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IDENTICAL GROUPS OF AUTOMORPHISMS OF THE PERMUTOHEDRON

Milan Studený

The paper concerns a remarkable polytope, known as the permutohedron, studied already in 1911 by Schoute. In general, the symmetry group of a polytope, which is the group of its automorphisms, does depend on the considered class of isomorphisms. One can distinguish isometries, affine isomorphisms, face-lattice order isomorphisms, and graph isomorphisms of a polytope. We show that, in case of the permutohedron, these four groups of automorphisms are (group) isomorphic; we also provide their group-theoretical description. Specifically, if N is the basic set, $|N| \geq 3$, then (each of) the automorphism group(s) is the direct product of the group of permutations of N (= the symmetric group S_N) and of a two-element group generated by central symmetry. The result confirms an unproved statement formulated by Crisman in 2011.

Keywords: polytope, permutohedron, automorphism, symmetry group

Classification: 20B25, 52B15

1. INTRODUCTION

Let us consider the real Euclidean space $\mathbb{R}^N$, where N is a non-empty finite index set, called the set of *basic variables*. Here we follow the usual mathematical notational convention: the symbol $\mathbb{R}^N$ denotes the set of functions from N to $\mathbb{R}$, that is, the set of real vectors whose components are indexed by elements of N. Then any (non-empty) polytope $\mathsf{P} \subseteq \mathbb{R}^N$ in this space can be assigned four groups of its automorphisms:

- those (Euclidean) *isometries* of P which map it onto itself,
- the bijective *affine isomorphisms* which map P onto itself,[1]
- the *order isomorphisms* which map the face lattice of P onto itself,[2]
- the *graph isomorphisms* which map vertices of P on themselves and preserve edges.

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[1] In our case of the permutohedron, the zero vector $\mathbf{0} \in \mathbb{R}^N$ is outside the affine hull $\mathrm{aff}(\mathsf{P})$ and these affine maps uniquely extend to linear self-transforms of the linear space $\mathrm{lin}(\mathsf{P}) \subseteq \mathbb{R}^N$ preserving P. Thus, affine isomorphisms can be interpreted as *linear transformations* of $\mathrm{lin}(\mathsf{P})$ preserving P.

[2] This corresponds to a well-known concept of *combinatorial equivalence* of polytopes [3, § 10], defined as the order-isomorphism of the lattices of their faces.

Indeed, in each of these four cases, the set of considered bijective transformations defines a group with the group operation being the composition of those transformations. As the respective automorphisms are uniquely determined by their restrictions to the set ext(P) of the vertices of the polytope P, all four groups are finite. Nonetheless, below, in Section 3, we give examples that the resulting automorphism groups of P, often called the *symmetry groups of* P, may differ, depending on the context, which is in our case the class of morphisms one has in mind.

The paper concerns a particular polytope called the permutohedron (in $\mathbb{R}^N$), briefly and informally described as the polytope, vertices of which have as their components permutations of the set $[n] := \{1, \ldots, n\}$ of integers between 1 and n, where $n := |N| \geq 1$.

More specifically, let us call any vector $(x_u)_{u \in N} \in \mathbb{R}^N$ such that $\{ x_u : u \in N \} = [n]$ a *rank vector* in $\mathbb{R}^N$ and define the *permutohedron* in $\mathbb{R}^N$, denoted by $\Pi(N)$, as the convex hull of all rank vectors in $\mathbb{R}^N$. Thus, if $N := \{a, b, c\}$ then both $(x_a, x_b, x_c) := (2, 1, 3)$ and $(y_a, y_b, y_c) := (1, 3, 2)$ are rank vectors in $\mathbb{R}^N$. Clearly, the number of rank vectors is $n!$ and all of them belong to the affine space $\{ y \in \mathbb{R}^N : \sum_{u \in N} y_u = \binom{n+1}{2} \}$. This appears to the affine hull of $\Pi(N)$ and, therefore, the dimension of $\Pi(N)$ is $n - 1$.

Note in this context that some authors dealing with related topics [1, 4, 6, 7, 8, 12], in their definitions of the permutohedron, accepted a seemingly harmless identification convention that N coincides with $[n]$, despite that such a convention is not necessary. On the contrary, such a highly specific convention leads to misunderstandings on the side of the reader. The point is that then the rank vectors become the self-bijections of $[n]$, that is, elements of the symmetric group $S_{[n]}$, and gain many other meanings, which mislead the reader. The artificially enforced group structure on them, thus, causes later identification problems (for more explanation see [10, § H]). To avoid these problems, in this paper, we assume that the set N of basic variables has no deeper mathematical structure, or, put it in another way, we intentionally regard N as an unordered set and distinguish it sharply from the ordered set $[n] = \{1, \ldots, n\}$, where $n = |N|$.

According to Ziegler [12, Example 0.10], the permutohedron was first studied by Schoute [9], already in 1911. This polytope exhibits several remarkable properties, see our summary in Section 5. In this paper, we utilize a combinatorial characterization of its faces provided by Billera and Sarangarajan in 1996 [1, Proposition 1.4].

A 2011 note [4] by Crisman contained a claim about what the symmetry group of the permutohedron is.[3] Essentially, the claim was that, for $n \geq 3$, the symmetry group is the direct (= commuting) product of two groups, of the symmetric group on n elements and of a two-element group generated by central symmetry. The given intuitive arguments concerned the cases $n = 3$ and $n = 4$ and a remark was made that generalizing to $n > 4$ is not hard, followed by a short vague hint.

That hint, however, implicitly assumed that the case of a general automorphism can be reduced to a special case of an automorphism that maps a particular square, which is a certain two-dimensional face of the permutohedron, onto itself. While it is more or less clear from three-dimensional geometric intuition that this reduction is possible in the case $n = 4$, the existence of such a reduction is far from being obvious if $n \geq 5$. So, the hint does not work as easily as anticipated by Crisman.

[3] Although the class of automorphisms under consideration was not specified there.

One has to take into consideration different types of edges in $\Pi(N)$ (see the numeral labels on the edges from Section 4.2). This is because the uniqueness-spreading procedure indicated in [4] may indeed fail in case of a 'wrong' starting edge;[4] see Example 4.4.

In this paper we give a complete proof of the conjecture by Crisman, which is based on some deeper observations on the respective permutohedral graph. Specifically, we show that, in all four considered contexts, the group of automorphisms has $2 \cdot n!$ elements and can be characterized as described below on this page.

