \wedge b) = D(0) = \emptyset$. This shows that $\mathcal{PI}_\alpha(L)$ is a Hausdorff space. □

Theorem 5.10. Let L be a De Morgan residuated lattice and $x^\perp$ has a complement in $\mathcal{I}_\alpha(L)$, for all $x \in L$. Then the following are equivalent:

(1) $\mathcal{PI}_\alpha(L)$ is a Hausdorff space;

(2) For two distinct prime α-ideals P and Q, there are $a, b \in L$ such that $a^\perp \subseteq P$ and $b^\perp \subseteq Q$ and there is no $T \in \mathcal{PI}_\alpha(L)$ such that $a \wedge b \notin T$.

Proof. (1) ⇒ (2) Let $\mathcal{PI}_\alpha(L)$ be a Hausdorff space and P, Q be two distinct prime α-ideals. Then there exist two open sets $D(a)$ and $D(b)$ such that $P \in D(a)$ and $Q \in D(b)$ and $D(a) \cap D(b) = \emptyset$. By Corollary 5.2(9), we have $P \in H(a^\perp)$ and $Q \in H(b^\perp)$, which implies that $a^\perp \subseteq P$ and $b^\perp \subseteq Q$. Assume that there exists a prime α-ideal T such that $a \wedge b \notin T$. From Corollary 5.2(5) it follows that $T \in D(a \wedge b) = D(a) \cap D(b)$, which contradicts the fact that $D(a) \cap D(b) = \emptyset$.

(2) ⇒ (1): Let P and Q be two distinct elements of $\mathcal{PI}_\alpha(L)$. Then there are $a, b \in L$ such that $a^\perp \subseteq P$ and $b^\perp \subseteq Q$. Since $a^\perp$ has a complement in $\mathcal{I}_\alpha(L)$, we have $a^\perp \vee_\alpha a^{\perp\perp} = L$ and so $L = a^\perp \vee_\alpha a^{\perp\perp} \subseteq P \vee_\alpha a^{\perp\perp}$. Hence $a^{\perp\perp} \nsubseteq P$ and so $P \in D(a^{\perp\perp}) = D(a)$. Similarly, $Q \in D(b)$. We claim that $D(a) \cap D(b) = \emptyset$. Assume that $D(a) \cap D(b) \neq \emptyset$. Then there exists a prime α-ideal T such that $T \in D(a) \cap D(b) = D(a \wedge b)$. This implies that $a \wedge b \notin T$, a contradiction. Hence, $\mathcal{PI}_\alpha(L)$ is a Hausdorff space. □

A characterization will be given for the space $\mathcal{PI}_\alpha(L)$ to be a regular space.

Theorem 5.11. Let L be a De Morgan residuated lattice. Then $\mathcal{PI}_\alpha(L)$ is a regular space if and only if for any $P \in \mathcal{PI}_\alpha(L)$ and $x \notin P$, there exist an ideal I of L and $y \in L$ such that $P \in D(y) \subseteq H(I) \subseteq D(x)$.

Proof. Let $\mathcal{PI}_\alpha(L)$ be a regular space. Suppose that $P \in \mathcal{PI}_\alpha(L)$ and $x \notin P$. Then $P \in D(x)$ and so $P \notin H(x)$. Since $\mathcal{PI}_\alpha(L)$ is a regular space, there are two disjoint open sets $D(y)$ and $D(I)$ such that $P \in D(y), H(x) \subseteq D(I)$ and $D(y) \cap D(I) = \emptyset$, for some $y \in L$ and an ideal I of L. Hence $D(y) \subseteq \mathcal{PI}_\alpha(L) \setminus D(I) = H(I)$. Therefore, $P \in D(y) \subseteq H(I) \subseteq D(x)$.

