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Priyapada Hembram; Nageswara Rao Vemuri

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PRESERVATION OF STRICT MONOTONICITY BY SOME GENERATING METHODS OF FUZZY IMPLICATIONS

Priyapada Hembram and Nageswara Rao Vemuri

Recently, in [20], the authors proposed the strict monotonicity for fuzzy implications and investigated various aspects of fuzzy implications such as properties, functional equations and families of fuzzy implications with respect to strict monotonicity. In this work, the preservation of strict monotonicity is examined for various generating methods of fuzzy implications that are available in the literature. More precisely, the generating methods such as the N-reciprocal method, the conjugacy classes, the aggregation of fuzzy implications, the sup -T composition, the lattice operations, the convex combinations, the $\triangledown$-composition, the $\circledast$-composition, the e-threshold generating method and the vertical e-threshold generating methods are considered and the conditions under which each of these methods preserve the strict monotonicity are investigated. Further, we investigate the strict monotonicity of some new classes of fuzzy implications proposed in the context of generalized hypothetical syllogism and prove that the aggregation of fuzzy implications coming from these classes satisfy the strict monotonicity always. As an interesting outcome of this study, we show that strict monotonicity is a criterion to characterize whether a given fuzzy implication is self-conjugate or not.

Keywords: strict monotonicity, aggregation functions, fuzzy implications, fuzzy negations, lattice operations, convex combinations, conjugacy classes

Classification: 20M32, 03B52

1. INTRODUCTION

Fuzzy implications are an extension of classical implication from $\{0,1\}$ setting to the multi-valued setting. They are defined on $[0,1]$ setting as follows:

Definition 1.1. (Baczyński et al. [5], Baczyński and Jayaram [9], Fodor and Roubens [17]) A function $I\colon [0,1]^2 \longrightarrow [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions: for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$,

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1,y) \geq I(x_2,y), \text{ i. e., } I(\,\cdot\,,y) \text{ is decreasing,} \tag{I1}$$

$$\text{if } y_1 \leq y_2, \text{ then } I(x,y_1) \leq I(x,y_2), \text{ i. e., } I(x,\,\cdot\,) \text{ is increasing,} \tag{I2}$$

$$I(0,0) = 1, I(1,1) = 1, I(1,0) = 0. \tag{I3}$$

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The set of fuzzy implications is denoted by $\mathbb{I}$.

Fuzzy implications have applications in various fields such as *fuzzy logic, artificial intelligence, approximate reasoning, machine learning, decision making, image processing* etc,. For the theoretical development and applicational value, note that, it is always essential to study the various aspects of fuzzy implications, such as properties, functional equations, different families and various algebraic structures that glean the deeper insights of fuzzy implications. Towards this, in the literature, many authors have studied fuzzy implications in different directions, for instance, contrapositive symmetry [4, 12, 16], distributivity [3, 10, 14], the law of importation [22, 29], the exchange principle [23, 44], the generalized hypothetical syllogism [11, 19, 21, 39], various families of fuzzy implications [7, 8, 13, 30, 31, 46, 47] etc,. However, there is always a need for systematic study of newly proposed properties, for instance, [20] and families of fuzzy implications, see [33, 34, 35, 36].

Among the recently proposed properties of fuzzy implications, the strict monotonicity is an important property that plays a significant role both in the theory and applications of fuzzy implications. Since fuzzy implications I are a generalization of classical implication, it is meaningful to expect the value of fuzzy implication I varying w.r.t. a change of antecedent or consequent value. Due to this fact, the strict monotonicity of fuzzy implications is proposed in [20] as follows:

Definition 1.2. (Hembram and Vemuri [20]) A fuzzy implication I on $[0,1]$ is called

(i) strictly decreasing in the first variable if for all $x, y \in [0,1]$ and $z \in [0,1)$,

$$x < y \text{ implies } I(x,z) > I(y,z). \tag{SM1}$$

(ii) strictly increasing in the second variable if for all $x \in (0,1]$ and $y, z \in [0,1]$,

$$y < z \text{ implies } I(x,y) < I(x,z). \tag{SM2}$$

(iii) strictly monotone (SM) if it is strictly decreasing in the first variable (SM1) and strictly increasing in the second variable (SM2).

Due to the contraposition symmetry in classical logic, an equivalent form of Definition 1.2 is presented below.

Remark 1.3. A fuzzy implication I on $[0,1]$ is called

(i) strictly decreasing in the first variable if for all $x, y \in [0,1]$ and $z \in [0,1)$,

$$I(x,z) = I(y,z) \text{ implies } x = y. \tag{SM1}$$

(ii) strictly increasing in the second variable if for all $x \in (0,1]$ and $y, z \in [0,1]$,

$$I(x,y) = I(x,z) \text{ implies } y = z. \tag{SM2}$$

Due to the equivalence between Definition 1.2 and Remark 1.3, we use them appropriately based on the task and convenience.

The strict monotonicity of basic fuzzy implications is presented in Table 1. Recall that in [20], the authors studied the strict monotonicity w.r.t. properties, functional equations and families of fuzzy implications.

Name	Formula	Strict monotonicity
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$	×
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$	×
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$	✓
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$	×
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$	×
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$	×
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$	(SM2)
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$	×
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x, y), & \text{if } x > y \end{cases}$	×
Least FI	$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$	×
Greatest FI	$I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$	×
Most Strict	$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$	(SM2)

Tab. 1. Strict monotonicity of basic fuzzy implications, see [9].

1.1. Motivation for this paper

Recall from Section 1.1 in [20] that strict monotonicity is an essential property for fuzzy implications that has to be studied w.r.t. various aspects of fuzzy implications. Regarding the preservation of strict monotonicity by various generating methods of fuzzy implications, we have the following observations.

(i) Recall that fuzzy implications are broadly classified into the following categories based on their ways of generation:

   (a) Fuzzy implications obtained from other fuzzy logical connectives (eg. (S,N)-, R-, QL-, D-implications, see [9]).

   (b) Fuzzy implications obtained from unary functions on $[0,1]$ (generators of continuous t-norms, continuous t-conorms) (eg. Yager's f,g-implications [46],

h-implications, (h, e)-implications [28] and k-implications [47]).

(c) Fuzzy implications with a specific expression (eg. fuzzy polynomial implications [33, 34], fuzzy (OP)-polynomial implications [35] and fuzzy rational implications [36]).

(d) Fuzzy implications obtained from fuzzy implications (eg. N-reciprocal method, conjugacy classes, lattice operations, convex combinations and sup-T composition [9], the $\circledast$ and the $\triangledown$-composition [40], the aggregation method [37] and e- threshold generated methods [30, 31]).

In [20], the authors have studied the strict monotonicity for fuzzy implications that belong to the categories (a)-(c) listed above. However, the same thing has not yet been done so far for the fuzzy implications that do come from category (d) listed above. This fact motivates us to study the strict monotonicity with respect to generating methods of fuzzy implications.

(ii) In [40], the authors have shown that the generating methods of fuzzy implications from fuzzy implications will lead to various algebraic structures on the set of fuzzy implications. In addition, in [41, 43], it was shown that these algebraic structures play a significant role in obtaining new perspectives and representations of various families of fuzzy implications. However, so far, the preservation of strict monotonicity by generating methods of fuzzy implications has not been done and consequently, there do not available any representation results similar to the ones obtained in [41] and any counterparts of the algebraic structures obtained for strict t-norms and strict t-conorms achieved in [45] and for strict negations in [9].

(iii) It is well known that fuzzy implications have many applications in diverse fields due to the various properties and functional equations. Thus, it is always essential to obtain fuzzy implications that satisfy these properties and functional equations. Towards this, many authors have investigated the preservation of these properties in different contexts such as contrapositive symmetry [4, 12, 16], distributivity [3, 10, 14], the law of importation [22, 29], the exchange principle [23, 44], the generalized hypothetical syllogism [11, 19, 21, 39] and also [2, 37, 42]. However, the preservation of strict monotonicity by various generating methods is not yet done, in spite of its significance mentioned in [20].

The above facts form the main motivation of this paper.

1.2. Objectives and approach of the paper

From motivation, it is clear that our objective in this paper is to study the strict monotonicity for fuzzy implications that do come from the category (d) listed above. In other words, we are interested in the study of strict monotonicity w.r.t. generating methods of fuzzy implications. There are only two ways to achieve the objectives of this paper, which are discussed below.

- Note that, towards achieving the objectives of this paper, firstly, one may propose new generating methods of fuzzy implications and then investigate the preservation of strict monotonicity w.r.t. these new methods. However, proposing a new

generating method without any stronger motivation and significant exploration becomes nothing but one of the troubling practices mentioned in [27]. Thus, in this work, we restrict ourselves to propose yet another generating method of fuzzy implications and investigate the strict monotonicity w.r.t. this method.

- Secondly, we choose to examine the preservation of strict monotonicity by each of the existing generating methods of fuzzy implications that fall under the category (d) of Section 1.1. Specifically, for each of the generating methods in the literature, we investigate the conditions on fuzzy implications and other involved functions so that the respective generating method preserves strict monotonicity.

1.3. The organization of the paper

In Section 2, we examine the preservation of strict monotonicity by generating methods of fuzzy implications that involve fuzzy implications and fuzzy negations on $[0, 1]$. Similarly, the preservation of strict monotonicity by the conjugacy classes and their generalizations is examined in Section 3. Then, the generating methods of fuzzy implications that involve fuzzy implications and aggregations on $[0, 1]$ are considered in Section 4 and their preservation of strict monotonicity is studied. Next, we consider the generating methods of fuzzy implications that involve only fuzzy implications and their preservation of strict monotonicity is investigated in Section 5. Further, in Section 6, we examine the strict monotonicity of some classes of fuzzy implications that are recently proposed and use them to construct fuzzy implications that satisfy the strict monotonicity. Finally, we present some concluding remarks in Section 7.

2. GENERATING METHODS OF FUZZY IMPLICATIONS USING FUZZY NEGATIONS

In this section, we investigate the preservation of strict monotonicity by generating methods of fuzzy implications that involve fuzzy negations and fuzzy implications. Towards this, we consider the N-reciprocal of fuzzy implications and examine the strict monotonicity of new fuzzy implications in terms of that of the old ones.

2.1. The N-reciprocal of fuzzy implications

In the following, we recall the definitions regarding fuzzy negations and N-reciprocal of fuzzy implications.

Definition 2.1. (Baczyński and Jayaram [9])

(i) A function $N : [0, 1] \to [0, 1]$ is called a *fuzzy negation* if it is a decreasing function such that $N(0) = 1$ and $N(1) = 0$.

(ii) A fuzzy negation is called strict if it is strictly decreasing and continuous.