First, any permutation of N, that is, any bijection $\pi : N \to N$ of N onto itself defines a linear isometric auto-transformation of $\mathbb{R}^N$:

$$\Psi_\pi : \quad (x_u)_{u\in N} \;\longmapsto\; (x_{\pi(u)})_{u\in N}\,. \tag{1}$$

To illustrate this isometry consider the case $N := \{a, b, c\}$ and the a-c-transposition $\pi : N \to N$ given by $\pi(a) := c$, $\pi(b) := b$, and $\pi(c) := a$. If one chooses to write the components of a vector x in $\mathbb{R}^N$ in the order (x_a, x_b, x_c) then this vector is assigned (by Ψ_π) the vector $(y_a, y_b, y_c) \in \mathbb{R}^N$, where $y_a = x_c$, $y_b = x_b$ and $y_c = x_a$. Thus, Ψ_π acts on the rank vectors as follows: $(1, 2, 3) \leftrightarrow (3, 2, 1)$, $(2, 1, 3) \leftrightarrow (3, 1, 2)$, $(1, 3, 2) \leftrightarrow (2, 3, 1)$. The formula (1) specifies a class of isometries indexed by elements of the symmetric group S_N, all of which map $\Pi(N)$ onto itself.

Additionally, the formula

$$\Psi_{\mathfrak{r}} : \quad (x_u)_{u\in N} \;\longmapsto\; \left(\frac{2}{n} \cdot \left\{\sum_{w\in N} x_w\right\} - x_u\right)_{u\in N} \tag{2}$$

defines another linear isometric auto-transformation of $\mathbb{R}^N$, interpretable as the axial symmetry,[5] which also maps $\Pi(N)$ onto itself. To illustrate this type of isometry consider again the case $N := \{a, b, c\}$, in which case $n = 3$. Then any $(x_a, x_b, x_c) \in \mathbb{R}^N$ is assigned (by $\Psi_{\mathfrak{r}}$) the vector $(y_a, y_b, y_c) \in \mathbb{R}^N$ where $y_a = \frac{1}{3} \cdot \{-x_a + 2 \cdot x_b + 2 \cdot x_c\}$, $y_b = \frac{1}{3} \cdot \{2 \cdot x_a - x_b + 2 \cdot x_c\}$, and $y_c = \frac{1}{3} \cdot \{2 \cdot x_a + 2 \cdot x_b - x_c\}$. In particular, $\Psi_{\mathfrak{r}}$ acts on the rank vectors as follows: $(1, 2, 3) \leftrightarrow (3, 2, 1)$, $(2, 1, 3) \leftrightarrow (2, 3, 1)$, $(1, 3, 2) \leftrightarrow (3, 1, 2)$. When restricted to $\mathrm{aff}(\Pi(N))$, $\Psi_{\mathfrak{r}}$ yields the central symmetry of $\Pi(N)$.

This central symmetry commutes with all the permutational transforms: $\Psi_\pi \circ \Psi_{\mathfrak{r}} = \Psi_{\mathfrak{r}} \circ \Psi_\pi$ for any $\pi \in S_N$, where $\circ$ denotes the composition. Our result is that, no matter which context one considers, Ψ is an automorphism of $\Pi(N)$ iff it either has the form Ψ_π or the form $\Psi_\pi \circ \Psi_{\mathfrak{r}}$ for some $\pi \in S_N$.

The cases $n \leq 2$ are degenerate. If $n = 1$ then $|\Pi(N)| = 1$ and the only automorphism is the identity, while if $n = 2$ then $\Pi(N)$ is a line segment and the only other automorphism besides the identity is the central symmetry, which, however, coincides with the only non-identical permutational automorphism. Once $n \geq 3$, the central symmetry strictly differs from the permutational symmetries.

The structure of the paper is as follows. In Section 2 we introduce our notation, define basic concepts, and recall those facts/results that we (later) utilize in the paper. In Section 3 we present observations about the relation of the considered automorphism

[4] Such a 'wrong' starting edge always exists for any $n = 2 \cdot k$ with $k \geq 2$.

[5] The axis is the line of constant vectors in $\mathbb{R}^N$.

groups, valid for a general (non-empty) polytope $\mathsf{P} \subseteq \mathbb{R}^N$; these establish a kind of hierarchy among them. In Section 4 we state our main result formally and present its technical proof, including auxiliary concepts and lemmas needed in the proof. In Section 5 we conclude with some remarks on the properties of the graph associated with the permutohedron. Appendix A is a kind of a tutorial in which we repeat the arguments for a celebrated result (we use) saying that, for a simple polytope, the structure of its graph already determines its face lattice.

2. PRELIMINARIES

As already noted, throughout the paper, the symbol N will denote a finite non-empty set of *basic variables*, intentionally considered to be an unordered set, while $n := |N| \geq 1$ will denote the number of its elements. We will consistently distinguish N from the totally ordered set of integers between 1 and n, denoted by $[n] := \{1, \ldots, n\}$. A special *reverse* self-transformation of $[n]$ defined by $\mathfrak{r}(i) := n + 1 - i$ for $i \in [n]$ will be used.

By an *enumeration* of N we will understand a bijective mapping $\varepsilon : [n] \to N$ from $[n]$ onto N. To specify ε we might occasionally use a record like $|\varepsilon(1)| \ldots |\varepsilon(n)|$, which is nothing but the corresponding ordered list of all elements of N. The collection of all enumerations of N will be denoted by $\Upsilon(N)$.

Every enumeration ε can equivalently be described by the corresponding *rank vector*, which is the vector $(\varepsilon_{-1}(u))_{u \in N}$ in $\mathbb{R}^N$, formally, the inverse $\varepsilon_{-1} : N \to [n]$ of the mapping $\varepsilon : [n] \to N$. We are going to denote it by ρ_ε in the geometric context of $\mathbb{R}^N$. The set of all rank vectors in $\mathbb{R}^N$ will be denoted by $\Re(N) := \{\, \rho_\varepsilon \,:\, \varepsilon \in \Upsilon(N) \,\}$.

The symbol $\delta(\star\star)$, where $\star\star$ is a predicate, will denote a zero-one function whose value is $+1$ if the statement $\star\star$ is true and 0 otherwise. We apply an infix notation with the composition operator: given mappings τ and ς, $(\tau \circ \varsigma)(\ell) := \tau(\varsigma(\ell))$ for any argument ℓ of ς. The inverse of a mapping ς will be denoted by ς_{-1}, using the lower index.[6]

2.1. Some lattice-theoretical concepts

A partially ordered set $(\mathfrak{L}, \preceq)$ is called a *lattice* [2, § I.4] if every two-element subset of $\mathfrak{L}$ has both the least upper bound, also named the *supremum*, and the greatest lower bound, also named the *infimum*. A finite lattice necessarily has the *least element* $\mathfrak{o}$, which satisfies $\mathfrak{o} \preceq \mathfrak{u}$ for any $\mathfrak{u} \in \mathfrak{L}$, and the *greatest elements*, that is, an element $\mathfrak{1} \in \mathfrak{L}$ satisfying $\mathfrak{l} \preceq \mathfrak{1}$ for any $\mathfrak{l} \in \mathfrak{L}$.