Conversely, let $P \in \mathcal{PI}_\alpha(L)$ and $H(J)$ be any closed set of $\mathcal{PI}_\alpha(L)$ such that $P \notin H(J)$ for an ideal J of L, which implies that $P \in \mathcal{PI}_\alpha(L) \setminus H(J) = D(J)$ and so $J \nsubseteq P$. Hence there is some $a \in J$ such that $a \notin P$. By hypothesis, there are an ideal I of L and $b \in L$ such that $P \in D(b) \subseteq H(I) \subseteq D(a)$. From $a \in J$, it follows that $H(J) \subseteq H(a)$ and so $H(I) \cap H(J) \subseteq D(a) \cap H(a) = \emptyset$, which implies that $H(J) \subseteq \mathcal{PI}_\alpha(L) \setminus H(I) = D(I)$. Since $D(b) \subseteq H(I)$, we get $D(b) \cap D(I) \subseteq H(I) \cap D(I) = \emptyset$. Thus, $\mathcal{PI}_\alpha(L)$ is a regular space. □

The following lemma provides some significant properties of $\mathcal{MI}(L)$ with the hull-kernel topology $\mathcal{T}_h$, which are similar to the ones proved by Henriksen et al. [9], Speed [31, 32] and Rasouli [28].

Proposition 5.12. Let L be a De Morgan residuated lattice. Then the following hold on $\mathcal{MI}(L)$:

(1) $D(x) = H(x^\perp)$ and $H(x) = D(x^\perp)$, for all $x \in L$;
(2) $H(x) = H(x^{\perp\perp})$ and $D(x) = D(x^{\perp\perp})$, for all $x \in L$;
(3) $x^\perp \cap y^\perp = z^\perp$ iff $H(x) \cap H(y) = H(z)$, for all $x, y, z \in L$;
(4) $x^{\perp\perp} = y^\perp$ iff $H(x) = H(y^\perp)$ iff $H(x) = D(y)$, for some $y \in L$;
(5) $x^\perp = \cap D(x)$, for all $x \in L$.

Proof. (1) This is an immediate consequence of Theorem 2.16(3).

(2) Let $M \in H(x)$. Then $x \in M$ and so $x^\perp \nsubseteq M$ by Theorem 2.16(3). Hence there exists $a \in x^\perp$ such that $a \notin M$. Consider $u \in (x^\perp)^\perp$. Then $u \wedge a = 0$ and so $u \in M$, which implies that $x^{\perp\perp} \subseteq M$. Thus $M \in H(x^{\perp\perp})$ and so $H(x) \subseteq H(x^{\perp\perp})$. Conversely, assume that $M \in H(x^{\perp\perp})$. Then $x^{\perp\perp} \subseteq M$. From Lemma 2.8(4) it follows that $x \in x^{\perp\perp} \subseteq M$, so $M \in H(x)$. This shows that $H(x^{\perp\perp}) \subseteq H(x)$. We conclude that $H(x^{\perp\perp}) = H(x)$ and so $D(x^{\perp\perp}) = D(x)$.

(3) Let $z^\perp = x^\perp \cap y^\perp$. By (1) and Proposition 5.1(7), we get $H(z) = D(z^\perp) = D(x^\perp \cap y^\perp) = D(x^\perp) \cap D(y^\perp) = H(x) \cap H(y)$. Conversely, let $H(z) = H(x) \cap H(y)$. By (1) and Proposition 5.1(7), we have $H(z) = D(z^\perp)$ and $H(x) \cap H(y) = D(x^\perp) \cap D(y^\perp) = D(x^\perp \cap y^\perp)$, so $D(z^\perp) = D(x^\perp \cap y^\perp)$. From Proposition 5.1(9) it follows that $z^\perp = x^\perp \cap y^\perp$.

(4) Applying (2) and Proposition 5.1(9), $x^{\perp\perp} = y^\perp$ iff $H(x) = H(y^\perp)$. Applying (1), $H(y^\perp) = D(y)$. Thus $x^{\perp\perp} = y^\perp$ iff $H(x) = D(y)$, for some $y \in L$.

