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A NOTE ON RESOLVING INCONSISTENCY OF FUZZY RELATIONAL EQUATIONS ON ŁUKASIEWICZ ALGEBRA

Pingke Li

The inconsistency of fuzzy relational equations is frequently encountered when modelling an inference system with fuzzy logic. A polynomial-time direct search procedure is designed to resolve the inconsistency for fuzzy relational equations on Łukasiewicz algebra by minimizing the maximum absolute deviation to the right-hand side vector. Alternatively, a reformulation approach is proposed by minimizing instead the sum of absolute deviations. It results in a polynomial-sized mixed integer linear programming problem and hence, may be readily solved with the aid of an off-the-shelf optimization solver. This reformulation approach is flexible in nature and can be adjusted to deal with some other problems related to the inconsistency of fuzzy relational equations.

Keywords: fuzzy relational equations, inconsistency, optimal approximation, mixed integer linear programming

Classification: 08A72, 15B15, 90C11

1. INTRODUCTION

Fuzzy relational equations, initially proposed by Sanchez [32] in 1970s, play an important role in approximate reasoning as a formulation of fuzzy if-then rules. As pointed out by Di Nola et al. [13] and Perfilieva and Novák [31], the consistency of fuzzy relational equations is crucial for the approximate dependence to be continuous when modelling a continuous dependency with fuzzy if-then rules.

In the theory of fuzzy logic, fuzzy relational equations may be defined using a general continuous triangular norm, of which the minimum, the product, and the Łukasiewicz triangular norm are the three most important prototypes, see, e. g., Hájek [17] and Klement et al. [18]. Fuzzy relational equations have been extensively investigated from both the theoretical and practical perspectives during the last several decades, see, e. g., Bělohlávek [4], De Baets [12] Li and Fang [24, 25], Peeva and Kyosev [30], and the references therein. Nevertheless, some issues related to the inconsistency of fuzzy relational equations are still open. This paper addresses these issues for fuzzy relational equations on Łukasiewicz algebra applying the ideas presented in Cimler et al. [9] and Li [20, 22].

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Łukasiewicz algebra concerned in this context may be viewed as an idempotent semiring $([0,1], \vee, \wedge_\ell)$ over the unit interval $[0,1]$ endowed with the maximum operation $\vee$ as "addition" and the Łukasiewicz triangular norm $\wedge_\ell$ as "multiplication," such that for all $a, b \in [0,1]$,

$$a \vee b = \max\{a, b\}$$

and

$$a \wedge_\ell b = \max\{a + b - 1, 0\}.$$

Note that $a \wedge_\ell (b + x) = a \wedge_\ell b + x$ for a positive value of x whenever $b + x \in [0,1]$ and $a + b \geq 1$. This truncated linearity property makes the Łukasiewicz triangular norm unique among all triangular norms.

A finite system of fuzzy relational equations defined over Łukasiewicz algebra may be formulated as

$$\bigvee_{j \in N} (a_{ij} \wedge_\ell x_j) = b_i, \quad \forall i \in M,$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ are two index sets, $a_{ij}, x_j, b_i \in [0,1]$ for each $i \in M$ and $j \in N$. Such a system of fuzzy relational equations may be written in its matrix form as

$$A \circ \boldsymbol{x} = \boldsymbol{b}$$

with $A = (a_{ij})_{m \times n}$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T$, respectively, and called a system of max-T_L equations for convenience. The symbol "$\circ$" is used to denote the composite operation using the maximum as "addition" and the Łukasiewicz triangular norm as "multiplication" (or the max-T_L composition for short) in a manner analogous to that used for matrices and vectors in linear algebra.

Within the framework of max-plus algebra, a modelling approach to discrete event systems, the equations of such type are defined analogously using the maximum and conventional addition for the composition and are commonly referred to as one-sided max-plus linear equations. The connections between max-T_L equations and max-plus equations have been discussed by Gavalec et al. [14] because the Łukasiewicz triangular norm can be regarded as a class of truncated linear functions. For a general discussion on max-plus algebra and its applications, the reader may refer to Baccelli et al. [1], Butkovič [5], and Gondran and Minoux [15].

Solving a system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ is to determine its solution set

$$S(A, \boldsymbol{b}) = \{\boldsymbol{x} \in [0,1]^n \mid A \circ \boldsymbol{x} = \boldsymbol{b}\}$$

for a given coefficient matrix A and a right-hand side vector $\boldsymbol{b}$. It is called consistent if $S(A, \boldsymbol{b}) \neq \emptyset$ and inconsistent otherwise. To test the consistency of a system $A \circ \boldsymbol{x} = \boldsymbol{b}$, a principal solution

$$\widehat{\boldsymbol{x}} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)^T$$

may be constructed as

$$\widehat{x}_j = \bigwedge_{i \in M} (a_{ij} \to_\ell b_i), \quad \forall j \in N,$$

where the notation $\wedge$ denotes the minimum operation

$$a \wedge b = \min\{a, b\}$$

and $\to_\ell$ denotes the residual implication with respect to the Łukasiewicz triangular norm, that is, for any $a, b \in [0, 1]$,

$$a \to_\ell b = \min\{1 - a + b, 1\}.$$

Note that $a \wedge_\ell x \leq b$ if and only if $x \leq a \to_\ell b$ and that $a \to_\ell b = 1$ whenever $a \leq b$. Besides, $a \to_\ell (b + x) = a \to_\ell b + x$ for a positive value of x whenever $a \geq b + x$. In a compact form, the principal solution may be denoted as

$$\widehat{\boldsymbol{x}} = A^T \triangleleft \boldsymbol{b}$$

where the symbol "$\triangleleft$" stands for the composite operation with the minimum as "addition" and the Łukasiewicz residual implication as "multiplication" (or the min-I_L composition for short).