A lattice $(\mathfrak{L}, \preceq)$ is *isomorphic* to a lattice $(\mathfrak{L}', \preceq')$ if there is a bijective mapping ι from $\mathfrak{L}$ onto $\mathfrak{L}'$, called the *order isomorphism*, which preserves the order, that is, $\mathfrak{l} \preceq \mathfrak{u}$ iff $\iota(\mathfrak{l}) \preceq' \iota(\mathfrak{u})$ for $\mathfrak{l}, \mathfrak{u} \in \mathfrak{L}$. Then both suprema and infima are preserved by the transformation ι, which means ι is also a *lattice isomorphism*; so, the lattices $(\mathfrak{L}, \preceq)$ and $(\mathfrak{L}', \preceq')$ are considered to be identical mathematical structures. In case $(\mathfrak{L}', \preceq') = (\mathfrak{L}, \preceq)$, we say that ι is an *order automorphism*.

Given a partially ordered set $(\mathfrak{L}, \preceq)$ and $\mathfrak{l}, \mathfrak{u} \in \mathfrak{L}$ such that $\mathfrak{l} \prec \mathfrak{u}$, which means $\mathfrak{l} \preceq \mathfrak{u}$ and $\mathfrak{l} \neq \mathfrak{u}$, and there is no $\mathfrak{e} \in \mathfrak{L}$ with $\mathfrak{l} \prec \mathfrak{e} \prec \mathfrak{u}$, we will write $\mathfrak{l} \prec\cdot\, \mathfrak{u}$ and say that $\mathfrak{l}$

[6] While the upper index will scarcely be used to denote the inverse element Ψ^{-1} of a group element Ψ. However, in case of the groups of transformations one typically has $\Psi^{-1} = \Psi_{-1}$.

is *covered* by $\mathfrak{u}$. A finite lattice $(\mathfrak{L}, \preceq)$ will be called *graded* [2, § I.3] if there exists a function $h : \mathfrak{L} \to \mathbb{Z}$, called a *height function*, which satisfies, for any $\mathfrak{l}, \mathfrak{u} \in \mathfrak{L}$,

$$\text{if } \mathfrak{l} \prec\cdot\, \mathfrak{u} \text{ then } h(\mathfrak{u}) = h(\mathfrak{l}) + 1.$$

The graded lattices can be characterized in terms of *chains*, which are (non-empty) totally ordered subsets of $\mathfrak{L}$; the *length* of such a chain is the number of its elements minus 1. A well-known fact is that a finite lattice is graded iff it satisfies the *Jordan-Dedekind chain condition* requiring that all maximal chains between the same endpoints have the same length [2, Lemma 1 in § I.3]. The function h which assigns to every $\mathfrak{u} \in \mathfrak{L}$ the (shared) length of maximal chains between $\mathfrak{o}$ and $\mathfrak{u}$ is then a (standardized) height function for the lattice $(\mathfrak{L}, \preceq)$.

2.2. Some graph-theoretical concepts

A finite undirected graph is a pair $G = (\mathsf{V}, \mathcal{E})$, where V is a non-empty finite set of its *nodes* and $\mathcal{E}$ a class of two-element subsets of V, called its *edges*.

A graph $G = (\mathsf{V}, \mathcal{E})$ is *isomorphic* to a graph $H = (\mathsf{V}', \mathcal{E}')$ if there is a bijective mapping Ψ from V onto V', called the *graph isomorphism*, transforming the edges, that is, $\{x, y\} \in \mathcal{E}$ iff $\{\Psi(x), \Psi(y)\} \in \mathcal{E}'$ for any $x, y \in \mathsf{V}$. It is a *graph automorphism* (of G) if $H = G$. Isomorphic graphs are considered to be identical mathematical structures.

An undirected graph G is called *regular* if the number of adjacent nodes to a given node is the same for all its nodes. It is called *connected* if, for every two of its nodes, there exists a path in G between those nodes.

2.3. Some polyhedral geometry

Concepts and basic facts from polyhedral geometry are explained in detail in textbooks on this topic [3, 12]. In this section, our notation and some definitions are recalled.

The symbol $\langle x, y\rangle := \sum_{u\in N} x_u \cdot y_u$, where $x = (x_u)_{u\in N} \in \mathbb{R}^N$ and $y = (y_u)_{u\in N} \in \mathbb{R}^N$, will denote the *scalar product* of vectors in $\mathbb{R}^N$; their Euclidean distance is

$$\lambda(x, y) := \sqrt{\sum_{u\in N} \{x_u - y_u\}^2}, \text{ which is nothing but } \sqrt{\langle x - y, x - y\rangle}\,.$$

The symbol $\mathsf{S} \oplus \mathsf{T} := \{\, x + y \;:\; x \in \mathsf{S} \;\&\; y \in \mathsf{T} \,\}$, where $\mathsf{S}, \mathsf{T} \subseteq \mathbb{R}^N$, will denote the Minkowski sum of two subsets of $\mathbb{R}^N$.

2.3.1. Affine subspaces and mappings

An affine combination of vectors in $\mathbb{R}^N$ is a real (finite) linear combination $\sum_j \alpha_j \cdot x^j$, $\alpha_j \in \mathbb{R}$, $x^j \in \mathbb{R}^N$, where $\sum_j \alpha_j = 1$. The *affine hull* of a set $\mathsf{S} \subseteq \mathbb{R}^N$, to be denoted by $\mathrm{aff}(\mathsf{S})$, is the collection of all affine combinations of vectors from S. It is always an *affine subspace* of $\mathbb{R}^N$, that is, a subset $\mathsf{A} \subseteq \mathbb{R}^N$ closed under affine combinations. A non-empty affine subspace is the same as a translate of a linear subspace of $\mathbb{R}^N$, that is, a set of the form $\mathsf{A} = \{x\} \oplus \mathsf{L}$, where $x \in \mathbb{R}^N$, $\mathsf{L} \subseteq \mathbb{R}^N$ is a linear subspace, [3, § 1, p. 6]; L is then uniquely determined by A while x is not unique.[7]

[7] Recall that a linear subspace is always non-empty by definition.

The dimension of such an affine space A is then the dimension of L. This allows one to introduce the *dimension* of any set $\mathsf{S} \subseteq \mathbb{R}^N$ as the dimension of its affine hull, denoted by $\dim(\mathsf{S})$. Note that, by a convention, the dimension of the empty set $\emptyset \subseteq \mathbb{R}^N$ is -1.