(iii) For a fuzzy implication I, the function $N_I : [0, 1] \to [0, 1]$ defined by

$$N_I(x) = I(x, 0), \qquad x \in [0, 1],$$

is called the natural negation of I or the negation induced by I.

Some examples of fuzzy negations w.r.t. strict monotonicity are presented in Table 2.

Formula	Strict monotonicity
$N_{\mathbf{C}}(x) = 1 - x, \quad x \in [0,1]$	✓
$N(x) = 1 - x^n, \quad x \in [0,1],\ n \in \mathbb{N}$	✓
$N(x) = \begin{cases} 1 - x^3, & \text{if } x \leq 0.5, \\ 1 - x^2, & \text{if } x > 0.5 \end{cases}$	✓
$N_{D1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in (0,1] \end{cases}$	×
$N_{D2}(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{if } x \in [0,1) \end{cases}$	×

Tab. 2. Examples of fuzzy negations w.r.t. strict monotonicity.

Definition 2.2. (Baczyński and Jayaram [9]) Let N be a fuzzy negation and I a fuzzy implication on $[0,1]$. A function $I_N\colon [0,1]^2 \to [0,1]$ defined by

$$I_N(x,y) = I(N(y), N(x)), \qquad x, y \in [0,1],$$

is called the *N-reciprocal* of I.

From Theorem 1.6.2 in [9], it follows that $I_N \in \mathbb{I}$. Now, in the following, we examine the strict monotonicity of N-reciprocal of fuzzy implications I_N in terms of that of I.

Theorem 2.3. Let I be a fuzzy implication and N a strict negation. Then the following statements are equivalent:

(i) I satisfies (SM1) ((SM2)).

(ii) I_N satisfies (SM2) ((SM1)).

P r o o f . Let $I \in \mathbb{I}$ and N be a strict negation. We prove only that I satisfies (SM1) if and only if I_N satisfies (SM2), as the proof of the other can be obtained similarly.

(i) $\Longrightarrow$ (ii). Let I satisfy (SM1) and $I_N(x,y) = I_N(x,z)$ for some $x \in (0,1]$ and $y, z \in [0,1]$. Now, we have the following cases:

- Let $x = 1$. Then, we have $I_N(1,y) = I_N(1,z)$. Then, by using the (SM1) of I and the strictness of N, we obtain that,

$$\begin{aligned} I(N(y), N(1)) = I(N(z), N(1)) &\Longleftrightarrow I(N(y), 0) = I(N(z), 0) \\ &\Longleftrightarrow N_I(N(y)) = N_I(N(z)) \Longleftrightarrow N(y) = N(z) \Longleftrightarrow y = z. \end{aligned}$$

- Let $0 < x < 1$. Since N is strict and $0 < x < 1$, it follows that $0 < N(x) < 1$. Then, $I(N(y), N(x)) = I(N(z), N(x))$ implies $N(y) = N(z)$ due to (SM1) of I and hence, $y = z$, follows from strictness of N.

(ii) $\Longrightarrow$ (i). Let I_N satisfy (SM2) and $I(x,z) = I(y,z)$ for some $x, y \in [0,1]$ and $z \in [0,1)$. Since N is strict, there exist unique $x', y' \in [0,1]$ and $z' \in (0,1]$ such that $N(x') = x$, $N(y') = y$ and $N(z') = z$. Then, we have,

$$\begin{aligned} I(x,z) = I(y,z) &\iff I(N(x'),N(z')) = I(N(y'),N(z')) \\ &\iff I_N(z',x') = I_N(z',y') \iff x' = y' \\ &\iff N(x') = N(y') \iff x = y. \end{aligned}$$

Thus, I satisfies (SM1).

This completes the proof. □

Remark 2.4. Note that Theorem 2.3 need not be true in the case N is not strict. For example, let $I = I_{\mathbf{RC}}$ which is strictly monotone fuzzy implication and N be the non-strict fuzzy negation given by

$$N(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.5, & \text{if } x \in (0, 0.5), \\ 0, & \text{if } x \in [0.5, 1]. \end{cases}$$

Then, it is easy to see that $I_N(0.1, 0.5) = 1 = I_N(0.1, 0.6)$, while $0.5 < 0.6$. This shows that I_N does not satisfy (SM2). Similarly, one can easily show that I_N does not satisfy (SM1) also and hence, I_N is not strictly monotone.

2.2. The generalized reciprocal fuzzy implications

In the following, we recall a natural generalization of Definition 2.2 and investigate the strict monotonicity of new fuzzy implications in terms of those of old fuzzy implications and fuzzy negations.

Definition 2.5. (Priyanka and Vemuri [38]) Let N_1, N_2 be two fuzzy negations and I a fuzzy implication. A function $I_{N_1,N_2} : [0,1]^2 \to [0,1]$ defined by

$$I_{N_1,N_2}(x,y) = I(N_1(y), N_2(x)), \qquad x, y \in [0,1],$$

is called the (N_1, N_2)- reciprocal of I.

In the following, we recall that the function I_{N_1,N_2} defined in Definition 2.5 is a fuzzy implication.

Theorem 2.6. (Priyanka and Vemuri [38]) If N_1, N_2 are fuzzy negations and I is a fuzzy implication on $[0,1]$, then I_{N_1,N_2} is again a fuzzy implication.

In the following, we investigate the strict monotonicity of I_{N_1,N_2} in terms of that of N_1, N_2 and I.

Theorem 2.7. Let N_1, N_2 be two strict fuzzy negations and I a fuzzy implication. Then, I satisfies (SM) if and only if I_{N_1,N_2} satisfies (SM).

P r o o f . Assume the hypothesis. Let I satisfy (SM1) and $I_{N_1,N_2}(x,y) = I_{N_1,N_2}(x,z)$, for some $x \in (0,1]$ and $y, z \in [0,1]$. Then, it follows that $I(N_1(y), N_2(x)) = I(N_1(z), N_2(x))$. Since N_2 is strict and $x > 0$, we have $N_2(x) < 1$. Thus, from (SM1) of I, it follows that $N_1(y) = N_1(z)$ and hence, $y = z$, due to the strictness of N_1. Thus, I_{N_1,N_2} satisfies (SM2).

Conversely, let I_{N_1,N_2} satisfy (SM2) and $I(x,z) = I(y,z)$, for some $x, y \in [0,1]$ and $z \in [0,1)$. Since $x, y \in [0,1], z \in [0,1)$ and N_1, N_2 are strict, there exist $x', y' \in [0,1]$ and $z' \in (0,1]$ such that $N_1(x') = x, N_1(y') = y$ and $N_2(z') = z$. Hence, we have $I(x,z) = I(y,z)$ implies that $I(N_1(x'), N_2(z')) = I(N_1(y'), N_2(z'))$ which further implies that $I_{N_1,N_2}(z', x') = I_{N_1,N_2}(z', y')$. Then from (SM2) of I_{N_1,N_2}, we get that $x' = y'$ which implies that $N_1(x') = N_1(y')$. i. e., $x = y$. Hence, I satisfies (SM1). Thus, we have shown that if N_1, N_2 are strict fuzzy negations then I satisfies (SM1) if and only if I_{N_1,N_2} satisfies (SM2). Similarly, it can be shown that I satisfies (SM2) if and only if I_{N_1,N_2} satisfies (SM1), under the same assumption. This completes the proof. □

Remark 2.8. Note that in Theorem 2.7, the strictness of fuzzy negations is important.

(i) In the following, we present an example that shows that if N_1 or N_2 or both are not strict then (SM) of I need not imply that of I_{N_1,N_2}. Towards this, let fuzzy negations N_1, N_2 and fuzzy implication I be defined respectively by

$$N_1(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.5, & \text{if } x \in (0,1), \\ 0, & \text{if } x = 1, \end{cases}$$

$N_2(x) = 1 - x$ for $x \in [0,1]$ and $I = I_{\mathbf{RC}}$. Then, clearly N_1 is not strict while N_2 is strict and I is also strictly monotone. Now, we get $I_{N_1,N_2}(0.4, 0.3) = 0.8 = I_{N_1,N_2}(0.4, 0.8)$, which implies that I_{N_1,N_2} does not satisfy (SM2).

(ii) In the following, we present an example which shows that (SM) of I_{N_1,N_2} need not imply that of I, in the case fuzzy negations N_1, N_2 are not strict. Let fuzzy negations N_1, N_2 and fuzzy implication I be defined respectively by

$$N_1(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.9 - 0.8x, & \text{if } x \in (0,1), \\ 0, & \text{if } x = 1, \end{cases}$$

$N_2(x) = 1 - x$ for all $x \in [0,1]$ and

$$I(x,y) = \begin{cases} 0, & \text{if } x \in [0.95, 1] \text{ and } y = 0, \\ 1 - x + xy, & \text{otherwise,} \end{cases} \qquad x, y \in [0,1].$$

It is clear that while the fuzzy negations N_1 and N_2 are strictly monotone (N_1 is not continuous and hence, not strict), fuzzy implication I is not strictly monotone (since, $I(0.96, 0) = 0 = I(0.97, 0)$). However, we get I_{N_1,N_2}, for all $x, y \in [0,1]$, as

follows:

$$I_{N_1,N_2}(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0, \\ 1 - x, & \text{if } x \in [0,1) \text{ and } y = 0, \\ 1 - 0.9x + 0.8xy, & \text{if } x \in [0,1] \text{ and } y \in (0,1), \\ 1, & \text{if } x \in [0,1] \text{ and } y = 1. \end{cases}$$

Now, it is easy to verify that I_{N_1,N_2} satisfies (SM).

3. GENERATING METHODS OF FUZZY IMPLICATIONS USING INCREASING BIJECTIONS ON $[0,1]$

In this section, we investigate the strict monotonicity of fuzzy implications generated by using fuzzy implications and increasing bijections on $[0,1]$. Let Φ denote the set of all increasing bijections on $[0,1]$. We recall the Φ-conjugates of fuzzy implications in the following and then examine their strict monotonicity.

Definition 3.1. (Baczyński and Jayaram [9], Drewniak [15]) For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-conjugate I_φ is defined as follows:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right), \qquad x,y \in [0,1].$$

Then it follows that I_φ is also a fuzzy implication and preserves many of the desirable properties of the fuzzy implication I, see Propositions 1.1.8 and 1.3.6 in [9]. Towards the preservation of strict monotonicity of fuzzy implications under conjugation, we have the following result.

Theorem 3.2. Let $I \in \mathbb{I}$ and $\varphi \in \Phi$. Then, I satisfies (SM1) (or (SM2) or (SM)) if and only if I_φ satisfies (SM1) (or (SM2) or (SM)).

Proof. Let $I \in \mathbb{I}$ and $\varphi \in \Phi$. We prove only that I satisfies (SM1) if and only if I_φ satisfies (SM1), as the proof of (SM2) can be obtained similarly.