(5) This is an immediate consequence of Corollary 2.17(2). □

Theorem 5.13. Let L be a De Morgan residuated lattice. Then $(\mathcal{MI}(L), \mathcal{T}_h)$ is a Hausdorff space.

Proof. Let M, N be two distinct prime minimal ideals of L. Then there is some $a \in M \setminus N$ and so $M \in H(a)$ and $N \in D(a)$. Applying Proposition 5.12(1), $M \in H(a) = D(a^\perp)$. From Corollary 5.2(7) it follows that $D(a) \cap D(a^\perp) = D(0) = \emptyset$. Hence $(\mathcal{MI}(L), \mathcal{T}_h)$ is a Hausdorff space. □

Lemma 5.14. Let L be a De Morgan residuated lattice. Then the following are equivalent for $x, y \in L$:

(1) $x^\perp = y^\perp$;
(2) $H(x) = H(y)$, where $H(\cdot)$ is the hull of $(\cdot)$ with respect to $\mathcal{MI}(L)$;
(3) $D(x) = D(y)$, where $D(\cdot)$ is the complement of $H(\cdot)$ in $\mathcal{MI}(L)$.

Proof. (1) ⇒ (2) Let $M \in H(x)$. From Theorem 2.16(3) it follows that $x^\perp \nsubseteq M$. By (1), $y^\perp \nsubseteq M$, which means that $M \in D(y^\perp)$. Applying Proposition 5.12(1) we have $M \in H(y)$. The converse is proved similarly.

(2) ⇒ (3) This is clear.

(3) ⇒ (1) By Proposition 5.12(5), we have $x^\perp = \cap D(x) = \cap D(y) = y^\perp$. □

The following results indicate some topological characterizations for α-ideals.

Corollary 5.15. Let L be a De Morgan residuated lattice and I an ideal of L. Then the following are equivalent for $x, y \in L$:

(1) I is an α-ideal of L;
(2) if $x^\perp = y^\perp$ and $x \in I$, then $y \in I$;
(3) if $H(x) = H(y)$ and $x \in I$, then $y \in I$, where $H(\cdot)$ is the hull of $(\cdot)$ with respect to $\mathcal{MI}(L)$;
(4) if $D(x) = D(y)$ and $x \in I$, then $y \in I$, where $D(\cdot)$ is the complement of $H(\cdot)$ in $\mathcal{MI}(L)$.

From Corollary 5.2(2) and (6) it follows that the set $\{H(x) \mid x \in L\}$ can be taken as a base for open sets on $\mathcal{PI}_\alpha(L)$, the resulting topology is called the dual hull-kernel topology and is denoted by $\mathcal{T}_d$, which was extensively discussed on various algebraic structures [9, 15, 31, 32]. In genenal, the hull-kernel topology is finer than the dual hull-kernel topology on $\mathcal{MI}(L)$. Some topological characterizations for quasi-complemented De Morgan residuated lattices are investigated below.

Theorem 5.16. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;
(2) $\mathcal{T}_h$ coincides with $\mathcal{T}_d$ on $\mathcal{MI}(L)$;
(3) $\mathcal{MI}(L)$ is compact in the hull-kernel topology $\mathcal{T}_h$;
(4) $\mathcal{D}(L) = Clop(\mathcal{MI}(L), \mathcal{T}_h)$, where $Clop(\mathcal{MI}(L), \mathcal{T}_h)$ is the set of all closed and open sets of $\mathcal{MI}(L)$;
(5) $\mathcal{D}(L)$ is a Boolean lattice.