It is well known that $A \circ \widehat{\boldsymbol{x}} \leq \boldsymbol{b}$ by the theory of residuation and hence $A \circ \boldsymbol{x} \leq \boldsymbol{b}$ if and only if $\boldsymbol{x} \leq \widehat{\boldsymbol{x}}$. Consequently, the consistency of $A \circ \boldsymbol{x} = \boldsymbol{b}$ is characterized by its principal solution, that is, $S(A, \boldsymbol{b}) \neq \emptyset$ if and only if $A \circ \widehat{\boldsymbol{x}} = \boldsymbol{b}$, in which case $\widehat{\boldsymbol{x}}$ is also the maximum solution in $S(A, \boldsymbol{b})$. However, the consistency of max-T_L equations, as well as other types of fuzzy relational equations, is sensitive to the perturbation or noise in the data. A system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ obtained during the modelling process via fuzzy logic may not necessarily possess an exact solution and need to be fine-tuned if the consistency is desired. In such a case, since $A \circ \widehat{\boldsymbol{x}} \leq \boldsymbol{b}$, the principal solution $\widehat{\boldsymbol{x}}$ induces only the best under approximation $A \circ \widehat{\boldsymbol{x}}$ to the right-hand side vector $\boldsymbol{b}$ in the sense that no vector lying between $A \circ \widehat{\boldsymbol{x}}$ and $\boldsymbol{b}$ would lead to a consistent system, which is not necessarily admissible for the modelling purpose.

In order to resolve the inconsistency of a system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$, whenever it presents, the right-hand vector $\boldsymbol{b}$ may be modified as slightly as possible, an intuitive idea proposed by Pedrycz [28, 29] originally for max-min equations. Formally, the problem concerned is to minimize $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_p$, or equivalently,

$$\begin{aligned} \min \quad & \|\boldsymbol{y} - \boldsymbol{b}\|_p \\ \text{s.t.} \quad & A \circ \boldsymbol{x} = \boldsymbol{y}, \end{aligned}$$

where $\|\cdot\|_p$ denotes the conventional L_p norm of vectors, e. g., $p = 1, 2$, or ∞. The research effort on this optimization problem has so far been devoted to max-min equations and max-plus equations particularly with respect to the L_∞ norm, while little is known for max-T_L equations except for a generic bisection search procedure developed by Li and Fang [27] and most recently an analytical approach presented by Baaj [2, 3].

For max-min equations, the most commonly used type of fuzzy relational equations, a polynomial-time procedure has been discovered by Cuninghame–Green and Cechlárová [11] and Cimler et al. [9] to obtain an optimal L_∞ approximation that is a vector to resolve the inconsistency with the least Chebyshev distance to the original right-hand side vector. An analogous procedure has also been developed by Cechlárová [6]

via the perturbation on the coefficient matrix. Besides, the bisection search procedure proposed by Li and Fang [27] works with the simultaneous perturbation on the coefficient matrix and right-hand side vector. However, when an optimal L_1 approximation is desired that is a vector to resolve the inconsistency with the least Manhattan distance to the original right-hand side vector, no algorithm has been developed to the best of our knowledge except for a reformulation approach recently documented by Li [22, 23].

For max-plus equations, an optimal L_∞ approximation may be explicitly constructed to resolve the inconsistency as demonstrated by Cuninghame–Green [10]. This direct approach has been generalized by Cechlárová and Cuninghame–Green [7] to the situation where both the coefficient matrix and right-hand side vector are allowed to be modified. Moreover, as shown by Li [20, 21], the optimal L_1 approximation may be obtained for max-plus equations by solving a mixed integer linear programming problem. Compared to the L_∞ approximation scenario, this reformulation approach is more flexible for the modelling purpose but requires much more computational effort.

This paper considers the problem of resolving the inconsistency, when it presents, of a system of max-T_L equations by modifying only the right-hand side vector. A polynomial-time direct search algorithm is designed to obtain an optimal L_∞ approximation for max-T_L equations applying an idea analogous to those for max-min equations in Cimler et al. [9], Cuninghame–Green and Cechlárová [11], and Li [22]. Besides, the reformulation approach in Li [20, 21] for max-plus equations is adjusted to obtain an optimal L_1 approximation for max-T_L equations taking advantage of the resemblance between these two types of equations.

The rest of this paper is organized as follows. Some results related to the consistency of max-T_L equations are presented in Section 2 and then followed by the description of the algorithm searching for an optimal L_∞ approximation. The reformulation procedure is demonstrated in Section 3 to transform the problem of minimizing $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$ into an equivalent polynomial-sized mixed integer linear programming problem. Some numerical examples are presented in both Section 2 and Section 3 for illustration and comparison. The conclusions are addressed in Section 4.

2. DIRECT METHOD FOR THE OPTIMAL L_∞ APPROXIMATION

In order to minimize $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_\infty$ when a system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ is inconsistent, it is intuitive to start with its principal solution $\widehat{\boldsymbol{x}} = A^T \triangleleft \boldsymbol{b}$ because it always holds that $A \circ \widehat{\boldsymbol{x}} \leq \boldsymbol{b}$ and $A \circ \widehat{\boldsymbol{x}}$ is the best under approximation so that no vector lying between $A \circ \widehat{\boldsymbol{x}}$ and $\boldsymbol{b}$ could restore the consistency. Denote

$$\delta = \tfrac{1}{2}\|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty.$$

It is straightforward that $\delta > 0$ since $S(A, \boldsymbol{b}) = \emptyset$ and

$$\min_{\boldsymbol{x} \in [0,1]^n} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_\infty \leq \|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty = 2\delta.$$

This provides an upper bound on the maximum absolute deviation for an optimal L_∞ approximation to the right-hand side vector.