($\Longrightarrow$) : Let I satisfy (SM1) and $I_\varphi(x,z) = I_\varphi(y,z)$ for some $x,y \in [0,1]$ and $z \in [0,1)$. Since φ is strictly increasing and $z < 1$, it follows that $\varphi(z) < 1$. Now, we get that

$$\begin{aligned} I_\varphi(x,z) = I_\varphi(y,z) &\Longleftrightarrow \varphi^{-1}\left(I(\varphi(x),\varphi(z))\right) = \varphi^{-1}\left(I(\varphi(y),\varphi(z))\right) \\ &\Longleftrightarrow I(\varphi(x),\varphi(z)) = I(\varphi(y),\varphi(z)) \\ &\Longleftrightarrow \varphi(x) = \varphi(y) \Longleftrightarrow x = y, \end{aligned}$$

due to the fact that I satisfies (SM1) and φ is an increasing bijection. Thus, I_φ satisfies (SM1).

($\Longleftarrow$) : Conversely, let I_φ satisfy (SM1) and $I(x,z) = I(y,z)$ for some $x,y \in [0,1]$ and $z \in [0,1)$. Since $\varphi \in \Phi$, there exist $x', y' \in [0,1]$ and $z' \in [0,1)$ such that $\varphi(x') = x$, $\varphi(y') = y$ and $\varphi(z') = z$. Then, we get that

$$I(x,z) = I(y,z) \Longleftrightarrow I(\varphi(x'),\varphi(z')) = I(\varphi(y'),\varphi(z'))$$

$$\iff \varphi^{-1}\left(I(\varphi(x'),\varphi(z'))\right) = \varphi^{-1}\left(I(\varphi(y'),\varphi(z'))\right)$$
$$\iff I_\varphi(x,z) = I_\varphi(y,z) \iff x = y,$$

due to the fact that I_φ satisfies (SM1). Thus, I satisfies (SM1).

This completes the proof. □

In the following, we show that the strict monotonicity is an important criterion for characterizing a class of fuzzy implications, namely, self-conjugate or invariant fuzzy implications, which are defined below.

Definition 3.3. (Baczyński and Jayaram [9]) A fuzzy implication I is called self-conjugate if $I_\varphi = I$, for all $\varphi \in \Phi$.

For more details about self-conjugate fuzzy implications and their algebraic structure, please refer to [9].

Proposition 3.4. Let I be a self-conjugate fuzzy implication. Then the range of the natural negation N_I of I is trivial. i. e., $I(x,0) = N_I(x) \in \{0,1\}$, for all $x \in [0,1]$.

Proof. Let $I \in \mathbb{I}$ be self-conjugate, i. e., $I_\varphi(x,y) = I(x,y)$, for all $x,y \in [0,1]$ and $\varphi \in \Phi$. If $y = 0$ then we get, for all $x \in [0,1]$, that

$$\begin{aligned} I_\varphi(x,0) = I(x,0) &\iff \varphi^{-1}\left(I(\varphi(x),\varphi(0))\right) = N_I(x) \\ &\iff \varphi^{-1}\left(I(\varphi(x),0)\right) = N_I(x) \\ &\iff N_I(x) = \varphi^{-1}(N_I(\varphi(x))). \end{aligned}$$

Thus, if I is a self-conjugate fuzzy implication then $N_I(x) = \varphi^{-1}(N_I(\varphi(x)))$, for all $x \in [0,1]$ and for all $\varphi \in \Phi$. Let, for some $m \in \mathbb{N}$, define $\varphi(x) = x^m$, for all $x \in [0,1]$. Then, clearly $\varphi \in \Phi$. Thus, we have $N_I(x^m) = (N_I(x))^m$, for all $x \in [0,1]$, which is a special case of the Cauchy's functional equation $f(xy) = f(x)f(y)$, see [1]. Then from the solution of the Cauchy's functional equation, it follows that the range of N_I is trivial, otherwise $N_I(x) = x^c$, for some $c > 0$, which is not a fuzzy negation. Thus, the range of N_I is trivial. □

Proposition 3.5. A self-conjugate fuzzy implication does not satisfy (SM1) and consequently, it is not strictly monotone.

Proof. Let $I \in \mathbb{I}$ be self-conjugate. Then from Proposition 3.4, it follows that the range of N_I is trivial and hence, N_I is not strictly decreasing, due to Proposition 2.7 in [20]. Thus, N_I does not satisfy (SM1). □

Remark 3.6. (i) Note from Proposition 3.5 that strict monotonicity is an important criterion to characterize whether a given fuzzy implication is self-conjugate or not.

(ii) From Proposition 3.5 note that a self-conjugate fuzzy implication does not satisfy (SM1). However, there are some self-conjugate fuzzy implications that satisfy (SM2) always, for instance, see $I_{\mathbf{D}}$.

(iii) Note that there are some self-conjugate fuzzy implications that satisfy neither (SM1) nor (SM2), for instance, $I_{\mathbf{0}}, I_{\mathbf{1}}, I_{\mathbf{GD}}$ and $I_{\mathbf{RS}}$, among the others.

3.1. The generalized conjugacy classes of fuzzy implications

In the context of conjugacy classes of fuzzy implications, in [43], the authors have proposed a natural generalization of Definition 3.1 and obtained a way of generating fuzzy implications from such ones and increasing bijections on $[0,1]$. They are defined as follows:

Definition 3.7. (Vemuri and Jayaram [43]) Let $I : [0,1]^2 \to [0,1]$ be a function and $\varphi, \psi, \mu \in \Phi$. The general bijective transformation $I_{\varphi,\psi,\mu} : [0,1]^2 \to [0,1]$ of I is defined as follows:

$$I_{\varphi,\psi,\mu}(x,y) = \varphi(I(\psi(x), \mu(y))) \quad x, y \in [0,1].$$

Lemma 3.8. (Vemuri and Jayaram [43]) Let $I : [0,1]^2 \to [0,1]$ be a function and $\varphi, \psi, \mu \in \Phi$. Then I is a fuzzy implication if and only if $I_{\varphi,\psi,\mu}$ is a fuzzy implication.

In the following, we investigate the preservation of strict monotonicity by the general bijective transformations.

Theorem 3.9. Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication and $\varphi, \psi, \mu \in \Phi$. Then I satisfies (SM1) ((SM2)) if and only if $I_{\varphi,\psi,\mu}$ satisfies (SM1) ((SM2)).

Proof. Assume the hypothesis. We prove the result for (SM1) only, as the proof of (SM2) follows analogously.

($\Longrightarrow$) : Let $x, y \in [0,1]$ and $z \in [0,1)$. Since, φ, ψ, μ are strictly increasing on $[0,1]$, it follows that $\psi(x), \psi(y) \in [0,1]$ and $\mu(z) \in [0,1)$. Then,

$$\begin{aligned} I_{\varphi,\psi,\mu}(x,z) = I_{\varphi,\psi,\mu}(y,z) &\Longleftrightarrow \varphi(I(\psi(x),\mu(z))) = \varphi(I(\psi(y),\mu(z))) \\ &\Longleftrightarrow I(\psi(x),\mu(z)) = I(\psi(y),\mu(z)) \\ &\Longleftrightarrow \psi(x) = \psi(y) \Longleftrightarrow x = y. \end{aligned}$$

Thus, $I_{\varphi,\psi,\mu}$ satisfies (SM1).

($\Longleftarrow$) : Let $x, y \in [0,1]$ and $z \in [0,1)$ be such that $I(x,z) = I(y,z)$. Since $\psi, \mu \in \Phi$, there exist $x', y' \in [0,1]$ and $z' \in [0,1)$ such that $x = \psi(x')$, $y = \psi(y')$ and $z = \mu(z')$. Then,

$$\begin{aligned} I(x,z) = I(y,z) &\Longleftrightarrow I(\psi(x'),\mu(z')) = I(\psi(y'),\mu(z')) \\ &\Longleftrightarrow \varphi(I(\psi(x'),\mu(z'))) = \varphi(I(\psi(y'),\mu(z'))) \\ &\Longleftrightarrow I_{\varphi,\psi,\mu}(x',z') = I_{\varphi,\psi,\mu}(y',z') \\ &\Longleftrightarrow x' = y' \Longleftrightarrow \psi(x') = \psi(y') \Longleftrightarrow x = y. \end{aligned}$$

Thus, I satisfies (SM1).

This completes the proof. □

Remark 3.10. From Theorem 3.9, it follows that, in the case $\varphi, \psi, \mu \in \Phi$, the fuzzy implication $I_{\varphi,\psi,\mu}$ preserves (SM1) ((SM2) or both). However, if $\varphi, \psi, \mu \notin \Phi$, then $I_{\varphi,\psi,\mu}$ may satisfy (SM1) ((SM2)) or may not. To see this, we have the following:

(i) Firstly, let $\psi(x) = x = \mu(x)$, for all $x \in [0,1]$ and ϕ be defined, for all $x \in [0,1]$, as

$$\phi(x) = \begin{cases} x, & \text{if } 0 \le x \le 0.2, \\ 0.2, & \text{if } 0.2 \le x < 0.8, \\ x^2, & \text{if } 0.8 \le x \le 1. \end{cases}$$

Clearly, $\psi, \mu \in \Phi$ while $\phi \notin \Phi$, as ϕ is not strictly increasing. Let $I = I_{\mathbf{RC}} \in \mathbb{I}$ which is strictly monotone. Then, from Definition 3.7, it follows, for all $x, y \in [0,1]$, that

$$I_{\varphi,\psi,\mu}(x,y) = \begin{cases} 1 - x + xy, & \text{if } 0 \le 1 - x + xy \le 0.2, \\ 0.2, & \text{if } 0.2 \le 1 - x + xy < 0.8, \\ (1 - x + xy)^2, & \text{if } 0.8 \le 1 - x + xy \le 1. \end{cases}$$

Now, it is easy to see that $I_{\varphi,\psi,\mu}(0.8, 0.1) = 0.2 = I_{\varphi,\psi,\mu}(0.7, 0.1)$ but $x = 0.8 > 0.7 = y$ and $z = 0.1$, which implies that $I_{\varphi,\psi,\mu}$ does not satisfy (SM1) and consequently, $I_{\varphi,\psi,\mu}$ does not satisfy (SM).