Proof. (1) $\Rightarrow$ (2) To prove $\mathcal{T}_h = \mathcal{T}_d$ on $\mathcal{MI}(L)$. It is equivalent to prove that $\{D(x) \mid x \in L\} = \{H(x) \mid x \in L\}$. On the one hand, for all $x \in L$, from Proposition 5.12((1)and (2)) and hypothesis, we have $D(x) = H(x^{\perp}) = H(y^{\perp\perp}) = H(y)$, for some $y \in L$. Thus $\{D(x) \mid x \in L\} \subseteq \{H(x) \mid x \in L\}$. On the other hand, from Proposition 5.12(4) and hypothesis, $H(x) = D(y)$, for some $y \in L$. This means that $\{H(x) \mid x \in L\} \subseteq \{D(x) \mid x \in L\}$. We conclude that $\{D(x) \mid x \in L\} = \{H(x) \mid x \in L\}$.

(2) $\Rightarrow$ (3) Assume that $\mathcal{T}_h = \mathcal{T}_d$ on $\mathcal{MI}(L)$. Since the set $\{D(x) \mid x \in L\}$ is a closed basis for the dual hull-kernel topology $\mathcal{T}_d$, by hypothesis, we have the set $\{D(x) \mid x \in L\}$ is also a closed basis for the hull-kernel topology $\mathcal{T}_h$. Let $\mathcal{F}$ be any family of closed sets which has the finite intersection property. This means that $\mathcal{F} = \{D(x) \mid x \in X\}$, for some $X \subseteq L$ and so $\bigcap_{i=1}^{n} D(x_i) \neq \emptyset$, for any family of finite elements $\{x_1, x_2, \cdots, x_n\} \subseteq X$. By Corollary 5.2(5), it follows that $D(x_1 \wedge x_2 \wedge \cdots \wedge x_n) \neq \emptyset$. This implies that $x_1 \wedge x_2 \wedge \cdots \wedge x_n \neq 0$. We denote by S_X the $\wedge$-closed subset of L generated by X. Thus $0 \notin S_X$ and so, by Zorn's Lemma, there is a $\wedge$-closed subset T_X that is maximal with respect to the property not containing 0. Therefore, $L \setminus T_X$ is a minimal prime monadic ideal of L. From $X \subseteq T_X$ and $T_X \cap (L \setminus T_X) = \emptyset$, it follows that $X \cap (L \setminus T_X) = \emptyset$. This shows that for all $x \in X$, $x \notin L \setminus T_X$ and so $L \setminus T_X \in D(x)$ for all $x \in X$, which means that $\cap\mathcal{F} = \cap\{D(x) \mid x \in X\} \neq \emptyset$. We conclude that $\mathcal{MI}(L)$ with the hull-kernel topology is a compact space.

(3) $\Rightarrow$ (4) Suppose that $(\mathcal{MI}(L), \mathcal{T}_h)$ is compact. Let $C \in Clop(\mathcal{MI}(L), \mathcal{T}_h)$. Then C is a closed set of $\mathcal{MI}(L)$. By hypothesis, C is compact and so there are finite elements $x_1, x_2, \cdots, x_n$ in L such that $C = \bigcup_{i=1}^{n} D(x_i)$. It follows from Corollary 5.2(6) that $C = \bigcup_{i=1}^{n} D(x_i) = D(x_1 \boxplus x_2 \boxplus \cdots \boxplus x_n)$ and so $C \in \mathcal{D}(L)$. Conversely, Assume $D(x) \in \mathcal{D}(L)$, for any $x \in L$. Then, Proposition 5.12(1), $D(x) = H(x^{\perp})$ and so $D(x)$ is also a closed subset of $\mathcal{MI}(L)$ because $H(x^{\perp})$ is a closed subset. We conclude that $\mathcal{D}(L) = Clop(\mathcal{MI}(L), \mathcal{T}_h)$.

(4) $\Rightarrow$ (5) For any $x \in L$, $H(x) = D(y)$, for some $y \in L$. Since $D(x) \cap H(x) = \emptyset$ and $D(x) \cup H(x) = \mathcal{MI}(L)$, then $D(x) \cap D(y) = \emptyset$ and $D(x) \cup D(y) = \mathcal{MI}(L)$, for some $y \in L$. Thus $\mathcal{D}(L)$ is a Boolean lattice.