On the other side, it holds that

$$A \circ ((\widehat{\boldsymbol{x}} + \delta) \wedge \boldsymbol{e}) \leq (A \circ \widehat{\boldsymbol{x}} + \delta) \wedge \boldsymbol{e}$$

due to the truncated linearity of the Łukasiewicz triangular norm, where $\boldsymbol{e}$ is the vector of all ones with its length being determined implicitly by the context. Note that the maximum or minimum of two vectors is calculated componentwise and the addition or subtraction between a vector and a scalar coincides with its counterpart in linear algebra. Consequently, it can be proved by contradiction that

$$\min_{\boldsymbol{x}\in[0,1]^n} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_\infty \geq \delta,$$

which identifies a lower bound on the maximum absolute deviation for an optimal L_∞ approximation. Otherwise, if

$$\min_{\boldsymbol{x}\in[0,1]^n} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_\infty = \delta' < \delta,$$

it follows that the system of max-T_L inequalities

$$(\boldsymbol{b} - \delta') \vee \boldsymbol{0} \leq A \circ \boldsymbol{x} \leq (\boldsymbol{b} + \delta') \wedge \boldsymbol{e}$$

is consistent with its maximum solution $A^T \triangleleft ((\boldsymbol{b} + \delta') \wedge \boldsymbol{e})$, where $\boldsymbol{0}$ is the vector of all zeros with a compatible length. Consequently, a modified principle solution may be constructed as

$$\widehat{\boldsymbol{x}}' = A^T \triangleleft ((\boldsymbol{b} + \delta') \wedge \boldsymbol{e}) - \delta'$$

so that $\widehat{\boldsymbol{x}}' \geq \boldsymbol{0}$ and

$$\boldsymbol{b} - 2\delta' \leq A \circ \widehat{\boldsymbol{x}}' \leq \boldsymbol{b}$$

due to the truncated linearity of the Łukasiewicz triangular norm. This implies that $\widehat{\boldsymbol{x}}' \leq \widehat{\boldsymbol{x}}$ and

$$\|A \circ \widehat{\boldsymbol{x}}' - \boldsymbol{b}\|_\infty \leq 2\delta' < 2\delta = \|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty,$$

which contradicts the fact that $A \circ \widehat{\boldsymbol{x}}$ is the best under approximation to the right-hand side vector $\boldsymbol{b}$.

Hence, it holds that

$$\delta \leq \min_{\boldsymbol{x}\in[0,1]^n} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_\infty \leq 2\delta$$

and one may try first the lifted principal solution

$$\widehat{\boldsymbol{x}}^\sharp = A^T \triangleleft ((\boldsymbol{b} + \delta) \wedge \boldsymbol{e})$$

as illustrated in Example 2.2, in order to obtain an optimal approximation that minimizes the maximum absolute deviation with respect to the right-hand side vector $\boldsymbol{b}$.

Remark 2.1. Such a lower bound property on the deviation level does not preserve for max-min equations as demonstrated by Cimler et al. [9] and Cuninghame–Green and Cechlárová [11], because the residual implication with respect to the minimum operation is not continuous.

Example 2.2. Consider the system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$, originally designed by Li and Fang [27], with

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

The associated principal solution is

$$\widehat{\boldsymbol{x}} = (0.7, 0.5, 1, 0.6)^T$$

while the system is inconsistent since

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.7 \\ 0.5 \\ 1 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.5 \\ 0.2 \end{pmatrix} \neq \begin{pmatrix} 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

It follows that

$$\delta = \tfrac{1}{2} \| A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b} \|_\infty = 0.5 \times 0.1 = 0.05$$

and

$$(\boldsymbol{b} + \delta) \wedge \boldsymbol{e} = (0.85, 0.65, 0.25)^T.$$

Consequently, the lifted principal solution becomes

$$\begin{pmatrix} 0.2 & 0.8 & 0.5 \\ 0.9 & 0.3 & 0.7 \\ 0.8 & 0.4 & 0.1 \\ 0.4 & 0.8 & 0.6 \end{pmatrix} \triangleleft \begin{pmatrix} 0.85 \\ 0.65 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.55 \\ 1 \\ 0.65 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.75 \\ 0.55 \\ 1 \\ 0.65 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.55 \\ 0.25 \end{pmatrix},$$

it indicates that for this particular instance the lifted principal solution

$$\widehat{\boldsymbol{x}}^\sharp = (0.75, 0.55, 1, 0.65)^T$$

does induce an optimal L_∞ approximation

$$\widehat{\boldsymbol{b}} = (0.8, 0.55, 0.25)^T$$

with $\| \widehat{\boldsymbol{b}} - \boldsymbol{b} \|_\infty = 0.05 = \delta$.

Note that the optimal L_∞ approximation to resolve the inconsistency of max-T_L equations is usually not unique as one may expect. For this instance,

$$\widehat{\boldsymbol{b}}' = (0.75, 0.55, 0.25)^T$$

is also an optimal L_∞ approximation such that $\|\widehat{\boldsymbol{b}}' - \boldsymbol{b}\|_\infty = 0.05 = \delta$ and

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.75 \\ 0.55 \\ 0.95 \\ 0.65 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.55 \\ 0.25 \end{pmatrix}.$$

Actually, any vector of the form

$$(0.75 + \varepsilon, 0.55, 0.25)^T$$

with $0 \leq \varepsilon \leq 0.05$ serves as an optimal L_∞ approximation for this particular instance since

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.75 \\ 0.55 \\ 0.95 + \varepsilon \\ 0.65 \end{pmatrix} = \begin{pmatrix} 0.75 + \varepsilon \\ 0.55 \\ 0.25 \end{pmatrix}.$$

However, to determine all the optimal L_∞ approximations to the right-hand side vector is in general not an easy task, which relies essentially on the geometrical structure of the space spanned by the columns in the coefficient matrix with max-T_L composition.

Example 2.2 illustrates that an optimal L_∞ approximation $A \circ \widehat{\boldsymbol{x}}^\sharp$ may be constructed directly via the lifted principal solution $\widehat{\boldsymbol{x}}^\sharp = A^T \triangleleft ((\boldsymbol{b}+\delta) \wedge \boldsymbol{e})$ when the lower bound $\delta = \frac{1}{2}\|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty$ is indeed admissible for resolving the inconsistency of max-T_L equations. Unfortunately, this lower bound may not be attainable in general. An extreme scenario is an inconsistent system such that $A \circ \boldsymbol{e} \leq \boldsymbol{b}$, for which the principal solution $\widehat{\boldsymbol{x}} = \boldsymbol{e}$ coincides with its lifted version and induces the optimal L_∞ approximation $A \circ \widehat{\boldsymbol{x}}$ with $\|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty = 2\delta$ reaching the upper bound of the deviation level.

Generally, when a system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ is inconsistent, a generic bisection search procedure developed by Li and Fang [27] may be launched to search the optimal L_∞ approximation by iteratively verifying the lifted principal solutions under different deviation levels of Chebyshev distance. This approach works as well when the coefficient matrix A and the right-hand side vector $\boldsymbol{b}$ are allowed to be modified simultaneously.