Let $\mathbf{0}$ denote the zero vector in $\mathbb{R}^N$. A (finite) collection of vectors $\{\, x^j \in \mathbb{R}^N \,:\, j \in J\}$ will be called *affinely independent* if the following implication holds:

$$\text{whenever} \quad \left[\sum_{j \in J} \alpha_j \cdot x^j = \mathbf{0} \;\;\&\;\; \sum_{j \in J} \alpha_j = 0 \right] \quad \text{then} \;\; [\, \alpha_j = 0 \text{ for every } j \in J \,].$$

A well-known criterion reduces affine independence to linear independence. Specifically, if $\{\, x^j \,:\, j \in J\}$ is affinely independent then, for any $i \in J$, $\{\, x^j - x^i \,:\, j \in J \setminus \{i\}\,\}$ is linearly independent and, conversely, if this set of vectors is linearly independent for some $i \in J$ then $\{\, x^j \,:\, j \in J\}$ is affinely independent; see [3, § 1, p. 7]. One can also always find an affinely independent generator of an affine space. Specifically, given $\mathsf{S} \subseteq \mathbb{R}^N$, there exists a finite affinely independent subset $\mathsf{G} \subseteq \mathsf{S}$ such that $\mathrm{aff}(\mathsf{S}) = \mathrm{aff}(\mathsf{G})$ and $|\mathsf{G}| = \dim(\mathsf{S}) + 1$; [3, § 1, p. 8, (d′)].

Given a non-empty affine subspace $\mathsf{A} \subseteq \mathbb{R}^N$ and (another) Euclidean space $\mathbb{R}^M$, a mapping $\Psi : \mathsf{A} \to \mathbb{R}^M$ will be called an *affine mapping* if

$$\Psi(\sum_j \alpha_j \cdot x^j) = \sum_j \alpha_j \cdot \Psi(x^j) \quad \text{for any affine combination of vectors } x^j \in \mathsf{A}.$$

When $\mathsf{A} = \{x\} \oplus \mathsf{L}$, a mapping $\Psi : \mathsf{A} \to \mathbb{R}^M$ is affine iff there exists a linear mapping $\Phi : \mathsf{L} \to \mathbb{R}^M$ and $y \in \mathbb{R}^M$ such that $\Psi(x + z) = y + \Phi(z)$ for all $z \in \mathsf{L}$; see [3, § 1, p. 8]. It follows from this observation and the linear criterion for affine independence that any affine mapping $\Psi : \mathsf{A} \to \mathbb{R}^M$, where $\dim(\mathsf{A}) = k$, is determined uniquely by its values on every (arbitrarily chosen) affinely independent subset $\mathsf{G} \subseteq \mathsf{A}$ such that $|\mathsf{G}| = k + 1$.

A convex combination is an affine combination where, moreover, $\alpha_j \geq 0$ for any j. Analogously, the *convex hull* of a set $\mathsf{S} \subseteq \mathbb{R}^N$, denoted by $\mathrm{conv}\,(\mathsf{S})$, is the collection of all convex combinations of vectors from S. It is always a *convex subset* of $\mathbb{R}^N$, that is, a subset $\mathsf{S} \subseteq \mathbb{R}^N$ closed under convex combinations.

2.3.2. Polytopes and their faces

A *polytope* (in $\mathbb{R}^N$) is the convex hull of a finite set of vectors in $\mathbb{R}^N$.[8] A polyhedron in $\mathbb{R}^N$ is a set of vectors $x \in \mathbb{R}^N$ specified by finitely many linear inequalities $\langle x, y\rangle \geq \beta$, where $y \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ are parameters of the inequalities. A fundamental result in polyhedral geometry says that a subset of $\mathbb{R}^N$ is a polytope iff it is a bounded polyhedron; see [12, Theorem 1.1] or [3, Theorem 9.2].

Given distinct $x, y \in \mathbb{R}^N$, the convex hull of $\{x, y\}$ is the *closed segment*, denoted by $[x, y]$, while the *open segment*, denoted by $]x, y[$, consists of convex combinations of x and y which have both coefficients non-zero:

$$]x, y[\;:=\; \{\, (1 - \alpha) \cdot x + \alpha \cdot y \;:\; 0 < \alpha < 1 \,\}.$$

[8] Note explicitly that we consider the empty set to be a polytope.

A *face* of a polytope $\mathsf{P} \subseteq \mathbb{R}^N$ is its convex subset $\mathsf{F} \subseteq \mathsf{P}$ such that one has $[x, z] \subseteq \mathsf{F}$ whenever $x, z \in \mathsf{P}$ and $]x, z[\cap \mathsf{F} \neq \emptyset$ [3, § 5]. Note that the reader might be familiar with a methodologically different definition of a face. The so-called exposed face of P is a subset $\mathsf{F} \subseteq \mathsf{P}$ consisting of vectors $x \in \mathsf{P}$ satisfying $\langle x, y\rangle = \beta$ for some $y \in \mathbb{R}^N$, $\beta \in \mathbb{R}$, such that $\langle z, y\rangle \geq \beta$ is a valid inequality for all $z \in \mathsf{P}$.[9] In case of a polytope P, these two conditions are, in fact, equivalent; see [3, Corollary 7.6].

Basic facts are that every face of a polytope is again a polytope [3, Theorem 7.3] and the number of faces of a polytope P is finite [3, Corollary 7.4]. Further relevant fact is that if $\mathsf{F} \subset \mathsf{F}'$ are distinct faces of P then $\dim(\mathsf{F}) < \dim(\mathsf{F}')$; see [3, Corollary 5.5]. The intersection of any collection of faces of P is a face of P [3, Theorem 5.9]. This implies that, for any $\mathsf{S} \subseteq \mathsf{P}$, there exists the least face F of P such that $\mathsf{S} \subseteq \mathsf{F}$; it will be called the *face generated by* S.

Faces of a polytope $\mathsf{P} \subseteq \mathbb{R}^N$ can be classified according to their dimensions. Thus, by a *vertex* of a polytope P will be called any such a vector $y \in \mathsf{P}$ that the singleton $\{y\}$ is a face of P (of the dimension 0). In other words, it means that y is an *extreme point* of P, which is a vector $y \in \mathsf{P}$ such that there is no open segment $]x, z[$ with distinct $x, z \in \mathsf{P}$ and $y \in]x, z[$. The set of vertices of P will be denoted by $\mathrm{ext}(\mathsf{P})$.