(ii) Secondly, let $\phi(x) = x = \psi(x)$, for all $x \in [0,1]$ and μ be defined, for all $x \in [0,1]$, as

$$\mu(x) = \begin{cases} x, & \text{if } 0 \le x \le 0.2, \\ 0.2, & \text{if } 0.2 \le x < 0.8, \\ x^2, & \text{if } 0.8 \le x \le 1. \end{cases}$$

Clearly, $\phi, \psi \in \Phi$ while $\mu \notin \Phi$, as μ is not strictly increasing. Let $I = I_{\mathbf{RC}}$ which is strictly monotone. Then, from Definition 3.7, it follows that

$$I_{\varphi,\psi,\mu}(x,y) = \begin{cases} 1 - x + xy, & \text{if } 0 \le y \le 0.2, \\ 1 - 0.8x, & \text{if } 0.2 < y < 0.8, \\ 1 - x + xy^2, & \text{if } 0.8 \le y \le 1. \end{cases}$$

Now, it is easy to see that $I_{\varphi,\psi,\mu}$ satisfies (SM1).

4. GENERATING METHODS OF FUZZY IMPLICATIONS THAT INVOLVE AGGREGATION FUNCTIONS

In this section, we examine the strict monotonicity of fuzzy implications that are obtained by using fuzzy implications and aggregation functions. In the literature, aggregation functions are defined as follows.

Definition 4.1. (Grabisch et al. [18]) A function $A : [0,1]^2 \to [0,1]$ is called a binary aggregation function if

(i) A is increasing in both variables,

(ii) $A(0,0) = 0$ and $A(1,1) = 1$.

For more details about binary aggregation functions such as properties, different classes and applications, please refer to [18].

In the following, we recall the generating method of fuzzy implications from such ones using aggregation functions.

Definition 4.2. (Vemuri and Jayaram [40]) Let A be a binary function on [0,1] and $I, J \in \mathbb{I}$. We define $(I \odot_A J)$ as

$$(I \odot_A J)(x, y) = A(I(x, y), J(x, y)), \qquad x, y \in [0, 1].$$

Theorem 4.3. (Vemuri and Jayaram [40]) Let A be an aggregation function on $[0, 1]$ and $I, J \in \mathbb{I}$. Then $(I \odot_A J)$ is a fuzzy implication on $[0, 1]$.

Towards the preservation of strict monotonicity by the generating method proposed in Definition 4.2, we require the following definition.

Definition 4.4. (Grabisch et al. [18]) An aggregation is called strictly monotone if it is strictly increasing in both variables.

Some examples of binary aggregation functions w.r.t. strict monotonicity are presented in Table 3.

Binary aggregation A	Formula of $A(x, y)$	Strict monotonicity
max	$\max(x, y)$	×
min	$\min(x, y)$	×
arithmetic mean	$\frac{x+y}{2}$	✓
weighted arithmetic mean	$w_1x + w_2y$, with $w_1, w_2 \in [0, 1]$ and $w_1 + w_2 = 1$	✓
geometric mean	$\sqrt{xy}$	×
weighted geometric mean	$x^{w_1}y^{w_2}$, with $w_1, w_2 \in [0, 1]$ and $w_1 + w_2 = 1$	×
harmonic mean	$\begin{cases} \frac{2xy}{x+y}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$	×

Tab. 3. Examples of aggregation functions w.r.t. strict monotonicity.

Theorem 4.5. Let A be a strictly monotone aggregation and I, J be fuzzy implications on $[0, 1]$. If I and J satisfy (SM1) ((SM2)) then $(I \odot_A J)$ satisfies the same.

Proof. Assume the hypothesis. We prove the result for (SM1) only since the proof for (SM2) follows similarly. Let I, J satisfy (SM1) and $x, y \in [0, 1]$ and $z \in [0, 1)$ be such that $x < y$. Since I and J satisfy (SM1), it follows that $I(x, z) > I(y, z)$ and $J(x, z) > J(y, z)$. Since A is strictly monotone, it implies further that

$$\begin{aligned}(I \odot_A J)(x, z) =& A(I(x, z), J(x, z)) > A(I(y, z), J(x, z)) \\ >& A(I(y, z), J(y, z)) = (I \odot_A J)(y, z).\end{aligned}$$

Thus, $(I \odot_A J)$ satisfies (SM1). □

Remark 4.6. Note that Theorem 4.5 need not hold true if the aggregation A is not strictly monotone. To see this,

(i) let $I \in \mathbb{I}$ be defined as

$$I(x,y) = \begin{cases} 1, & \text{if } y = 1, \\ 1-x, & \text{if } y < 1, \end{cases} \tag{1}$$

and $J = I_{\mathbf{RC}}$. Then, it is easy to see that I, J satisfy (SM1) and aggregation A defined by

$$A(x,y) = \begin{cases} 1, & \text{if } x > 0.2 \text{ or } y > 0.3, \\ 0, & \text{otherwise,} \end{cases}$$

is not strictly monotone. Also, from $(I \odot_A J)(0.3, 0.4) = 0 = (I \odot_A J)(0.35, 0.4)$, it follows that $(I \odot_A J)$ does not satisfy (SM1).

(ii) let $I = I_{\mathbf{YG}}$, $J = I_{\mathbf{D}}$ which satisfy (SM2) and aggregation A is defined by

$$A(x,y) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly, A is not strictly monotone and $(I \odot_A J)$ does not satisfy (SM2) as $(I \odot_A J)(0.9, 0.4) = 0 = (I \odot_A J)(0.9, 0.2)$.

Note that Theorem 4.5 discusses the strict monotonicity of fuzzy implications $(I \odot_A J)$, for a general aggregation function A. However, in the literature, aggregation functions are classified into different classes based on the properties they satisfy. Now, in the following, we consider A coming from some of these classes and for each of these cases, we examine the strict monotonicity of $(I \odot_A J)$.

4.1. Using polynomial aggregation functions

Recall that among the different classes of aggregation functions, some aggregations are expressed as polynomials on $[0, 1]$. They are defined as follows:

Definition 4.7. (Massanet et al. [32]) Let n $\in \mathbb{N}$. A binary operator $A : [0,1]^2 \to [0,1]$ is called a binary polynomial aggregation function of degree n if it is a binary aggregation function and its expression is given by

$$A(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j,$$

for all $x, y \in [0,1]$, $a_{ij} \in \mathbb{R}$ and there exist some $0 \le i, j \le n$ with $i + j = n$ such that $a_{ij} \neq 0$.

Table 4 presents some examples of binary polynomial aggregation functions.

Binary polynomial aggregation A	Formula of $A(x, y)$
arithmetic mean	$\frac{x+y}{2}$
weighted arithmetic mean	$w_1x + w_2y$, with $w_1, w_2 \in [0, 1]$ and $w_1 + w_2 = 1$
the product t-norm $T_{\mathbf{P}}$	$x \cdot y$
the probabilistic sum t-conorm $S_{\mathbf{P}}$	$x + y - x \cdot y$

Tab. 4. Examples of binary polynomial aggregations, see [32].

In the following, we investigate the relationship between binary polynomial aggregation functions and strict monotonicity. Before doing this, we have the following example.

Remark 4.8. Note that while the binary aggregation $A(x, y) = \frac{x+y}{2}$, for all $x, y \in [0, 1]$ is strictly monotone, the aggregation $A(x, y) = x$, for all $x, y \in [0, 1]$, is not strictly monotone. Thus, it follows that every binary polynomial aggregation function need not be strictly monotone.

In the following, we investigate the relation between strict monotonicity and annihilators of a binary aggregation function defined on $[0, 1]$.

Lemma 4.9. Let A be a binary aggregation function on $[0, 1]$. If A satisfies (SM2)((SM1)) then A has no left (right) annihilators.

P r o o f . Let A be a binary aggregation function on $[0, 1]$ and A satisfy (SM2). Assume that $a \in [0, 1]$ is a left annihilator of A. Then, it implies that $A(a, x) = a$ for all $x \in [0, 1]$, which is a contradiction that A satisfies (SM2). Thus, A has no left annihilator. The proof of the other can be obtained similarly. □

Remark 4.10. Note that the converse of Lemma 4.9 need not be true always. To see this, let $a \in (0, 1)$ be fixed arbitrarily and A be defined for all $x, y \in [0, 1]$ by,

$$A(x, y) = \begin{cases} 0, & \text{if } x = 0 = y, \\ 1, & \text{if } x = 1 = y, \\ \frac{a}{2}, & \text{if } x, y \in [0, a] \text{ and } x \neq 0 \neq y, \\ a, & \text{otherwise.} \end{cases} \tag{2}$$

Then, clearly A is a binary aggregation on $[0, 1]$ that has no left and right annihilators. However, clearly A satisfies neither (SM2) nor (SM1).

In the following, we show that the converse of Lemma 4.9 is true for binary polynomial aggregation functions in a more general setting. Towards this, we propose the following definition.

Definition 4.11. (i) An element $x \in [0, 1]$ is called a generalized left annihilator of a binary aggregation A if there exists a function k on $[0, 1]$ such that $A(x, y) = k(x)$, for all $y \in [0, 1]$.

(ii) An element $y \in [0,1]$ is called a generalized right annihilator of a binary aggregation A if there exists a function k on $[0,1]$ such that $A(x,y) = k(y)$, for all $x \in [0,1]$.

(iii) An element $a \in [0,1]$ is called a generalized annihilator of a binary aggregation A if it is both generalized left annihilator and generalized right annihilator.

Remark 4.12. (i) Every (left, right) annihilator of a binary aggregation A is a generalized (left, right) annihilator. Due to this fact, a stronger form Lemma 4.9 says that if A satisfies (SM1)((SM2)) then A has no generalized right (left) annihilators.

(ii) Let $A(x,y) = x^2$, for all $x, y \in [0,1]$. Then, clearly A is a binary aggregation on $[0,1]$. Note that, while every $x \in [0,1]$ is a generalized left annihilator of A, only 0 and 1 are the left annihilators of A. Thus, every generalized left (right) annihilator of A need not be a left (right) annihilator of A.

Theorem 4.13. Let A be a binary polynomial aggregation function on $[0,1]$. Then A satisfies (SM2)((SM1)) if and only if A has no generalized left (right) annihilators.

P r o o f. Let A be a binary polynomial aggregation of degree n.

($\Longrightarrow$)**:** Follows from Remark 4.12(i).

($\Longleftarrow$)**:** Let A have no generalized left annihilators. We claim that A satisfies (SM2). Suppose there exist $x, y, z \in [0,1]$ such that $A(x,y) = A(x,z)$ with $y < z$. Since A is increasing in both variables, there exists a function k on $[0,1]$ such that that $A(x,t) = k(x)$, for all $t \in [y,z]$. Define a polynomial $P_x(t) := A(x,t) - k(x)$, by extending $A(x,t)$ to $\mathbb{R}$, the set of real numbers, with the same expression. Then P_x is a polynomial of degree, say m with $0 \leq m \leq n$.