Nevertheless, by exploiting the properties of the Łukasiewicz triangular norm within the composite operation, a polynomial-time iterative procedure is proposed here as an alternative method to obtain the optimal L_∞ approximation. This direct search procedure, analogous to those developed for max-min equations by Cimler et al. [9], Cuninghame–Green and Cechlárová [11], and Li [22], determines first the minimum deviation level with the aid of a perturbation matrix and then constructs an optimal L_∞ approximation using the corresponding lifted principal solution.

A deviation level $d \in [\delta, 2\delta]$ on the right-hand side vector $\boldsymbol{b}$ is admissible in presence of inconsistency if and only if there is a solution $\boldsymbol{x}$ such that

$$(\boldsymbol{b} - d) \vee \boldsymbol{0} \leq A \circ \boldsymbol{x} \leq (\boldsymbol{b} + d) \wedge \boldsymbol{e}.$$

Consequently, the minimum deviation level $\widehat{d}$ is simply the greatest lower bound of these admissible deviation levels, which itself is admissible since the involved max-T_L and min-I_L compositions are monotone and continuous. That is, the set of all such admissible deviation levels is a closed interval $[\widehat{d}, 2\delta]$.

Under a given deviation level d, since $A \circ \boldsymbol{x} \leq (\boldsymbol{b} + d) \wedge \boldsymbol{e}$ if and only if $\boldsymbol{x} \leq A^T \triangleleft ((\boldsymbol{b} + d) \wedge \boldsymbol{e})$, it suffices to verify $A \circ \widehat{\boldsymbol{x}}^\sharp(d) \geq (\boldsymbol{b} - d) \vee \boldsymbol{0}$ with the corresponding lifted principal solution $\widehat{\boldsymbol{x}}^\sharp(d) = A^T \triangleleft ((\boldsymbol{b} + d) \wedge \boldsymbol{e})$. This requires that for each $i \in M$ there exists at least one active index $j \in N$ of the variables for the ith inequality, by which it means that

$$a_{ij} \wedge_\ell \widehat{x}_j^\sharp(d) \geq (b_i - d) \vee 0,$$

where

$$\widehat{x}_j^\sharp(d) = \bigwedge_{i \in M} \big(a_{ij} \to_\ell ((b_i + d) \wedge 1)\big)$$

is the jth component of the lifted principal solution $\widehat{\boldsymbol{x}}^\sharp(d)$ with respect to the deviation level d. Therefore, one may calculate the least perturbation required for each fixed pair of $i \in M$ and $j \in N$, denoted by d_{ij}, such that the jth variable is active for the ith inequality.

However, if $d_{ij} \geq 2\delta$, it is not necessary to calculate the value of d_{ij} accurately because it is already known that $\widehat{d} \leq 2\delta$. In such a case, one may set $d_{ij} = 2\delta$ directly to save some computational effort.

Once these values have been obtained and organized as a perturbation matrix $D = (d_{ij})_{m \times n}$, it is evident that the value of $\min_{j \in N} d_{ij}$ is the least perturbation required for the ith inequality such that there is at least one active index of the variables, and consequently, the minimum deviation level to resolve the inconsistency is determined as

$$\widehat{d} = \max_{i \in M} \min_{j \in N} d_{ij}.$$

Such an idea has also been applied to solve a class of max-separable optimization problems encountered in fuzzy algebra and max-plus algebra, see, e. g., Li and Fang [26], Tharwat and Zimmermann [33], Zimmermann [34], and the references therein.

In order to obtain the perturbation matrix, it is necessary to deal with the coupling relation between the deviation level and the lifted principal solution for each pair of $i \in M$ and $j \in N$. Denote, for each $j \in N$, the index set

$$M_j(d) = \{k \in M \mid a_{kj} > (b_k + d) \wedge 1\}$$

with respect to the deviation level d such that $\delta \leq d \leq 2\delta$. Note that $b_k + d < 1$ and $a_{kj} > b_k + d$ if $k \in M_j(d)$ and hence,

$$\widehat{x}_j^\sharp(d) = \bigwedge_{k \in M_j(d)} \big(a_{kj} \to_\ell (b_k + d)\big) = \bigwedge_{k \in M_j(d)} \big\{ - a_{kj} + b_k \big\} + 1 + d.$$

It follows that $\widehat{x}_j^\sharp(d) < 1$ if $M_j(d) \neq \emptyset$ and $\widehat{x}_j^\sharp(d) = 1$ otherwise. Furthermore, for given $i \in M$ and $j \in N$,

$$a_{ij} \wedge_\ell \widehat{x}_j^\sharp(d) \geq (b_i - d) \vee 0$$

if and only if

$$a_{ij} \geq (b_i - d) \vee 0$$

and

$$\big(a_{ij} + \min_{k \in M_j(d)} \big\{ - a_{kj} + b_k \big\} + d\big) \vee 0 \geq (b_i - d) \vee 0,$$

which reduces to

$$a_{ij} + \min_{k \in M_j(d)} \big\{ -a_{kj} + b_k \big\} + d \geq b_i - d$$

whenever $b_i - d > 0$. Note that, if $b_i - d \leq 0$, it holds trivially that

$$a_{ij} \wedge_\ell \widehat{x}_j^\sharp(d) \geq (b_i - d) \vee 0$$

and hence, $d_{ij} = \delta$ for all $j \in N$ if $b_i - \delta \leq 0$.

Consequently, in order to activate variable $x_j = \widehat{x}_j^\sharp(d)$ for the ith inequality under the condition $b_i - d > 0$, it requires both

$$d \geq b_i - a_{ij}$$

and

$$d \geq \tfrac{1}{2}\big(b_i - a_{ij} + \max_{k \in M_j(d)} \big\{a_{kj} - b_k\big\}\big),$$

which defines a lower bound of d provided that the index set $M_j(d)$ remains unchanged, i. e.,

$$d < \min_{k \in M_j(d)} \big\{a_{kj} - b_k\big\}.$$

In case of incompatibility, these two boundary values should be recalculated by increasing the value of d to the incumbent upper bound, if it does not exceed 2δ, and hence shrinking the index set $M_j(d)$. The least perturbation d_{ij} is then determined accordingly with a sophisticated implementation of this procedure in computational complexity of $O(m)$.