A consequence of famous *Krein-Milman theorem* is that every polytope P equals to the convex hull of the set of its vertices: $\mathsf{P} = \mathrm{conv}\,(\mathrm{ext}(\mathsf{P}))$; see [3, Theorem 7.2(c)] or [12, Proposition 2.2(i)]. Because $\mathrm{ext}(\mathsf{F}) = \mathrm{ext}(\mathsf{P}) \cap \mathsf{F}$ for any face F of P [3, Theorem 7.3], any face F of P is uniquely determined by a subset of $\mathrm{ext}(\mathsf{P})$, namely by $\mathrm{ext}(\mathsf{F})$.

We will say that a set $\mathsf{S} \subseteq \mathrm{ext}(\mathsf{P})$ of vertices of P *represents a face* if $\mathsf{S} = \mathrm{ext}(\mathsf{F})$ for a face F of P. A particular closure-like operation on $\mathrm{ext}(\mathsf{P})$ will be used in this paper: given $\mathsf{S} \subseteq \mathrm{ext}(\mathsf{P})$, the set of vertices of the face of P generated by S will called the *facial closure* of S and denoted by $\mathrm{fac}(\mathsf{S})$. It is the least superset of S representing a face.

A *geometric edge* of a polytope P is a closed segment $[x, z] \subseteq \mathsf{P}$ in P which forms a face of P (of the dimension 1); then necessarily $x, z \in \mathrm{ext}(\mathsf{P})$. The set of *neighbours* of a vertex x of P is then defined by

$$\mathrm{ne}_{\mathsf{P}}(x) \;:=\; \{\, z \in \mathrm{ext}(\mathsf{P}) \;:\; [x, z] \text{ is a geometric edge of } \mathsf{P} \,\}\,.$$

Every face F of a non-empty polytope P with $\dim(\mathsf{F}) = \dim(\mathsf{P}) - 1$ is called (its) *facet*. A well-known fact is that facets yield inequality description of P within $\mathrm{aff}(\mathsf{P})$ (unique up to positive multiples of inequalities if $\mathrm{aff}(\mathsf{P}) = \mathbb{R}^N$); see [3, Theorem 8.2].

The *relative interior* of a (non-empty) polytope $\mathsf{P} \subseteq \mathbb{R}^N$, denoted by $\mathrm{relint}(\mathsf{P})$, is the topological interior of P relative to (the induced) Euclidean topology on $\mathrm{aff}(\mathsf{P})$; it is non-empty if P is non-empty. A (non-empty) polytope $\mathsf{P} \subseteq \mathbb{R}^N$ of the dimension $d \geq 0$ is called *simple* if every its vertex has precisely d neighbours; see [3, § 12].

2.3.3. Face lattice and graph of a polytope

A fundamental fact is that the collection of faces of a polytope P, ordered by the inclusion relation $\subseteq$, is a finite lattice. It is called the *face lattice* of P and denoted by $\mathfrak{F}(\mathsf{P})$. The infimum of two faces $\mathsf{F}_1, \mathsf{F}_2 \in \mathfrak{F}(\mathsf{P})$ is their intersection $\mathsf{F}_1 \cap \mathsf{F}_2$ and the supremum is

[9] Observe that the empty set is always an exposed face of any polytope P: take $y := \mathsf{0}$ and $\beta := -1$.

the face generated by their union $\mathsf{F}_1 \cup \mathsf{F}_2$. In fact, $\mathfrak{F}(\mathsf{P})$ is a graded lattice, where the dimension of faces plays the role of a height function; see [12, Theorem 2.7(i)].

The *graph* of a (non-empty) polytope P, denoted by $G(\mathsf{P})$, is an undirected graph having the vertex set $\mathsf{V} := \mathrm{ext}(\mathsf{P})$ as the set of its nodes. Distinct nodes $x, y \in \mathsf{V}$ are adjacent in $G(\mathsf{P})$ iff the segment $[x, y]$ is a geometric edge (= 1-dimensional face) of P. Clearly, the definition implies that the face lattice of P determines its graph.

2.3.4. Permutohedron

The formerly given definition of the *permutohedron* in $\mathbb{R}^N$, denoted by $\Pi(N)$, was that it is the convex hull of the set of rank vectors for all enumerations of N:

$$\Pi(N) := \mathrm{conv}\,(\,\Re(N)\,) \;=\; \mathrm{conv}\,(\,\{\,\rho_\varepsilon \;:\; \varepsilon \in \Upsilon(N)\,\}\,)\,.$$

Because enumerations are closer to combinatorial interpretation (than the rank vectors themselves) it appears to be convenient to identify/label the vertices of $\Pi(N)$ with these. It was also mentioned that the affine hull of $\Pi(N)$ is $\{\, y \in \mathbb{R}^N \;:\; \sum_{u\in N} y_u = \binom{n+1}{2}\,\}$, implying that the dimension of $\Pi(N)$ is $n-1$.

2.3.5. Face lattice of the permutohedron

Here we present an elegant *combinatorial description* of (non-empty) faces of the permutohedron $\Pi(N)$, mentioned in [12, p. 18] and proved in [1, § 1, Proposition 1.4].

Every non-empty face of the permutohedron $\Pi(N)$ uniquely corresponds to an ordered partition of the basic set N into non-empty blocks. We will use the symbol $|\,A_1|\ldots|A_m\,|$ to denote such a partition into m blocks, $1 \le m \le n$. Note that this notation is consistent with our records for enumerations. In fact, the ordered partition serves as a compact description of the set of those enumerations of N which *encode the vertices* of the corresponding face of $\Pi(N)$ – the vertices are the respective rank vectors.

It is the set of those enumerations $\varepsilon \in \Upsilon(N)$ which are consistent with $|\,A_1|\ldots|A_m\,|$. Specifically, these are the ordered lists of elements of N where first (all) the elements of A_1 are listed, then (all) the elements of A_2 follow and so on. Thus, any face of $\Pi(N)$ also corresponds to a special subset of $\Upsilon(N)$. The dimension of the face described by $|\,A_1|\ldots|A_m\,|$ is $n-m$ then [1, Proposition 1.3].