(i) Let $0 < m \leq n$: In this case, note that P_x is non-constant polynomial of degree m. Thus, from the fundamental theorem of algebra, P_x has at most m roots in $\mathbb{R}$. However, recall that for every $t \in [y,z]$, we have $A(x,t) - k(x) = 0$, which implies that P_x has uncountable number of roots in $\mathbb{R}$. This is a contradiction.

(ii) Let $m = 0$. Then, P_x is a constant. Now, we have two sub-cases:

- If P_x is a non-zero constant on $\mathbb{R}$, then P_x has no roots in $\mathbb{R}$, which is a contradiction to the assumption that $A(x,t) = k(x)$, for all $t \in [y,z]$.
- If $P_x \equiv 0$, then $A(x,t) = k(x)$, for all $t \in \mathbb{R}$ and in particular, $A(x,t) = k(x)$, for all $t \in [0,1]$. This implies that x is a generalized left annihilator of A, a contradiction to our assumption.

Thus, in all the cases, we have a contradiction to the assumption that $A(x,y) = A(x,z)$ for some $x, y, z \in [0,1]$ with $y < z$. Hence, A satisfies (SM2). The proof of other case follows similarly.

This completes the proof. □

Finally, the preservation of (SM2) ((SM1)) by the generating method ($I \odot_A J$) for the polynomial aggregations is presented in the following.

Proposition 4.14. Let A be a polynomial aggregation on $[0,1]$ that has no generalized left (right) annihilators. If the fuzzy implications I, J satisfy (SM2) ((SM1)) then $(I \odot_A J)$ also satisfies the same.

Proof. Follows from Theorems 4.13 and 4.5. □

Remark 4.15. Note that the condition A is polynomial aggregation in Proposition 4.14 is only a sufficient condition but need not be necessary. For instance, let $A(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{2}$, which is not a polynomial. However, it is easy to see that A is strictly monotone and consequently, $(I \odot_A J)$ is also strictly monotone for all strictly monotone fuzzy implications I and J.

4.2. Using aggregations that are mean

Recall that, in the literature, the mean is an important aggregation function that has been used widely. In this section, we investigate the strict monotonicity of fuzzy implications obtained via Definition 4.2, in which the aggregation A is a mean. In the following, we recall two important classes of means.

Definition 4.16. (Grabisch et al. [18])

(i) An aggregation function A is called a weighted arithmetic mean if there exist some $w_1, w_2 \in [0,1]$ with $w_1 + w_2 = 1$ such that $A(x,y) = w_1 x + w_2 y$, for all $x, y \in [0,1]$.

(ii) An aggregation function A is called a weighted geometric mean if there exist some $w_1, w_2 \in [0,1]$ with $w_1 + w_2 = 1$ such that $A(x,y) = x^{w_1} \cdot y^{w_2}$, for all $x, y \in [0,1]$.

In the following, we consider $(I \odot_A J)$ defined as in Definition 4.2 for aggregation A in two cases, namely, weighted arithmetic mean and weighted geometric mean, and investigate the strict monotonicity of $(I \odot_A J)$ in terms of that of I and J. Before doing so, note that in case the aggregation A is the weighted arithmetic mean, the fuzzy implication $(I \odot_A J)$ is nothing but the classical convex combination of fuzzy implications I and J, which is defined in [9] and recalled below.

Definition 4.17. (Baczyński and Jayaram [9]) Convex combination of fuzzy implications I, J is defined as

$$K(x,y) = \lambda I(x,y) + (1-\lambda) J(x,y), \qquad x, y \in [0,1],\ \lambda \in [0,1].$$

Recall from [9] that the convex combination of fuzzy implications is always a fuzzy implication. Now, in the following, we study the strict monotonicity of $(I \odot_A J)$ in terms of that of I and J in two cases for the aggregation A.

Theorem 4.18. Let I, J be fuzzy implications on $[0,1]$. Then the following statements hold true.

(i) If I and J are strictly monotone then the convex combination (or the weighted arithmetic mean) of I and J is also strictly monotone.

(ii) If I and J are strictly monotone then the weighted geometric mean of I and J is also strictly monotone.

Proof.

(i) Note that the convex combination is a binary polynomial aggregation on $[0,1]$ of degree 1 and it has neither left nor right annihilators. Due to this fact, the proof of this part directly follows from Theorem 4.14.

(ii) Let A be the weighted geometric mean defined, for all $x, y \in [0,1]$, by $A(x,y) = x^{w_1} y^{w_2}$, for some $w_1, w_2 \in [0,1]$ with $w_1 + w_2 = 1$. Then, $(I \odot_A J)$ becomes

$$(I \odot_A J)(x,y) = A(I(x,y), J(x,y)) = (I(x,y))^{w_1} (J(x,y))^{w_2}, \qquad x, y \in [0,1].$$

We prove only (SM2), as the proof of (SM1) follows similarly. Let I and J satisfy (SM2). Let $x \in (0,1]$ and $y, z \in [0,1]$ be such that $y < z$. Then, from (SM2) of I and J, we get that, $I(x,y) < I(x,z)$ and $J(x,y) < J(x,z)$. Thus, we have $(I(x,y))^{w_1} < (I(x,z))^{w_1}$ and $(J(x,y))^{w_2} < (J(x,z))^{w_2}$, which implies $(I(x,y))^{w_1} \cdot (J(x,y))^{w_2} < (I(x,z))^{w_1} \cdot (J(x,z))^{w_2}$. i. e., $(I \odot_A J)(x,y) < (I \odot_A J)(x,z)$. Thus, $(I \odot_A J)$ satisfies (SM2).

This completes the proof. □

Remark 4.19. Recall from Definition 4.4 and Table 4.1 in [18] that the weighted arithmetic mean and the weighted geometric mean are a special case of weighted quasi-arithmetic means. Further, similar to Theorem 4.18, it can be proved that if I and J are strictly monotone then the weighted quasi-arithmetic mean of I and J is also strictly monotone.

4.3. Using aggregation functions that are fuzzy logic connectives

In the literature, there are some aggregation functions that are also fuzzy logic connectives and play an important role in fuzzy logic too. Among this class of aggregation functions, the most significant fuzzy logic connectives are t-norms and t-conorms, which are defined as follows.

Definition 4.20. (Baczyński and Jayaram [9], Klement et al. [24])

(i) A function $T\colon [0,1]^2 \to [0,1]$ is called **triangular norm** (t-norm) if T is commutative, associative, increasing in both variables and having 1 as identity.

(ii) A function $S\colon [0,1]^2 \to [0,1]$ is called **triangular conorm** (t-conorm) if S is commutative, associative, increasing in both variables and having 0 as identity.

Now, in the following, we recall the strict monotonicity of t-norms and t-conorms.

Definition 4.21. (Baczyński and Jayaram [9], Klement et al. [24])

(i) A t-norm T is called strictly monotone if $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.

(ii) A t-conorm S is called strictly monotone if $S(x,y) < S(x,z)$ whenever $x < 1$ and $y < z$.

Some examples of basic t-norms and t-conorms w.r.t. strict monotonicity are presented in Tables 5 and 6, respectively.

Remark 4.22. Recall that t-norms and t-conorms are also binary aggregation functions on $[0,1]$. However, from Definitions 4.4 and 4.21, it follows that the strict monotonicity of t-norms and t-conorms is slightly different from that of general binary aggregation functions on $[0,1]$ in the sense that the t-norms (t-conorms) satisfying Definition 4.21(i) (Definition 4.21(ii)) cannot be strictly monotone aggregation functions because they have an annihilator.

Name	Formula	Strict monotonicity
minimum	$T_{\mathbf{M}}(x,y) = \min(x,y)$	×
algebraic product	$T_{\mathbf{P}}(x,y) = xy$	✓
Łukasiewicz	$T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$	×
drastic product	$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$	×
nilpotent minimum	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$	×

Tab. 5. Examples of basic t-norms w.r.t. strict monotonicity, see [24].

Name	Formula	Strict monotonicity
maximum	$S_{\mathbf{M}}(x,y) = \max(x,y)$	×
probabilistic sum	$S_{\mathbf{P}}(x,y) = x+y-xy$	✓
Łukasiewicz	$S_{\mathbf{LK}}(x,y) = \min(x+y,1)$	×
drastic sum	$S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$	×
nilpotent maximum	$S_{\mathbf{nM}}(x,y) = \begin{cases} 1, & \text{if } x+y \geq 1 \\ \max(x,y), & \text{otherwise} \end{cases}$	×

Tab. 6. Examples of basic t-conorms w.r.t. strict monotonicity, [24].

In the following, we examine the strict monotonicity of $(I \odot_A J)$ in the case A is either a strictly monotone t-norm or A is strictly monotone t-conorm.

Theorem 4.23. Let I, J be strictly monotone fuzzy implications and A be a strictly monotone t-norm(strictly monotone t-conorm) on $[0,1]$. Then $(I \odot_A J)$ is also strictly monotone.

P r o o f . Let I, J be strictly monotone fuzzy implications and A be a strictly monotone t-norm on $[0,1]$.

(i) We show that $(I \odot_A J)$ satisfies (SM2). Let $x, y, z \in [0,1]$ be such that $y < z$. Note that from the strict monotonicity of I, J and Lemma 2.3(ii) in [20], it follows that

$$I(x,y) = 0 \iff (x,y) = (1,0) \iff J(x,y) = 0. \tag{3}$$

- Let $I(x,y) = 0$. Then, from Eq.(3), it follows that $x = 1$ and $y = 0$ and hence, we have also $J(x,y) = 0$. Hence, $(I \odot_A J)(x,y) = (I \odot_A J)(1,0) = 0$. Since $y < z$ and $y = 0$, we have $I(x,z) = I(1,z) > 0$ and also, $J(x,z) > 0$. Consequently, $(I \odot_A J)(x,z) > 0$, due to the fact that A has no zero divisors, follows from Proposition 2.11(iii) in [24]. Thus, $(I \odot_A J)(x,z) > 0 = (I \odot_A J)(x,y)$ and hence, $(I \odot_A J)$ satisfies (SM2).
- Let $I(x,y) > 0$. Then, from Eq. (3), it follows that $(x,y) \neq (1,0)$ and hence, $J(x,y) > 0$. Then clearly, from (SM2) of I and J, we obtain that $0 < I(x,y) < I(x,z)$ and $0 < J(x,y) < J(x,z)$. Finally, from the strict monotonicity of the t-norm A, it follows directly that, $0 < A(I(x,y), J(x,y)) < A(I(x,z), J(x,z))$. Thus, we have $(I \odot_A J)(x,y) < (I \odot_A J)(x,z)$, which implies that $(I \odot_A J)$ satisfies (SM2).

Thus, in both cases, we have shown that $(I \odot_A J)$ satisfies (SM2).