Subsequently, the calculation of the perturbation matrix D can be completed with computational complexity of $O(m^2 n)$. The technical details of the whole procedure are summarized in Algorithm 1.

Algorithm 1 Calculation of the perturbation matrix

Input: the coefficient matrix A, right-hand side vector $\boldsymbol{b}$, lower bound δ
Output: the perturbation matrix $D = (d_{ij})_{m \times n}$

for $i \in M$, $j \in N$ **do**
  Set $d = \max(\delta, b_i - a_{ij})$ and $d_{ij} = d$.
  while $d < 1$ **do**
    Determine $M_j(d) = \{k \in M \mid a_{kj} > (b_k + d) \wedge 1\}$.
    Calculate $\underline{d} = \tfrac{1}{2}\big(b_i - a_{ij} + \max_{k \in M_j(d)} \{a_{kj} - b_k\}\big)$.
    Calculate $\overline{d} = \min_{k \in M_j(d)} \{a_{kj} - b_k\}$.
    if $\underline{d} \geq \overline{d}$ and $\overline{d} \leq 2\delta$ **then**
      Set $d = \overline{d}$.
    else
      Set $d_{ij} = \max\{\underline{d}, d\} \wedge 2\delta$ and $d = 1$.
    end if
  end while
end for
return $D = (d_{ij})_{m \times n}$

Example 2.3. Consider the system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

The principal solution for this instance is

$$\widehat{\boldsymbol{x}} = (0.8, 0.6, 1, 0.7)^T$$

but the system is inconsistent since

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.8 \\ 0.6 \\ 1 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.3 \end{pmatrix} \neq \begin{pmatrix} 0.8 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

Hence,

$$\delta = \tfrac{1}{2} \| A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b} \|_\infty = 0.5 \times 0.2 = 0.1$$

and

$$(\boldsymbol{b} + \delta) \wedge \boldsymbol{e} = (0.9, 0.7, 0.4)^T.$$

The lifted principal solution is then calculated with respect to $(\boldsymbol{b} + \delta) \wedge \boldsymbol{e}$ as

$$\begin{pmatrix} 0.2 & 0.5 & 0.5 \\ 0.9 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0.1 \\ 0.4 & 0.8 & 0.6 \end{pmatrix} \triangleleft \begin{pmatrix} 0.9 \\ 0.7 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.7 \\ 1 \\ 0.8 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.9 \\ 0.7 \\ 1 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \end{pmatrix} \not\geq \begin{pmatrix} 0.7 \\ 0.5 \\ 0.2 \end{pmatrix}.$$

This means that the lower bound $\delta = 0.1$ of the deviation level is inadmissible for this instance. By resorting to Algorithm 1, the perturbation matrix turns out to be

$$D = \begin{pmatrix} 0.2 & 0.15 & 0.2 & 0.2 \\ 0.15 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \end{pmatrix}$$

and hence, the least perturbation is

$$\widehat{d} = 0.15.$$

As a result, the lifted principal solution, with respect to $(\boldsymbol{b} + \widehat{d}) \wedge \boldsymbol{e}$, becomes

$$\begin{pmatrix} 0.2 & 0.5 & 0.5 \\ 0.9 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0.1 \\ 0.4 & 0.8 & 0.6 \end{pmatrix} \triangleleft \begin{pmatrix} 0.95 \\ 0.75 \\ 0.45 \end{pmatrix} = \begin{pmatrix} 0.95 \\ 0.75 \\ 1 \\ 0.85 \end{pmatrix}$$

and the corresponding optimal L_∞ approximation $\widehat{\boldsymbol{b}}$ is

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.95 \\ 0.75 \\ 1 \\ 0.85 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.65 \\ 0.45 \end{pmatrix}$$

such that $\|\widehat{\boldsymbol{b}} - \boldsymbol{b}\|_\infty = 0.15 = \widehat{d}$. Besides, any vector of the form

$$(0.65, 0.45 + \varepsilon, 0.45)^T$$

with $0 \leq \varepsilon \leq 0.2$ serves as an optimal L_∞ approximation for this case since

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.95 \\ 0.75 \\ 1 \\ 0.65 + \varepsilon \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.45 + \varepsilon \\ 0.45 \end{pmatrix}.$$

Remark 2.4. Although, as pointed out by Gavalec et al. [14], a system of max-T_L equations can be reformulated into a system of max-plus equations, there are some subtle differences between these two types of equations when the issues related to inconsistency are concerned. Note that the lower bound counterpart of the deviation level for the optimal L_∞ approximation can always be achieved when dealing with the inconsistency of one-sided max-plus linear equations, see, e. g., Butkovič [5] and Cuninghame–Green [10].

3. REFORMULATION METHOD FOR THE OPTIMAL L_1 APPROXIMATION

As illustrated in Section 2, an optimal L_∞ approximation may result in a consistent system of max-T_L equations with most, if not all, components in the right-hand side vector changed. However, it is desired in some situations that the components to be modified in the right-hand side vector are as few as possible. This property may be partially captured by the optimal L_1 approximation that minimizes $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$, the sum of absolute deviations to the right-hand side vector. Although this issue has not been explicitly tackled for fuzzy relational equations over Łukasiewicz algebra, it turns out that the reformulation approach developed by Li [20, 21] for max-plus equations can be adjusted to obtain the optimal L_1 approximation for inconsistent max-T_L equations, taking advantage of the truncated linearity of the Łukasiewicz triangular norm.