In particular, if $n \ge 2$, the facets of $\Pi(N)$ are described by (ordered) partitions of N into two non-empty blocks and correspond to proper subsets of N. The polyhedral description of the permutohedron in $\mathbb{R}^N$ is as follows [1, Proposition 1.2]:

$$\Pi(N) \;=\; \left\{ z \in \mathbb{R}^N \;:\; \sum_{v\in N} z_v = \binom{n+1}{2} \;\&\; \sum_{u\in S} z_u \ge \binom{|S|+1}{2} \quad \text{for every } S \subseteq N \right\}.$$

2.4. Some group-theoretical concepts

Let $(\mathbb{G}, \odot)$ be a group, say a finite one, and $\mathbb{H}_1, \mathbb{H}_2 \subseteq \mathbb{G}$ two of its subgroups. Then a subgroup $\mathbb{H}$ of $\mathbb{G}$ is called the (internal) *direct product* of $\mathbb{H}_1$ and $\mathbb{H}_2$ if

- the intersection $\mathbb{H}_1 \cap \mathbb{H}_2$ contains solely the identity element of $(\mathbb{G}, \odot)$,
- any element of $\mathbb{H}$ has the form $\Psi_1 \odot \Psi_2$ for some $\Psi_1 \in \mathbb{H}_1$ and $\Psi_2 \in \mathbb{H}_2$,[10]
- $\mathbb{H}_1$ and $\mathbb{H}_2$ *commute*, that is, $\Psi_1 \odot \Psi_2 = \Psi_2 \odot \Psi_1$ for any $\Psi_1 \in \mathbb{H}_1$ and $\Psi_2 \in \mathbb{H}_2$.

We say that groups $(\mathbb{G}, \odot)$ and $(\mathbb{H}, \circledast)$ are *isomorphic* if there exists a bijective mapping ι from $\mathbb{G}$ onto $\mathbb{H}$, called the *group isomorhism*, which respects the group operations, that is, $\iota(g_1 \odot g_2) = \iota(g_1) \circledast \iota(g_2)$ for any $g_1, g_2 \in \mathbb{G}$. Then the inverse of ι is also a group isomorphism and the groups are considered to be identical mathematical structures.

3. HIERARCHY OF CONSIDERED AUTOMORPHISM GROUPS

In section we show that, given a non-empty polytope P in $\mathbb{R}^N$, there is a kind of hierarchy among the four groups of its automorphisms mentioned in Section 1. Specifically,

- the isometries form a subgroup of the group of affine self-transformations,
- the group of affine self-transformations can be embedded into the group of order isomorphisms of the face lattice of P and interpreted as its subgroup, and
- the order isomorphisms can be analogously interpreted as graph isomorphisms.

We also give examples that the inclusions can be strict for a general polytope P.

3.1. Isometries are affine isomorphisms

Given a non-empty domain $\mathsf{D} \subseteq \mathbb{R}^N$, a mapping $\Psi : \mathsf{D} \to \mathbb{R}^N$ will be called an *isometry* if $\lambda(\Psi(x), \Psi(y)) = \lambda(x, y)$ for any $x, y \in \mathsf{D}$. Clearly, since λ is a metric, Ψ must be an injective mapping then, and can be viewed as a bijection from D onto $\Psi(\mathsf{D}) \subseteq \mathbb{R}^N$.

By an *affine isomorphism* we will name an injective mapping $\Psi : \mathsf{D} \to \mathbb{R}^N$ from a non-empty domain $\mathsf{D} \subseteq \mathbb{R}^N$ which respects affine combinations:

$$\Psi\left(\sum_j \alpha_j \cdot x^j\right) = \sum_j \alpha_j \cdot \Psi(x^j) \quad \text{whenever } \textstyle\sum_j \alpha_j = 1,\ x^j \in \mathsf{D}, \text{ and } \sum_j \alpha_j \cdot x^j \in \mathsf{D},$$

viewed as a bijection onto $\Psi(\mathsf{D})$. If $\Psi(\mathsf{D}) = \mathsf{D}$ then Ψ is an *affine automorphism* (of D).

Lemma 3.1. Every isometry Ψ having a non-empty polytope $\mathsf{P} \subseteq \mathbb{R}^N$ as its domain uniquely extends to an isometry Φ on $\mathrm{aff}(\mathsf{P})$ and this Φ is an affine isomorphism. Hence, Ψ is an affine isomorphism from P onto a polytope $\Psi(\mathsf{P})$ such that $\dim(\Psi(\mathsf{P})) = \dim(\mathsf{P})$.

In fact, the arguments below imply that the isometries on non-empty affine subspaces are just those affine mappings where the matrix defining the linear part is orthogonal.

P r o o f . The first auxiliary observation concerns the case of convex sets.

[10] In which case $\Psi_1 \in \mathbb{H}_1$ and $\Psi_2 \in \mathbb{H}_2$ are uniquely determined by the first condition because $\Psi_1 \odot \Psi_2 = \Phi_1 \odot \Phi_2$ for $\Phi_1 \in \mathbb{H}_1$ and $\Phi_2 \in \mathbb{H}_2$ implies $\Phi_1^{-1} \odot \Psi_1 = \Phi_2 \odot \Psi_2^{-1} \in \mathbb{H}_1 \cap \mathbb{H}_2$.

I. If $\Psi : \mathsf{D} \to \mathbb{R}^N$ is an isometry on a non-empty convex set $\mathsf{D} \subseteq \mathbb{R}^N$ then

$$\forall\, x, y \in \mathsf{D} \;\; \forall\, \alpha \in [0,1] \quad \Psi(\,(1-\alpha)\cdot x + \alpha \cdot y\,) = (1-\alpha)\cdot \Psi(x) \,+\, \alpha \cdot \Psi(y)\,.$$

Indeed, if $x, y \in \mathbb{R}^N$, $x \neq y$, then the vector $z_\alpha := (1-\alpha)\cdot x + \alpha \cdot y$ is the only $z \in \mathbb{R}^N$ with $\lambda(x,z) = \alpha \cdot \lambda(x,y)$ and $\lambda(z,y) = (1-\alpha)\cdot\lambda(x,y)$.[11] Because Ψ is an isometry, $\Psi(z_\alpha)$ have to coincide with the only vector $z' \in \mathbb{R}^N$ with $\lambda(\Psi(x), z') = \alpha\cdot\lambda(\Psi(x),\Psi(y))$ and $\lambda(z', \Psi(y)) = (1-\alpha)\cdot\lambda(\Psi(x),\Psi(y))$, which is $z' = (1-\alpha)\cdot\Psi(x) \,+\, \alpha\cdot\Psi(y)$.

The second auxiliary observation deals with the case of zero vector included.

II. If $\Psi : \mathsf{D} \to \mathbb{R}^N$ is an isometry on a convex set $\mathsf{D} \subseteq \mathbb{R}^N$ such that $\mathbf{0} \in \mathsf{D}$ and $\Psi(\mathbf{0}) = \mathbf{0}$ then Ψ preserves the scalar product: $\langle \Psi(x), \Psi(y)\rangle = \langle x, y\rangle$ whenever $x, y, x+y \in \mathsf{D}$.