(ii) Similarly, it can be shown that $(I \odot_A J)$ satisfies (SM1).

The proof of this result in the case A is a strictly monotone t-conorm can be obtained similarly. □

Remark 4.24. (i) Note that Theorem 4.23 need not hold true if A is not strictly monotone. To see this, let the aggregation A and fuzzy implications I, J be defined, respectively, by $A = T_{\mathbf{LK}}$, $I = I_{\mathbf{RC}}$ and $J(x,y) = 1 - x^2 + x^2y^2$, for all $x, y \in [0,1]$. It is clear that while A is a t-norm that is not strictly monotone, the fuzzy implications I and J are strictly monotone. Now, it is easy to see that

$$(I \odot_A J)(1, 0.1) = 0 = (I \odot_A J)(1, 0.2),$$

with $0.1 < 0.2$. This implies that $(I \odot_A J)$ does not satisfy (SM2).

(ii) In the above point, we have shown that if the t-norm (t-conorm) A is not strictly monotone then the composition $(I \odot_A J)$ of strictly monotone fuzzy implications need not be again strictly monotone. However, there are some t-norms (t-conorms) which are not strictly monotone but the composition $(I \odot_A J)$ of strictly monotone fuzzy implications is again strictly monotone. We prove this fact for t-norm $T_{\mathbf{M}}$ and t-conorm $S_{\mathbf{M}}$ in the following.

4.3.1. Lattice operations on fuzzy implications

In the literature, it was shown that the point-wise operations max and min will lead to generate fuzzy implications and a lattice structure on the set of fuzzy implications. In the following, we recall this result.

Theorem 4.25. (Baczyński and Jayaram [9]) The family $(\mathbb{I}, \leq)$ is a complete, completely distributive lattice with the lattice operations

$$(I \vee J)(x, y) := \max(I(x, y), J(x, y)), \qquad x, y \in [0, 1],$$
$$(I \wedge J)(x, y) := \min(I(x, y), J(x, y)), \qquad x, y \in [0, 1],$$

where $I, J \in \mathbb{I}$.

It is needless to mention that the compositions $I \vee J$ and $I \wedge J$ are a special cases of the composition $(I \odot_A J)$, where $A = S_{\mathbf{M}}$ and $A = T_{\mathbf{M}}$, respectively. Now, as mentioned in Remark 4.24(ii), in the following, we prove that fuzzy implications $I \vee J$ and $I \wedge J$ are strictly monotone for all strictly monotone fuzzy implications I and J.

Theorem 4.26. If I and J are strictly monotone fuzzy implications, then $I \vee J$ and $I \wedge J$ are also strictly monotone.

P r o o f . Let I and J be strictly monotone fuzzy implications. We prove only that $I \vee J$ is strictly monotone as the proof of strict monotonicity of $I \wedge J$ follows in a similar way. Let $x, y \in [0, 1]$, $z \in [0, 1)$ be arbitrarily fixed such that $(I \vee J)(x, z) = (I \vee J)(y, z)$. i. e., $\max(I(x, z), J(x, z)) = \max(I(y, z), J(y, z))$. Now, we have the following four cases:

(i) Let $I(x, z) \geq J(x, z)$ & $I(y, z) \geq J(y, z)$. Then, it follows that $I(x, z) = I(y, z)$. Now, (SM1) of I implies $x = y$.

(ii) Let $I(x, z) \leq J(x, z)$ & $I(y, z) \leq J(y, z)$. Then, it follows that $J(x, z) = J(y, z)$, then (SM1) of J implies $x = y$.

(iii) Let $I(x, z) \geq J(x, z)$ & $I(y, z) \leq J(y, z)$. Then, it follows that $I(x, z) = J(y, z)$.

  (a) Assume that $x < y$. The (SM1) of J implies that $J(x, z) > J(y, z)$. As $I(x, z) = J(y, z)$, we get $J(x, z) > I(x, z)$, which is a contradiction.

  (b) Similarly, we get a contradiction in the case $x > y$.

  Thus, we have $x = y$.

(iv) The case $I(x, z) \leq J(x, z)$ & $I(y, z) \geq J(y, z)$ is similar to the case (iii).

Hence, in all the cases, we proved that $x = y$. Therefore, $I \vee J$ satisfies (SM1). Similarly, one can prove that $I \vee J$ satisfies (SM2).

Thus, $I \vee J$ is strictly monotone. □

4.3.2. Sup-T composition of fuzzy implications

In [6], the authors proposed the sup-T composition of fuzzy implications and investigated some algebraic aspects of fuzzy implications. We recall the sup-T composition of fuzzy implications in the following.

Definition 4.27. (Baczyński et al. [6], Baczyński and Jayaram [9]) Let $I, J \in \mathbb{I}$ and T be a t-norm. Then sup-T composition of I, J is given as follows:

$$(I \overset{T}{\circ} J)(x,y) = \sup_{t\in[0,1]} (T(I(x,t), J(t,y))), \qquad x,y \in [0,1].$$

Theorem 4.28. (Baczyński et al. [6], Baczyński and Jayaram [9]) Let T be a t-norm and I, J be any fuzzy implications. Then, $I \overset{T}{\circ} J \in \mathbb{I}$ if and only if $(I \overset{T}{\circ} J)(1,0) = 0$.

Towards the strict monotonicity of the sup-T composition of fuzzy implications, we present the following remark.

Remark 4.29. Note that, if I and J are fuzzy implications that satisfy (SM1)((SM2)), then $(I \overset{T}{\circ} J)$ need not satisfy the same always. To see this,

(i) let $I, J \in \mathbb{I}$ be defined as in Remark 4.6(i). Clearly, I and J satisfy (SM1). Let T be any t-norm. Then, it follows that $(I \overset{T}{\circ} J)(x,y) = \max(1-x, y)$, for all $x, y \in [0,1]$. Now, it is easy to see that, $(I \overset{T}{\circ} J)(0.2, 0.9) = 0.9 = (I \overset{T}{\circ} J)(0.3, 0.9)$ which implies that $(I \overset{T}{\circ} J)$ does not satisfy (SM1).

(ii) let $I, J \in \mathbb{I}$ be defined by $I = I_{\mathbf{RC}}$ and $J = I_{\mathbf{D}}$, respectively. Then clearly I and J satisfy (SM2). Let T be any t-norm. Then, it follows that, $(I \overset{T}{\circ} J)(x,y) = \max(1-x, y)$, for all $x, y \in [0,1]$. Now, it is easy to see that, $(I \overset{T}{\circ} J)(0.2, 0.3) = 0.8 = (I \overset{T}{\circ} J)(0.2, 0.4)$, which implies that $(I \overset{T}{\circ} J)$ does not satisfy (SM2).

From Remark 4.29, it follows that sup-T composition of strictly monotone fuzzy implications I, J need not satisfy the strict monotonicity always, thus leading to the following problem.

Problem 4.30. Characterize the strictly monotone fuzzy implications I, J and t-norm T such that sup-T composition $(I \overset{T}{\circ} J)$ satisfies the strict monotonicity.

4.4. Using n-ary aggregation functions

Recall that aggregation functions are, in general, n-ary functions defined on $[0,1]$. Due to this fact, Definition 4.2 of $(I \odot_A J)$ has been generalized naturally in [37] to n-arguments by taking A to be an n-ary aggregation function and n- fuzzy implications $I_1, I_2, \ldots, I_n$. Before recalling this generalization, we recall the n-ary aggregation function in the following.

Definition 4.31. (Grabisch et al. [18]) An n-ary aggregation function is a function $f : [0,1]^n \to [0,1]$ of $n > 1$ arguments that maps the (n-dimensional) unit cube on to the interval, with the following properties

(i) $f(\underbrace{0, 0, \ldots, 0}_{n-\text{times}}) = 0$ and $f(\underbrace{1, 1, \ldots, 1}_{n-\text{times}}) = 1$.

(ii) $\mathbf{x} \leq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$.

Definition 4.32. (Reiser et al. [37]) Let $A : [0,1]^n \to [0,1]$ be an n-ary aggregation function and $I_1, I_2, \ldots, I_n$ be fuzzy implications on $[0,1]$. Define a mapping $A_{I_1,I_2,\ldots,I_n} : [0,1]^2 \to [0,1]$ by

$$A_{I_1,I_2,\ldots,I_n}(x,y) = A(I_1(x,y), I_2(x,y), \ldots, I_n(x,y)), \qquad x, y \in [0,1].$$

Theorem 4.33. (Reiser et al. [37]) Let $A : [0,1]^n \to [0,1]$ be an n-ary aggregation function and $I_1, I_2, \ldots, I_n$ be fuzzy implications on $[0,1]$. Then $A_{I_1,I_2,\ldots,I_n}$ is a fuzzy implication.

We examine the strict monotonicity of $A_{I_1,I_2,\ldots,I_n}$ in terms of that of fuzzy implications $I_1, I_2, \ldots, I_n$ and aggregation function A defined on $[0,1]$. Before doing so, recall that an n-ary aggregation function is strictly monotone if it is strictly monotone in each variable.

Theorem 4.34. Let $A : [0,1]^n \to [0,1]$ be an n-ary strictly monotone aggregation function and $I_1, I_2, \ldots, I_n$ be strictly monotone fuzzy implications on $[0,1]$. Then $A_{I_1,I_2,\ldots,I_n}$ is also a strictly monotone fuzzy implication on $[0,1]$.

P r o o f . The proof is similar to that of Theorem 4.5. □

Remark 4.35. Note that Theorem 4.34 is a natural generalization of $(I \odot_A J)$ to n-arguments. Due to this fact, a direct generalization of all the contents of Sections 4.1, 4.2 and 4.3 still hold true for the mapping $A_{I_1,I_2,\ldots,I_n}$ defined in Definition 4.32.

5. GENERATING METHODS OF FUZZY IMPLICATIONS THAT INVOLVE ONLY FUZZY IMPLICATIONS

In the literature, note that, there are some generating methods of fuzzy implications that involve only fuzzy implications and no other functions and parameters. For more details about these generating methods, please refer to [30, 31, 40].

5.1. The $\triangledown$-composition of fuzzy implications

In [40], the authors proposed a novel way of generating fuzzy implications by using fuzzy implications, namely the $\triangledown$-composition method and showed that $(\mathbb{I}, \triangledown)$ is a semigroup. In the following, we recall this method.

Definition 5.1. (Vemuri and Jayaram [40]) For fuzzy implications I and J, define the function $I \triangledown J$ as

$$(I \triangledown J)(x,y) = I(J(y,x), J(x,y)), \qquad x, y \in [0,1].$$

Theorem 5.2. (Vemuri and Jayaram [40]) For fuzzy implications I and J, the function $I \triangledown J$ is a fuzzy implication on $[0,1]$, i. e., $I \triangledown J \in \mathbb{I}$.