For an inconsistent system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ with $\widehat{\boldsymbol{x}} = A^T \triangleleft \boldsymbol{b}$, note that

$$A \circ (\widehat{\boldsymbol{x}} \vee \boldsymbol{x}) = (A \circ \widehat{\boldsymbol{x}}) \vee (A \circ \boldsymbol{x})$$

for any $\boldsymbol{x}$ such that $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{e}$, according to the distributive law with respect to the max-T_L composition. Since $A \circ \widehat{\boldsymbol{x}} \leq \boldsymbol{b}$, it follows that

$$\|A \circ (\widehat{\boldsymbol{x}} \vee \boldsymbol{x}) - \boldsymbol{b}\|_1 \leq \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$$

because the inequality relation holds componentwise for the absolute deviations under comparison. Consequently, since $\widehat{\boldsymbol{x}} \leq \widehat{\boldsymbol{x}} \vee \boldsymbol{x} \leq \boldsymbol{e}$, the restriction $\widehat{\boldsymbol{x}} \leq \boldsymbol{x} \leq \boldsymbol{e}$ may be imposed to reduce the searching space for the optimal L_1 approximation in order to resolve the inconsistency by minimizing $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$, that is,

$$\min_{\boldsymbol{x} \in [0,1]^n} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1 = \min_{\widehat{\boldsymbol{x}} \leq \boldsymbol{x} \leq \boldsymbol{e}} \|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1.$$

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^T$ be a nonnegative vector so that the constraint $\widehat{\boldsymbol{x}} \leq \boldsymbol{x} \leq \boldsymbol{e}$ may be written as

$$\widehat{\boldsymbol{x}} + \boldsymbol{s} \leq \boldsymbol{e}.$$

Let $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)^T$ be a second nonnegative vector to quantify the absolute deviations in the right-hand side vector such that

$$\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1 = \|\boldsymbol{t}\|_1 = \boldsymbol{e}^T \boldsymbol{t}.$$

The problem of minimizing $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$ subject to $\widehat{\boldsymbol{x}} \leq \boldsymbol{x} \leq \boldsymbol{e}$ may be formulated equivalently as

$$\begin{aligned} \min \quad & \boldsymbol{e}^T \boldsymbol{t} \\ \text{s.t.} \quad & \\ & (-\boldsymbol{t} + \boldsymbol{b}) \vee \boldsymbol{0} \leq A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \leq (\boldsymbol{t} + \boldsymbol{b}) \wedge \boldsymbol{e}, \\ & \boldsymbol{t} \geq \boldsymbol{0},\ \boldsymbol{0} \leq \boldsymbol{s} \leq \boldsymbol{e} - \widehat{\boldsymbol{x}}. \end{aligned}$$

Since $A \circ \widehat{\boldsymbol{x}} \leq A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \leq A \circ \boldsymbol{e}$ and $\boldsymbol{t}$ is nonnegative, the relaxed constraint

$$-\boldsymbol{t} + \boldsymbol{b} \leq A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \leq \boldsymbol{t} + \boldsymbol{b}$$

may be used instead and further decoupled and simplified by some formulation tricks routinely used in discrete and combinatorial optimization.

Note that the constraint $A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \leq \boldsymbol{t} + \boldsymbol{b}$ stands for

$$\max_{j \in N} \left\{ \max\{a_{ij} + \widehat{x}_j + s_j - 1, 0\} \right\} \leq t_i + b_i, \quad \forall i \in M.$$

It is equivalent, in this context, to the system of linear inequalities

$$a_{ij} + \widehat{x}_j + s_j - 1 \leq t_i + b_i, \quad \forall i \in M,\ j \in N.$$

Besides, the constraint $A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \geq -\boldsymbol{t} + \boldsymbol{b}$ stands for

$$\max_{j \in N} \left\{ \max\{a_{ij} + \widehat{x}_j + s_j - 1, 0\} \right\} \geq -t_i + b_i, \quad \forall i \in M.$$

This implies that for each $i \in M$, either $0 \geq -t_i + b_i$ or there exists at least one active index $j \in N$ of the variables such that

$$a_{ij} + \widehat{x}_j + s_j - 1 \geq -t_i + b_i.$$

Using a total of $m(n+1)$ auxiliary binary variables to signal these active indices, the constraint $A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \geq -\boldsymbol{t} + \boldsymbol{b}$ can be reformulated as

$$a_{ij} + \widehat{x}_j + s_j - 1 + K(1 - z_{ij}) \geq -t_i + b_i, \quad \forall i \in M,\ j \in N,$$

where K is a sufficiently large constant, e. g., $K = 2$ in this context, along with the side constraints

$$\begin{aligned} & K(1 - z_{i0}) \geq -t_i + b_i, \quad \forall i \in M, \\ & \textstyle\sum_{j \in N \cup \{0\}} z_{ij} = 1, \quad \forall i \in M, \\ & z_{ij} \in \{0, 1\}, \quad \forall i \in M,\ j \in N \cup \{0\}. \end{aligned}$$

With this alternative formulation, for each $i \in M$, either $z_{i0} = 1$ such that $0 \geq -t_i + b_i$ or $z_{ij} = 1$ for some $j \in N$ such that $a_{ij} + \widehat{x}_j + s_j - 1 \geq -t_i + b_i$, which implies $A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \geq -\boldsymbol{t} + \boldsymbol{b}$.

By this manner, the problem of minimizing $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_1$ may be reformulated into a polynomial-sized mixed integer linear programming problem as

$$\begin{aligned} \min \quad & t_1 + t_2 + \cdots + t_m \\ \text{s.t.} \quad & \\ & -t_i + s_j \leq -a_{ij} - \widehat{x}_j + b_i + 1, \quad \forall i \in M,\ j \in N, \\ & t_i + s_j + K(1 - z_{ij}) \geq -a_{ij} - \widehat{x}_j + b_i + 1, \quad \forall i \in M,\ j \in N, \\ & t_i + K(1 - z_{i0}) \geq b_i, \quad \forall i \in M, \\ & \textstyle\sum_{j \in N \cup \{0\}} z_{ij} = 1, \quad \forall i \in M, \\ & t_i \geq 0, \quad \forall i \in M, \\ & 0 \leq s_j \leq 1 - \widehat{x}_j, \quad \forall j \in N, \\ & z_{ij} \in \{0, 1\}, \quad \forall i \in M,\ j \in N \cup \{0\}. \end{aligned}$$

It may be solved to optimality on median or even large size instances via an off-the-shelf solver for mixed integer linear programming without recourse to any particular solution method, although it contains much more decision variables and constraints than its original compact form.