Indeed, to observe $\Psi(x+y) = \Psi(x) + \Psi(y)$ whenever $x, y, x+y \in \mathsf{D}$ one can apply Step I. twice: to express $\Psi(\frac{1}{2}\cdot x + \frac{1}{2}\cdot y)$ on basis $x, y \in \mathsf{D}$ and then on basis $\mathbf{0}, x+y \in \mathsf{D}$. Then realize that the formula $2\cdot\langle x, y\rangle = \lambda(\mathbf{0}, x+y)^2 - \lambda(\mathbf{0},x)^2 - \lambda(\mathbf{0},y)^2$ is valid for arbitrary $x, y \in \mathbb{R}^N$ and write using the assumption that Ψ is an isometry

$$\begin{aligned} 2\cdot\langle x, y\rangle &= \lambda(\mathbf{0}, x+y)^2 - \lambda(\mathbf{0},x)^2 - \lambda(\mathbf{0},y)^2 \\ &= \lambda(\Psi(\mathbf{0}), \Psi(x+y))^2 - \lambda(\Psi(\mathbf{0}),\Psi(x))^2 - \lambda(\Psi(\mathbf{0}),\Psi(y))^2 \\ &= \lambda(\mathbf{0}, \Psi(x)+\Psi(y))^2 - \lambda(\mathbf{0},\Psi(x))^2 - \lambda(\mathbf{0},\Psi(y))^2 = 2\cdot\langle\Psi(x),\Psi(y)\rangle\,. \end{aligned}$$

The third auxiliary observation provides a sufficient condition for an isometry.

III. If $\Phi : \mathsf{L} \to \mathbb{R}^N$ is a linear function defined on a linear subspace $\mathsf{L} \subseteq \mathbb{R}^N$ such that there exists a linear basis $\{\, e^i : i \in I \,\}$ of L and $\varepsilon > 0$ satisfying

$$\langle e^i, e^j\rangle = \varepsilon\cdot\delta(i=j) = \langle\Phi(e^i),\Phi(e^j)\rangle \qquad \text{for } i, j \in I,\,[12]$$

then Φ is an isometry on L.

Note that $\dim(\mathsf{L}) \geq 1$ by the assumption; let us denote $f^i := \Phi(e^i) \in \mathbb{R}^N$ for $i \in I$ and consider the norm defined by $\|z\| = \sqrt{\langle z, z\rangle}$ for $z \in \mathbb{R}^N$. To show that Φ is an isometry it is enough to verify $\|\Phi(z)\| = \|z\|$ for any $z \in \mathsf{L}$, since $\lambda(x,y) = \|x-y\|$ for $x, y \in \mathbb{R}^N$. Given $z \in \mathsf{L}$ consider the unique linear combination $z = \sum_{i\in I}\alpha_i\cdot e^i$ and write

$$\begin{aligned} \|z\|^2 = \left\langle \sum_{i\in I}\alpha_i\cdot e^i, \sum_{j\in I}\alpha_j\cdot e^j\right\rangle &= \sum_{i\in I}\sum_{j\in I}\alpha_i\cdot\alpha_j\cdot\langle e^i, e^j\rangle \\ &= \sum_{i\in I}\sum_{j\in I}\alpha_i\cdot\alpha_j\cdot\varepsilon\cdot\delta(i=j) = \varepsilon\cdot\sum_{i\in I}\alpha_i^2\,. \end{aligned}$$

As $\Phi(z) = \sum_{i\in I}\alpha_i\cdot f^i$, similar writing with vectors f^i yields $\|\Phi(z)\|^2 = \varepsilon\cdot\sum_{i\in I}\alpha_i^2$.

Note that one can additionally assume that $\mathbf{0} \in \mathrm{relint}(\mathsf{P})$ and $\Psi(\mathbf{0}) = \mathbf{0}$, without loss of generality. Indeed, take and fix $z \in \mathrm{relint}(\mathsf{P})$ and consider a polytope $\mathsf{P}' := \{-z\} \oplus \mathsf{P}$ in place of P and the function $\Psi'(y) := \Psi(z+y) - \Psi(z)$ for $y \in \mathsf{P}'$ in place of Ψ. Given $x, y \in \mathsf{P}'$, the writing $\lambda(\Psi'(x),\Psi'(y)) = \lambda(\Psi(z+x),\Psi(z+y)) = \lambda(z+x, z+y) = \lambda(x,y)$

[11] This reduces to a two-dimensional task: consider the affine hull of x, y, and hypothetical $z \neq z_\alpha$.

[12] This condition, in fact, ensures that the orthogonality is preserved by Φ.

shows that Ψ' is an isometry. Additionally, given its isometric extension Φ' on $\mathrm{aff}(\mathsf{P}') = \{-z\} \oplus \mathrm{aff}(\mathsf{P})$, the formula $\Phi(x) := \Psi(z) + \Phi'(-z+x)$ for $x \in \mathrm{aff}(\mathsf{P})$ defines the respective isometric extension of Ψ, by analogous arguments.

In this particular case, the affine hull $\mathsf{L} := \mathrm{aff}(\mathsf{P})$ is a linear subspace of $\mathbb{R}^N$. If $\mathsf{L} = \{\mathbf{0}\}$ there is nothing to prove. Thus, assume $\dim(\mathsf{L}) \geq 1$ in the sequel. Then one can consider such an Euclidean ball B around $\mathbf{0}$ in L that $\mathsf{B} \subseteq \mathrm{relint}(\mathsf{P})$ and construct a such linear basis $\{ e^i : i \in I \}$ of L that both $e^i + e^j \in \mathsf{B} \subseteq \mathsf{P}$ and $\langle e^i, e^j \rangle = \varepsilon \cdot \delta(i = j)$ for all $i, j \in I$ and some $\varepsilon > 0$.[13] Because any $z \in \mathsf{L}$ is a unique affine combination $z = (1 - \sum_{i \in I} \alpha_i) \cdot \mathbf{0} + \sum_{i \in I} \alpha_i \cdot e^i$ one can put $\Phi(z) := \sum_{i \in I} \alpha_i \cdot \Psi(e^i)$, defining a linear/affine function on L. By application of Step II. one has $\langle \Psi(e^i), \Psi(e^j) \rangle = \langle e^i, e^j \rangle$ for any pair $i, j \in I$. The assumptions of Step III. are thus fulfilled for Φ, implying that Φ is an isometry on L. It remains to show $\Psi(z) = \Phi(z)$ for all $z \in \mathsf{P}$.