Theorem 5.3. (Vemuri and Jayaram [40]) $(\mathbb{I}, \triangledown)$ forms a semigroup.

In the following, we examine the strict monotonicity of fuzzy implication $I \triangledown J$ in terms of that of I and J.

Theorem 5.4. Let I, J be strictly monotone fuzzy implications. Then $I \triangledown J$ is also strictly monotone.

Proof.

Let $I, J \in \mathbb{I}$ be strictly monotone. We prove that $I \triangledown J$ satisfies (SM).

(i) We prove here that $I \triangledown J$ satisfies (SM2). Let $x \in (0, 1]$ and $y, z \in [0, 1]$ with $y < z$.

- Let $x = 1$. Then, we have,

$$(I \triangledown J)(1, y) = I(J(y, 1), J(1, y)) = I(1, J(1, y)),$$
$$(I \triangledown J)(1, z) = I(J(z, 1), J(1, z)) = I(1, J(1, z)).$$

  Thus, $(I \triangledown J)(1, y) = I(1, J(1, y)) < I(1, J(1, z)) = (I \triangledown J)(1, z)$, follows due to the fact that I and J satisfy (SM2).

- Let $x \in (0, 1)$. Then due to (SM1) and (SM2) of I and J, we have

$$I(x, y) < I(x, z) \text{ and } I(z, x) < I(y, x),$$
$$J(x, y) < J(x, z) \text{ and } J(z, x) < J(y, x).$$

$$\begin{aligned} \text{Now, } (I \triangledown J)(x, y) &= I(J(y, x), J(x, y)) < I(J(z, x), J(x, y)) \\ &< I(J(z, x), J(x, z)) = (I \triangledown J)(x, z), \end{aligned}$$

  which implies that $I \triangledown J$ satisfies (SM2).

(ii) Similarly, it can be proved that $I \triangledown J$ satisfies (SM1).

Thus, $I \triangledown J$ is strictly monotone. □

Due to the fact that $(\mathbb{I}, \triangledown)$ forms a semigroup, every fuzzy implication I can be composed to itself n-times to obtain fuzzy implications. The self-composition of I w.r.t. $\triangledown$ is defined as follows.

Definition 5.5. (Vemuri and Jayaram [40]) Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the nth power of I w.r.t the binary operation $\triangledown$ as follows:

$$\text{For n=1,} \qquad I_{\triangledown}^{[n]}(x, y) = I(x, y),$$

$$\text{and for } n \geq 1, \; I_{\triangledown}^{[n]}(x, y) = I\left(x, I_{\triangledown}^{[n-1]}(x, y)\right) = I_{\triangledown}^{[n-1]}\left(x, I(x, y)\right), \qquad x, y \in [0, 1].$$

Corollary 5.6. If I is a strictly monotone fuzzy implication then $I_{\triangledown}^{[n]}$ is strictly monotone, for all $n \in \mathbb{N}$.

Proof. Follows from Theorem 5.4. □

5.2. The $\circledast$-composition of fuzzy implications

In [40], the authors proposed another novel generating method of fuzzy implications without using any functions and explored this method in [40, 41, 42]. It is defined as follows.

Definition 5.7. (Vemuri and Jayaram [40]) For any $I, J \in \mathbb{I}$, we define $I \circledast J \colon [0,1]^2 \longrightarrow [0,1]$ as

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x, y \in [0,1].$$

Theorem 5.8. (Vemuri and Jayaram [40]) For all $I, J \in \mathbb{I}$, we have $I \circledast J \in \mathbb{I}$ and $(\mathbb{I}, \circledast)$ is a monoid.

Now, in the following, we investigate the strict monotonicity of $I \circledast J$ in terms of those of I and J.

Theorem 5.9. If the fuzzy implications I and J are strictly monotone then so is $I \circledast J$.

Proof. Let $I, J \in \mathbb{I}$ be strictly monotone. We show that $I \circledast J$ also satisfies strict monotonicity.

(i) Let $I, J \in \mathbb{I}$ satisfy (SM2) and $x \in (0,1]$ and $y, z \in [0,1]$ with $y < z$. Then, we have $J(x,y) < J(x,z)$, follows due to (SM2) of J. Then, we get that $(I \circledast J)(x,y) = I(x, J(x,y)) < I(x, J(x,z)) = (I \circledast J)(x,z)$, which implies that $I \circledast J$ satisfies (SM2).

(ii) Similarly, we can prove that if $I, J \in \mathbb{I}$ satisfy (SM1) then $I \circledast J$ also satisfies the same.

Thus, if I and J are strictly monotone then $I \circledast J$ is also strictly monotone. □

Due to the fact that $(\mathbb{I}, \circledast)$ is a monoid, a fuzzy implication I can be composed to itself n times to obtain new fuzzy implications. The self-composition of I w.r.t. $\circledast$ is defined as follows.

Definition 5.10. (Vemuri and Jayaram [40]) Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the nth power of I w.r.t. the operation $\circledast$ as follows:

$$\text{For n=1,} \qquad I_{\circledast}^{[n]}(x,y) = I(x,y),$$

and for $n \geq 2$,
$$I_{\circledast}^{[n]}(x,y) = I\left(x, I_{\circledast}^{[n-1]}(x,y)\right) = I_{\circledast}^{[n-1]}\left(x, I(x,y)\right), \quad x, y \in [0,1].$$

Corollary 5.11. If $I \in \mathbb{I}$ is strictly monotone then $I_{\circledast}^{[n]}$ is strictly monotone, for all $n \in \mathbb{N}$.

Proof. Follows from Theorem 5.9. □

5.3. The threshold generating method of fuzzy implications

In [30], the authors proposed a new construction method of fuzzy implications from two given ones and characterized the family of h-implications using this method. This method is defined as follows.

Definition 5.12. (Massanet and Torrens [30]) Let I_1, I_2 be two fuzzy implications on $[0,1]$ and $e \in (0,1)$. The function $I_{I_1-I_2} : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$, by

$$I_{I_1-I_2}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ e \cdot I_1\Big(x, \dfrac{y}{e}\Big), & \text{if } x > 0 \text{ and } y \le e, \\ e + (1-e) \cdot I_2\Big(x, \dfrac{y-e}{1-e}\Big), & \text{if } x > 0 \text{ and } y > e, \end{cases}$$

is called the e-threshold generated implication from I_1 and I_2.

Remark 5.13. Note that from Proposition 4 in [30], it follows that for any fuzzy implications I_1 and I_2, we have $I_{I_1-I_2}(x,e) = e$, for all $x > 0$. This implies that the fuzzy implication $I_{I_1-I_2}$ does not satisfy (SM1).

In the following, we investigate (SM2) of $I_{I_1-I_2}$ in terms of that of I_1 and I_2.

Theorem 5.14. If $I_1, I_2 \in \mathbb{I}$ satisfy (SM2) then the e-threshold generated fuzzy implication $I_{I_1-I_2}$ also satisfies (SM2).

P r o o f . Let $I_1, I_2 \in \mathbb{I}$ satisfy (SM2) and $x \in (0,1], y, z \in [0,1]$ be such that $I_{I_1-I_2}(x,y) = I_{I_1-I_2}(x,z)$. Note that, we have the following possibilities w.r.t. y and z: $y, z \in [0,e]$ or $y, z \in (e,1]$ or y, z belong to the different intervals.

(i) Let $y, z \in [0,e]$. Then, from $I_{I_1-I_2}(x,y) = I_{I_1-I_2}(x,z)$, it follows that $e \cdot I_1\Big(x, \dfrac{y}{e}\Big) = e \cdot I_1\Big(x, \dfrac{z}{e}\Big)$, which implies that $y = z$, due to (SM2) of I_1.

(ii) Let $y, z \in (e,1]$. Then, from $I_{I_1-I_2}(x,y) = I_{I_1-I_2}(x,z)$, it follows that $e + (1-e) \cdot I_2\Big(x, \dfrac{y-e}{1-e}\Big) = e + (1-e) \cdot I_2\Big(x, \dfrac{z-e}{1-e}\Big)$, which implies that $y = z$, due to (SM2) of I_2 and $e \in (0,1)$.

(iii) Let y, z belong to the different intervals and suppose $y \in [0,e]$ and $z \in (e,1]$. Then, from Proposition 4 in [30], it follows that $I_{I_1-I_2}(x,y) \le e$ and $I_{I_1-I_2}(x,z) \ge e$, which implies that $I_{I_1-I_2}(x,y) = e = I_{I_1-I_2}(x,z)$, due to the assumption $I_{I_1-I_2}(x,y) = I_{I_1-I_2}(x,z)$. Now, $I_{I_1-I_2}(x,z) = e$ implies that $e + (1-e) \cdot I_2\Big(x, \dfrac{z-e}{1-e}\Big) = e$ and consequently, $(1-e) \cdot I_2\Big(x, \dfrac{z-e}{1-e}\Big) = 0$. Thus, $I_2\Big(x, \dfrac{z-e}{1-e}\Big) = 0$. Now we claim that $\dfrac{z-e}{1-e} = 0$. Assume that $\dfrac{z-e}{1-e} > 0$. Then since $I_2 \in \mathbb{I}$, we have $I_2\Big(x, \dfrac{z-e}{1-e}\Big) \ge I_2\Big(1, \dfrac{z-e}{1-e}\Big)$. Again, since I_2 satisfies (SM2) and $\dfrac{z-e}{1-e} > 0$,

therefore $I_2\Big(1, \dfrac{z-e}{1-e}\Big) > I_2\Big(1,0\Big)$. This implies that $I_2\Big(x, \dfrac{z-e}{1-e}\Big) > I_2\Big(1,0\Big)$, which again implies that $0 > 0$, a contradiction. Hence, $\dfrac{z-e}{1-e} = 0$. Consequently, $z = e$. This is a contradiction to the assumption $z \in (e, 1]$. Thus, y, z cannot belong to the different intervals simultaneously.

From the above cases, it follows that $I_{I_1-I_2}$ satisfies (SM2). □

5.4. The vertical e-threshold generating method of fuzzy implications

In [31], the authors proposed another generating method of fuzzy implication, similar to the one defined in Definition 5.12.

Definition 5.15. (Massanet and Torrens [31]) Let I_1, I_2 be two fuzzy implications on $[0,1]$ and $e \in (0,1)$. The function $I_{I_1|I_2} : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$, by

$$I_{I_1|I_2}(x,y) = \begin{cases} e + (1-e)\cdot I_1\Big(\dfrac{x}{e}, y\Big), & \text{if } x < e \text{ and } y < 1, \\ e \cdot I_2\Big(\dfrac{x-e}{1-e}, y\Big), & \text{if } x \geq e \text{ and } y < 1, \\ 1, & \text{if } y = 1, \end{cases}$$

is called the vertical e-threshold generated implication from I_1 and I_2.