Besides, some redundant inequalities in the formulation may be identified to reduce the instance size during the preprocessing stage. For example, a recognition of $a_{ij} \leq b_i$ for some $i \in M$ and $j \in N$ implies $a_{ij} + \widehat{x}_j + s_j - 1 \leq t_i + b_i$ in this context. This means for each $i \in M$, only a subgroup of inequalities defined over $N_i = \{j \in N \mid a_{ij} > b_i\}$ need to be concerned.

Remark 3.1. It is worth noting that if the nonnegative vector $\boldsymbol{t}$ is restricted to be binary, i. e., $t_i \in \{0, 1\}$ for all $i \in M$, the resulted optimization problem, inevitably NP-hard, aims to find a maximum cardinality consistent subsystem of max-T_L equations. This problem has also been addressed by Cechlárová and Diko [8] for equations defined over maxmin algebra and maxgroup algebra (e. g., max-plus algebra), two typical scenarios of extremal algebra.

Remark 3.2. The requirement $A \circ (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \geq -\boldsymbol{t} + \boldsymbol{b}$ consists of the constraints of a disjunctive-like form and may be reformulated in an alternative manner as $t_i \geq b_i z_{i0}$ and $a_{ij} + \widehat{x}_j + s_j + t_i \geq (b_i + 1) z_{ij}$ for each pair of $i \in M$ and $j \in N$ along with the cardinality constraints $\sum_{j \in N \cup \{0\}} z_{ij} = 1$ for all $i \in M$.

Example 3.3. Consider the inconsistent system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ presented in Example 2.2 where

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}.$$

If an optimal L_1 approximation to the right-hand side vector is desired, the corresponding mixed integer linear programming problem is formulated, with $K = 2$, as

$$\begin{aligned}
\min \quad & t_1 + t_2 + t_3 \\
\text{s.t.} \quad & \\
& -t_1 + s_1 \leq 0.9, \quad -t_1 + s_2 \leq 0.4, \quad -t_1 + s_3 \leq 0, \quad -t_1 + s_4 \leq 0.8, \\
& -t_2 + s_1 \leq 0.1, \quad -t_2 + s_2 \leq 0.8, \quad -t_2 + s_3 \leq 0.2, \quad -t_2 + s_4 \leq 0.2, \\
& -t_3 + s_1 \leq 0, \quad -t_3 + s_2 \leq 0, \quad -t_3 + s_3 \leq 0.1, \quad -t_3 + s_4 \leq 0, \\
& t_1 + s_1 - 2z_{11} \geq -1.1, \quad t_1 + s_2 - 2z_{12} \geq -1.6, \\
& \quad t_1 + s_3 - 2z_{13} \geq -2, \quad t_1 + s_4 - 2z_{14} \geq -1.2, \\
& t_2 + s_1 - 2z_{21} \geq -1.9, \quad t_2 + s_2 - 2z_{22} \geq -1.2, \\
& \quad t_2 + s_3 - 2z_{23} \geq -1.8, \quad t_2 + s_4 - 2z_{24} \geq -1.8, \\
& t_3 + s_1 - 2z_{31} \geq -2, \quad t_3 + s_2 - 2z_{32} \geq -2, \\
& \quad t_3 + s_3 - 2z_{33} \geq -1.9, \quad t_3 + s_4 - 2z_{34} \geq -2, \\
& t_1 - 2z_{10} \geq -1.2, \quad t_2 - 2z_{20} \geq -1.4, \quad t_3 - 2z_{30} \geq -1.8, \\
& z_{10} + z_{11} + z_{12} + z_{13} + z_{14} = 1, \quad z_{20} + z_{21} + z_{22} + z_{23} + z_{24} = 1, \\
& \quad z_{30} + z_{31} + z_{32} + z_{33} + z_{34} = 1, \\
& s_1 \leq 0.3, \quad s_2 \leq 0.5, \quad s_3 \leq 0, \quad s_4 \leq 0.4, \\
& t_1, t_2, t_3, s_1, s_2, s_3, s_4 \geq 0, \quad z_{10}, z_{11}, \ldots, z_{34} \in \{0, 1\}.
\end{aligned}$$

By calling the linear programming solver `lp_solve` 5.5 using `lpSolve` package in `R` 3.0.1, an optimal solution is obtained with

$$\boldsymbol{t}^* = (0, 0.1, 0)^T, \quad \boldsymbol{s}^* = (0, 0, 0, 0)^T.$$

This indicates that for this instance the best under approximation

$$A \circ \widehat{\boldsymbol{x}} = (0.8, 0.5, 0.2)^T$$

also serves as an optimal L_1 approximation with $\|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_1 = 0.1$ to reach the consistency.

Note that the optimal L_∞ approximation

$$\widehat{\boldsymbol{b}} = (0.8, 0.55, 0.25)^T$$

obtained in Example 2.2 is also an optimal L_1 approximation to the right-hand side vector since $\|\widehat{\boldsymbol{b}} - \boldsymbol{b}\|_1 = 0.1$. Actually, for this particular instance, any vector of the form

$$(0.8, 0.5 + \varepsilon, 0.2 + \varepsilon)^T$$

with $0 \leq \varepsilon \leq 0.1$ defines an optimal L_1 approximation to the right-hand side vector since

$$\begin{pmatrix} 0.2 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.7 + \varepsilon \\ 0.5 + \varepsilon \\ 1 \\ 0.6 + \varepsilon \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.5 + \varepsilon \\ 0.2 + \varepsilon \end{pmatrix}.$$

For $\varepsilon = 0.05$, it becomes simultaneously an optimal L_∞ approximation to resolve the inconsistency.

Example 3.4. Consider the inconsistent system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ presented in Example 2.3 where

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

Analogously, by solving the corresponding mixed integer linear programming problem, an optimal solution is identified with

$$\boldsymbol{t}^* = (0.2, 0, 0.1)^T, \quad \boldsymbol{s}^* = (0, 0.1, 0, 0.1)^T.$$

The associated optimal L_1 approximation $\boldsymbol{b}^*$ is constructed accordingly as

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.8 \\ 0.7 \\ 1 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \end{pmatrix}$$

such that $\|\boldsymbol{b}^* - \boldsymbol{b}\|_1 = 0.3$.