(5) $\Rightarrow$ (1) For any $x \in L$, $D(x) \cap D(y) = \emptyset$ and $D(x) \cup D(y) = Min(L)$, for some $y \in L$. Applying Corollary 5.2(5) and (6), we have $D(x \wedge y) = \emptyset$ and $D(x \boxplus y) = \mathcal{MI}(L)$, respectively. From $D(x \wedge y) = \emptyset$, it follows that $x \wedge y \in \cap Min(L) = \{0\}$ and so $x \wedge y = 0$. On the other hand, from $D(x \boxplus y) = \mathcal{MI}(L)$, we have $x \boxplus y \notin M$, for all $M \in \mathcal{MI}(L)$. It follows from Theorem 2.16(3) that $(x \boxplus y)^{\perp} \subseteq M$, for all $M \in \mathcal{MI}(L)$. This means that $(x \boxplus y)^{\perp} \subseteq \cap\mathcal{MI}(L) = \{0\}$. It follows from Proposition 4.3(2) that L is quasi-complemented. □

Theorem 5.17. Let L be a De Morgan residuated lattice. Then the following are equivalent:

(1) L is quasi-complemented;
(2) for all $x, y \in L$, there is $a \in x^{\perp}$ such that $D(y) \cap (\mathcal{MI}(L) \setminus D(x)) = D(y \wedge a)$;
(3) $\mathcal{PI}_{\alpha}(L) = \mathcal{MI}(L)$;
(4) for any $x \in L$, $D(x)$ is closed in $(\mathcal{PI}_{\alpha}(L), \mathcal{T}_h)$;
(5) $\mathcal{PI}_{\alpha}(L)$ is a Hausdorff space;
(6) for any $x \in L$, $H(x)$ is compact as a subset of $\mathcal{MI}(L)$.

P r o o f. (1) $\Rightarrow$ (2) Suppose that L is quasi-complemented. For all $x \in L$, there is $a \in L$ such that $x \wedge a = 0$ and $(x \boxplus a)^{\perp} = \{0\}$. Let $P \in D(y) \cap (\mathcal{MI}(L) \setminus D(x))$. Then $y \notin P$ and $x \in P$. If $a \in P$, then $x \boxplus a \in P$. By Propositions 5.1(1) and 5.12(2), we have $D(x \boxplus a) = D((x \boxplus a)^{\perp\perp}) = D(0^{\perp}) = D(L) \subseteq D(P)$. By Proposition 5.1(8), $L \subseteq P$, a contradiction. Hence $a \notin P$. Since P is prime, by $y \notin P$ and $a \notin P$, we have $y \wedge a \notin P$, which implies that $P \in D(y \wedge a)$. Conversely, assume that $P \in D(y \wedge a)$. Then $y \wedge a \notin P$, which means that $y \notin P$ and $a \notin P$ as P is a down set. Since $x \wedge a = 0 \in P$, we have $x \in P$. Hence $P \in D(y) \cap H(x)$. Therefore, $D(y) \cap (\mathcal{MI}(L) \setminus D(x)) = D(y \wedge a)$.

(2) $\Rightarrow$ (3) Clearly, $\mathcal{MI}(L) \subseteq \mathcal{PI}_{\alpha}(L)$. Conversely, let $P \in \mathcal{PI}_{\alpha}(L)$. Suppose that $x \in P$ and $y \notin P$. By hypothesis, there is $a \in x^{\perp}$ such that $D(y) \cap (\mathcal{MI}(L) \setminus D(x)) = D(y \wedge a)$. Hence $P \in D(y \wedge a)$ and so $y \wedge a \notin P$, which implies that $a \notin P$. Since $a \in x^{\perp}$, we get $x \in 0(P)$. Thus $P \subseteq 0(P)$. By Theorem 2.16(2), P is minimal. Therefore, $\mathcal{PI}_{\alpha}(L) = \mathcal{MI}(L)$.