Remark 5.16. From Proposition 4 in [31], note that, for any two fuzzy implications I_1 and I_2, we have $I_{I_1|I_2}(e,y) = e$, for all $y \in [0,1)$. This implies that $I_{I_1|I_2}$ does not satisfy (SM2).

However, in the following, we examine (SM1) for the fuzzy implication $I_{I_1|I_2}$ in terms of that of I_1 and I_2.

Theorem 5.17. If $I_1, I_2 \in \mathbb{I}$ satisfy (SM1) then the vertical e-threshold generated fuzzy implication $I_{I_1|I_2}$ also satisfies (SM1).

Proof. Let $I_1, I_2 \in \mathbb{I}$ satisfy (SM1) and $x, y \in [0,1], z \in [0,1)$ be such that $I_{I_1|I_2}(x,z) = I_{I_1|I_2}(y,z)$. Note that, we have the following possibilities w.r.t. x and y: $x, y \in [0,e)$ or $x, y \in [e,1]$ or x, y belong to the different intervals.

(i) Let $x, y \in [0,e)$. Then, from $I_{I_1|I_2}(x,z) = I_{I_1|I_2}(y,z)$, it follows that $e + (1-e)\cdot I_1\Big(\dfrac{x}{e}, z\Big) = e + (1-e)\cdot I_1\Big(\dfrac{y}{e}, z\Big)$, which implies that $x = y$, due to (SM1) of I_1 and $e \in (0,1)$.

(ii) Let $x, y \in [e,1]$. Then, it follows from $I_{I_1|I_2}(x,z) = I_{I_1|I_2}(y,z)$, that $e \cdot I_2\Big(\dfrac{x-e}{1-e}, z\Big) = e \cdot I_2\Big(\dfrac{y-e}{1-e}, z\Big)$, which implies that $y = z$, due to (SM1) of I_2.

(iii) Let x, y belong to the different intervals and suppose $x \in [0,e)$ and $y \in [e,1]$. Then, from Proposition 4 in [31], it follows that $I_{I_1|I_2}(x,z) \geq e$ and $I_{I_1|I_2}(y,z) \leq e$, which implies that $I_{I_1|I_2}(x,z) = e = I_{I_1|I_2}(y,z)$, due to the assumption $I_{I_1|I_2}(x,z) =$

$I_{I_1|I_2}(y,z)$. Now, $I_{I_1|I_2}(x,z) = e$ implies that $e = e + (1-e)\cdot I_1\Big(\frac{x}{e}, z\Big)$ and consequently, $I_1\Big(\frac{x}{e}, z\Big) = 0$. Thus, $\frac{x}{e} = 1$ and $z = 0$, follows due to Lemma 2.3(ii) in [20]. Hence, $x = e$ and $z = 0$, a contradiction to the assumption $x \in [0, e)$. Thus, x, y cannot belong to the different intervals simultaneously.

From the above cases, it follows that $I_{I_1|I_2}$ satisfies (SM1). □

6. STRICT MONOTONICITY OF SOME NEW CLASSES OF FUZZY IMPLICATIONS

Recall that in [39] the author proposed some new classes of fuzzy implications that satisfy the generalized hypothetical syllogism. In this section, we investigate the strict monotonicity for some of these fuzzy implications and use them to generate, via $(I \odot_A J)$, fuzzy implications that satisfy the strict monotonicity. Towards this, we recall the following definition from [39].

Definition 6.1. (Vemuri [39]) Let N be a strict fuzzy negation on $[0,1]$ and $\psi \in \Phi$. Define $I, J\colon [0,1]^2 \longrightarrow [0,1]$ as follows:

$$I(x,y) = \begin{cases} 1, & \text{if } y = 1, \\ N(x), & \text{if } y < 1, \end{cases} \tag{4}$$

$$J(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \psi(y), & \text{if } x > 0. \end{cases} \tag{5}$$

Lemma 6.2. (Vemuri [39]) The functions I, J defined as in Definition 6.1 are fuzzy implications.

Let us denote by

- $\mathbb{I}_{\mathbb{N}}$ - the set of fuzzy implications of the form (4) for some strict fuzzy negation N.
- $\mathbb{S}$ - the set of fuzzy implications of the form (5) for some increasing bijection $\psi \in \Phi$.

Now, in the following, we discuss the strict monotonicity of fuzzy implications coming from $\mathbb{I}_{\mathbb{N}} \cup \mathbb{S}$.

Theorem 6.3. Let I be a fuzzy implication on $[0,1]$. Then the following statements hold true:

(i) If $I \in \mathbb{I}_{\mathbb{N}}$ then I satisfies (SM1).

(ii) If $I \in \mathbb{S}$ then I satisfies (SM2).

P r o o f . Let I be a fuzzy implication on $[0,1]$.

(i) Let $I \in \mathbb{I}_{\mathbb{N}}$. Then, I is of the form (4) for some strict fuzzy negation N. Assume that $I(x,z) = I(y,z)$ for some $x, y \in [0,1]$ and $z < 1$. Then by definition of I, we have, $N(x) = N(y)$. Since N is strict, it follows directly that $x = y$. Hence, I satisfies (SM1).

(ii) Let $I \in \mathbb{S}$. Then, I is of the form (5) for some $\psi \in \Phi$. Assume that $I(x,y) = I(x,z)$ for some $x \in (0,1]$ and $y, z \in [0,1]$. Then by definition of I, we have, $\psi(y) = \psi(z)$. Since ψ is an increasing bijection, it follows that $y = z$. Hence, I satisfies (SM2).

This completes the proof. □

From Theorem 6.3, note that, if $I \in \mathbb{I}_{\mathbb{N}}$ (or $\mathbb{S}$) then I satisfies (SM1) (or (SM2)) only but not strict monotonicity. However, in the following, using the $(I \odot_A J)$ composition, we construct fuzzy implications that satisfy the strict monotonicity.

Theorem 6.4. Let A be a strictly monotone aggregation function and I, J be two fuzzy implications. Then the following statements hold true.

(i) If $I, J \in \mathbb{I}_{\mathbb{N}}$ then $(I \odot_A J)$ satisfies (SM1).

(ii) If $I, J \in \mathbb{S}$ then $(I \odot_A J)$ satisfies (SM2).

(iii) If $I \in \mathbb{I}_{\mathbb{N}}$ and $J \in \mathbb{S}$ then $(I \odot_A J)$ is strictly monotone.

P r o o f . Assume the hypothesis. We prove only (iii), as the proof of (i) and (ii) can be obtained similarly. Let $I \in \mathbb{I}_{\mathbb{N}}$ and $J \in \mathbb{S}$. Then I and J are of the forms given in Eqs.(4) and (5) respectively, for some strict negation N and increasing bijection ψ. Then, the fuzzy implication $(I \odot_A J)$ is given by

$$(I \odot_A J)(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ A(N(x), \psi(y)), & \text{if } x > 0 \text{ and } y < 1, \end{cases} \qquad x, y \in [0,1]. \tag{6}$$

Now, we show that $(I \odot_A J)$ satisfies (SM2). Let $x > 0$ and $y, z \in [0,1]$ be such that $(I \odot_A J)(x,y) = (I \odot_A J)(x,z)$. Then, we get that $A(N(x), \psi(y)) = A(N(x), \psi(z))$, which implies that $\psi(y) = \psi(z)$, due to the strict monotonicity of A and N. Then, finally $\psi(y) = \psi(z)$ implies that $y = z$, following the strictness of ψ. Thus, $(I \odot_A J)$ satisfies (SM2). The proof $(I \odot_A J)$ satisfies (SM1) follows similarly. Thus, the fuzzy implication $(I \odot_A J)$ satisfies (SM). □

Remark 6.5. Note that Theorem 6.4 need not hold true if A is not strictly monotone. To see this, let $A(x,y) = \max(x,y)$, for all $x, y \in [0,1]$ and $I, J \in \mathbb{I}$ be given by Eqs.(4) and (5) where $N(x) = 1 - x$ and $\psi(x) = x$, for all $x \in [0,1]$. Then, it is easy to obtain that $(I \odot_A J)(x,y) = \max(1-x, y) = I_{\mathbf{KD}}(x,y)$, for all $x, y \in [0,1]$. Then, it follows from Remark 2.4 in [20] that $I_{\mathbf{KD}}$ is not strictly monotone.

7. CONCLUDING REMARKS

In [20], the authors proposed the strict monotonicity for fuzzy implications and studied the same w.r.t. various properties, functional equations and families of fuzzy implications. In this work, we have investigated the preservation of strict monotonicity by different generating methods of fuzzy implications that are available in the literature.

In particular, we have investigated the conditions under which the N-reciprocal method, φ-conjugates of fuzzy implications and their natural generalizations preserve the strict monotonicity. As an interesting outcome of this study, we proved that self conjugate fuzzy implications are not strictly monotone, thus the strict monotonicity becomes an important criterion to characterize the class of self conjugate fuzzy implications. Further, we have shown that the generating methods of fuzzy implications such as the lattice operations, the convex combinations, the $\triangledown$-composition and the $\circledast$-composition always preserve the strict monotonicity. While the $(I \odot_A J)$-composition of fuzzy implications preserves the strict monotonicity for strictly monotone aggregation functions A, we showed that sup-T compositions of fuzzy implications need not preserve the same. Moreover, we also proved that the e-threshold method of strictly monotone fuzzy implications satisfies (SM2) only and the vertical e-threshold method of strictly monotone fuzzy implications satisfies (SM1) only. Finally, the strict monotonicity of some of the fuzzy implications proposed in [39] is investigated and some fuzzy implications satisfying the strict monotonicity are obtained via the $(I \odot_A J)$-composition, in the case A is strictly monotone aggregation.

Note that, in this work and [20], we have studied various properties, functional equations and families of fuzzy implications w.r.t. strict monotonicity and investigated the preservation of the same by different generating methods of fuzzy implications. Also, it is to be emphasized that so far, we have achieved some hitherto unknown insights of fuzzy implications without considering any additional properties/conditions along with the strict monotonicity. However, it is always worth considering some properties such as continuity along with strict monotonicity and exploring some representations results of some sub-classes of fuzzy implications, similar to the ones obtained for strict fuzzy negations, strict t-norms and strict t-conorms, please see [9, 24] for more details.

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DECLARATION OF CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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Priyapada Hembram, School of Mathematics and Statistics, University of Hyderabad. India.

e-mail: priyapadahembramsb@gmail.com

Nageswara Rao Vemuri, School of Mathematics and Statistics, University of Hyderabad. India.

e-mail: nrvemuriz@uohyd.ac.in