Note that for this case the principal solution $\widehat{\boldsymbol{x}}$ induces as well an optimal L_1 approximation

$$A \circ \widehat{\boldsymbol{x}} = (0.6, 0.5, 0.3)^T$$

since $\|A \circ \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_1 = 0.3$. Actually, any vector of the form

$$(0.6, 0.5 + \varepsilon, 0.3 + \varepsilon)^T$$

with $0 \leq \varepsilon \leq 0.1$ serves as an optimal L_1 approximation to the right-hand side vector such that

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.8 + \varepsilon \\ 0.6 + \varepsilon \\ 1 \\ 0.7 + \varepsilon \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 + \varepsilon \\ 0.3 + \varepsilon \end{pmatrix},$$

of which, however, none is an optimal L_∞ approximation at the same time. Interestingly, this instance admits another group of optimal L_1 approximations in the form of

$$(0.6 + \varepsilon, 0.6, 0.4 + \varepsilon)^T$$

with $0 \leq \varepsilon \leq 0.1$, since

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.9 + \varepsilon \\ 0.7 + \varepsilon \\ 1 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.6 + \varepsilon \\ 0.6 \\ 0.4 + \varepsilon \end{pmatrix}.$$

For $\varepsilon = 0.05$, the vector

$$\boldsymbol{b}^* = (0.65, 0.6, 0.45)^T$$

becomes simultaneously an optimal L_1 and optimal L_∞ approximation to the right-hand side vector $\boldsymbol{b} = (0.8, 0.6, 0.3)^T$ to resolve the inconsistency.

It happens in Examples 3.3 and 3.4 that the principal solution $\widehat{\boldsymbol{x}}$ induces an optimal L_1 approximation $A \circ \widehat{\boldsymbol{x}}$ and there exists a simultaneously optimal L_1 and optimal L_∞ approximation to the right-hand side vector when the concerned system $A \circ \boldsymbol{x} = \boldsymbol{b}$ is inconsistent. It is in general not the case as illustrated by the following example.

Example 3.5. Consider the system of max-T_L equations $A \circ \boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.8 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}.$$

The principal solution associated with this instance is

$$\widehat{\boldsymbol{x}} = (0.9, 0.7, 1, 0.6)^T$$

and hence,

$$A \circ \widehat{\boldsymbol{x}} = (0.6, 0.4, 0.4)^T.$$

It can be verified that

$$\widehat{\boldsymbol{b}} = (0.7, 0.5, 0.5)^T$$

is an optimal L_∞ approximation with

$$\|\widehat{\boldsymbol{b}} - \boldsymbol{b}\|_\infty = 0.1 = \tfrac{1}{2}\|A\widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty$$

which is actually the unique one for this instance and results in a right-hand side vector with all three components changed. However, the optimal L_1 approximation for this instance, by solving the corresponding mixed integer linear programming problem, is

$$\boldsymbol{b}^* = (0.8, 0.6, 0.6)^T,$$

which is also unique, such that

$$\begin{pmatrix} 0.2 & 0.9 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0.4 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.8 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0.9 \\ 1 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.6 \end{pmatrix}$$

and $\|\boldsymbol{b}^* - \boldsymbol{b}\|_1 = 0.2$. Note that the optimal L_1 approximation $\boldsymbol{b}^*$ differs only in the third component compared to the original vector $\boldsymbol{b}$ and is not identical to $A \circ \widehat{\boldsymbol{x}}$. This suggests that for this instance there is no vector being simultaneously the optimal L_1 and optimal L_∞ approximation to the right-hand side vector. It also illustrates that the inconsistency of max-T_L equations may possibly be resolved, with the optimal L_1 approximation, by modifying few components in the right-hand side vector, while the optimal L_∞ approximation could results in a right-hand side vector with many components changed.

Remark 3.6. It is a natural idea to combine the L_1 norm and L_∞ norm to measure the deviation in the right-hand side vector when resolving the inconsistency of max-T_L equations. A possible approach is to search for a vector with the minimum sum of absolute deviations among all the optimal L_∞ approximations, which may be implemented by introducing additional bound constraints into the reformulated mixed integer programming problem. Another approach, as performed by Li [19] for a special case of max-min equations, is to use a convex combination of these two vector norms to measure the deviations, which can be reformulated as well into a mixed integer linear programming problem. Besides, the polyhedral norms, e. g., the largest-k norm, are also interesting alternatives, which are linear programming representable as addressed by Gotoh and Uryasev [16] and include the L_1 norm and L_∞ norm as the special limiting cases.

Remark 3.7. When an optimal L_2 approximation is desired that has the least sum of squared deviations to the right-hand side vector, this reformulation approach works analogously and transforms the problem of minimizing $\|A \circ \boldsymbol{x} - \boldsymbol{b}\|_2$ into a mixed integer quadratic programming problem with the exact same set of variables and constraints demonstrated in this section.

Remark 3.8. The role of the Łukasiewicz triangular norm is remarkable in the theory of triangular norms because any nilpotent triangular norm is isomorphic to the Łukasiewicz triangular norm. This means that theoretically one may focus on max-T_L equations for any type of fuzzy relational equations with the maximum and nilpotent triangular norm composition. However, there are subtle differences when the inconsistency issues are concerned. The reformulation approach introduced in this section may not result in a mixed integer linear programming problem.

4. CONCLUSION

The problem of inconsistency resolving is concerned in this article for a finite system of fuzzy relational equations with the maximum and Łukasiewicz triangular norm composition. A polynomial-time direct search algorithm is developed to obtain a consistent system with the least perturbation on the right-hand side vector with respect to the maximum absolute deviation. A reformulation approach is also proposed by minimizing the sum of absolute deviations to the right-hand side vector and results in a polynomial-sized mixed integer linear programming problem, which requires more computational effort but may be solved to optimality by some well developed techniques or off-the-shelf software packages. This reformulation approach is somewhat more flexible for modelling purposes and works as well for a variety of deviation measures, e. g., the sum of squared deviations and the convex combination of the maximum absolute deviation and sum of absolute deviations. It may be extended as well, as demonstrated by Li [22], to handle the linear optimization problem over the approximate solutions of fuzzy relational equations.

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