(3) $\Rightarrow$ (4) Let $\mathcal{PI}_{\alpha}(L) = \mathcal{MI}(L)$. Applying Proposition 5.12(1), for any $x \in L$ $D(x) = H(x^{\perp})$. Then $D(x)$ is closed in $(\mathcal{PI}_{\alpha}(L), \mathcal{T}_h)$.

(4) $\Rightarrow$ (5) Let P, Q be two distinct prime α-ideals of L. Then there is some $a \in P \setminus Q$, which shows that $P \in \mathcal{PI}_{\alpha}(L) \setminus D(a)$ and $Q \in D(a)$. By hypothesis, $\mathcal{PI}_{\alpha}(L) \setminus D(a)$ is an open set. Thus $\mathcal{PI}_{\alpha}(L)$ is a Hausdorff space.

(5) $\Rightarrow$ (6) Clearly, $\mathcal{MI}(L) \subseteq \mathcal{P}_{\alpha}(L)$. Suppose that $P \in \mathcal{P}_{\alpha}(L)$ but $P \notin \mathcal{MI}(L)$. Then $M \subseteq P$ and $M \neq P$, for some $M \in \mathcal{MI}(L)$. Since $\mathcal{PI}_{\alpha}(L)$ is Hausdorff, there are two disjoint basic open sets $D(a)$ and $D(b)$ such that $M \in D(a)$ and $P \in D(b)$. By Corollary 5.2(2) and (5), we obtain that $a \wedge b = 0$. Since $M \in D(a)$, we get $a \notin M$, so $b \in M \subseteq P$, a contradiction. This shows that $P \in \mathcal{MI}(L)$. Hence $\mathcal{MI}(L) = \mathcal{P}_{\alpha}(L)$. Therefore, $\mathcal{PI}_{\alpha}(L)$ is a compact Hausdorff space, which implies that $H(x)$ is compact as a subset of $\mathcal{MI}(L)$.

(6) $\Rightarrow$ (1) From Corollary 5.2(7) it follows that $\emptyset = D(x) \cap D(x^{\perp})$, for all $x \in L$. Applying Proposition 5.12(1), we have $\emptyset = H(x) \cap H(x^{\perp}) = H(x) \cap H(\bigcup_{t \in x^{\perp}} t) = H(x) \cap (\bigcap_{t \in x^{\perp}} H(t))$. By hypothesis, we deduce that $H(x) \cap H(t_1) \cap \cdots \cap H(t_n) = \emptyset$, for $t_1, \cdots, t_n \in x^{\perp}$, which implies that $H(x) \cap H(t_1 \boxplus \cdots \boxplus t_n) = \emptyset$, for $t_1, \cdots, t_n \in x^{\perp}$. Put $y = t_1 \boxplus \cdots \boxplus t_n$ and so $y \in x^{\perp}$, we have $H(x) = D(y)$. Applying Proposition 5.12(4), $x^{\perp\perp} = y^{\perp}$, for $y \in x^{\perp}$. □

6. CONCLUSIONS

Inspired by the works of Cornish [5] and Holdon [11, 12], we extended the notion of α-ideals to De Morgan residuated lattices and some equivalent conditions were derived for an ideal to be an α-ideal. It was proved that the lattice of all α-ideals under a reasonable join operation is a complete Heyting algebra. Using annihilator operators, quasi-complemented De Morgan residuated lattices were defined. Some algebraic and topological characterizations were derived for quasi-complemented De Morgan residuated lattices. The hull-kernel topology on prime α-ideals in De Morgan residuated lattices was investigated. It was demonstrated that the space of prime α-ideals under the hull-kernel topology is a spectral space. Several conditions were obtained for a De Morgan residuated lattice to be quasi-complemented.

In our further work, we shall introduce and study monadic ideals in monadic residuated lattices [24, 37, 38, 39] as a natural generalization of ideals in residuated lattices.

Moreover, we shall attempt to establish the relationship between monadic ideals and monadic filters in monadic residuated lattices.

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