To this end realize that Ψ is affine on the intersection of P with any line in L. Specifically, if $x, y \in \mathsf{P}$, $x \neq y$, then $\mathsf{P} \cap \mathrm{lin}(\{x, y\})$ is a closed segment $[x', y']$ with $x', y' \in \mathsf{P}$, $x' \neq y'$, and one can apply Step I. to x' and y' to observe that. In particular, to show $\Psi = \Phi$ on $[x', y']$ it is enough to verify that Ψ and Φ coincide at two distinct points of $[x', y']$. We apply this principle repeatedly to show $\Psi(z) = \Phi(z)$ for $z \in \mathsf{P}$ by induction on k, the number of $\alpha_i \neq 0$ in the unique linear combination $z = \sum_{i \in I} \alpha_i \cdot e^i$.

If $k \leq 1$, that is, $z \in \mathrm{lin}(e^i) \cap \mathsf{P}$ for some $i \in I$, then the two points are $\mathbf{0}$ and e^i. If $k \geq 2$ then in place of z consider such a multiple $\hat{z} := \delta \cdot z$, $0 < \delta \leq 1$, for which $\hat{z} \in \mathsf{B}$. Write $\hat{z} = \sum_{i \in I} \hat{\alpha}_i \cdot e^i$, take $j \in I$ with $\hat{\alpha}_j \neq 0$, and define $\hat{x} := \sum_{i \in I, i \neq j} \hat{\alpha}_i \cdot e^i$ and $\hat{y} := \hat{\alpha}_j \cdot e^j$.[14] The point is that then $\hat{x}, \hat{y} \in \mathsf{B} \subseteq \mathsf{P}$ and, by the induction premise, $\Psi(\hat{x}) = \Phi(\hat{x})$ and $\Psi(\hat{y}) = \Phi(\hat{y})$. Hence, by the principle, $\Psi = \Phi$ at $\frac{1}{2} \cdot \hat{x} + \frac{1}{2} \cdot \hat{y} = \frac{1}{2} \cdot \hat{z}$. Apply the principle again with $\mathbf{0}$ and $\frac{1}{2} \cdot \hat{z}$ to get $\Psi(z) = \Phi(z)$, finishing the induction step.

Since Φ is a linear/affine bijection on L, the image $\Phi(\mathsf{P}) = \Psi(\mathsf{P})$ of P is a polytope. Moreover, $\Phi(\mathrm{aff}(\mathsf{P})) = \Phi(\mathsf{L}) = \mathrm{aff}(\Phi(\mathsf{P}))$ gives $\dim(\Psi(\mathsf{P})) = \dim(\Phi(\mathsf{P})) = \dim(\mathsf{P})$. □

Remark 3.2. Analogous arguments as given in the (end of the) proof of Lemma 3.1 can be used to show that any affine isomorphism Ψ on a non-empty polytope P uniquely extends to an affine isomorphism Φ on $\mathrm{aff}(\mathsf{P})$.

In our case of an isometry Ψ of a non-empty polytope $\mathsf{P} \subseteq \mathbb{R}^N$ onto itself, Lemma 3.1 implies that Ψ is an affine automorphism of P. Since the composition of isometries is an isometry, the considered isometries form a subgroup of the group of affine automorphisms of P. The inclusion might be, however, strict as later Example 3.4 shows.

3.2. Affine isomorphisms define order isomorphisms

We observe now that any affine automorphism of a polytope P yields an order automorphism of the face lattice $\mathfrak{F}(\mathsf{P})$ (see Section 2.3.3).

Lemma 3.3. Given an affine isomorphism $\Psi : \mathsf{P} \to \mathbb{R}^N$ having a non-empty polytope $\mathsf{P} \subseteq \mathbb{R}^N$ as its domain, the image $\Psi(\mathsf{P})$ is a polytope in $\mathbb{R}^N$.
The mapping $\iota : \mathsf{F} \subseteq \mathsf{P} \mapsto \Psi(\mathsf{F})$ restricted to $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$ is then an order isomorphism from the face lattice $\mathfrak{F}(\mathsf{P})$ onto the face lattice $\mathfrak{F}(\Psi(\mathsf{P}))$ (see Section 2.1).

[13] Basically, this is a 'small' multiple of an orthogonal basis of L.
[14] In fact, this is a kind of orthogonal decomposition of $\hat{z}$.

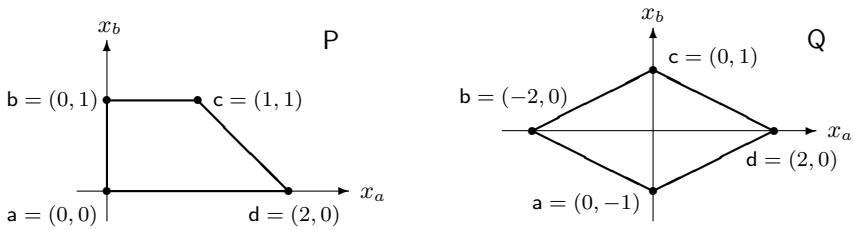


Fig. 1. Polytopes with different automorphism groups.

P r o o f . Every affine mapping respects affine combinations (see definition in Section 2.3.1). Therefore, it also respects convex combinations and, thus, maps convex sets to convex sets. Hence, the claim that $\Psi(\mathsf{P})$ is a polytope is evident.

Because Ψ, viewed as a mapping defined on the power set of P, is inclusion-preserving and both $\mathfrak{F}(\mathsf{P})$ and $\mathfrak{F}(\Psi(\mathsf{P}))$ are ordered by inclusion, one only needs to verify that the image $\Psi(\mathsf{F})$ of any $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$ is a face of $\Psi(\mathsf{P})$. Because Ψ maps convex sets to convex sets, it also maps closed segments to closed segments and open segments to open segments. Therefore, the definition of a face we gave in Section 2.3.2 nicely transforms from F to $\Psi(\mathsf{F})$, showing that $\Psi(\mathsf{F})$ is a face of $\Psi(\mathsf{P})$ once F is a face of P. □

In particular, any affine automorphism Ψ of a non-empty polytope P, defines, by Lemma 3.3, an order automorphism of $\mathfrak{F}(\mathsf{P})$, as $\Psi(\mathsf{P}) = \mathsf{P}$. Clearly, in case $\Psi = \Psi_1 \circ \Psi_2$ the order automorphism of $\mathfrak{F}(\mathsf{P})$ defined by Ψ is the composition of those defined by Ψ_1 and Ψ_2. Hence, the group of affine automorphisms of P is embedded as a subgroup into the group of order automorphisms of $\mathfrak{F}(\mathsf{P})$. Nonetheless, the embedding might be strict as the following example shows.

Example 3.4. Consider the case $N := \{a, b\}$ and the trapezoid $\mathsf{P} \subseteq \mathbb{R}^N$ shown in the left picture of Figure 1. The group of affine automorphisms of P has two elements, the identity and the one defined by $\Psi(x_a, x_b) := (2 - x_a - x_b, x_b)$ for $x_a, x_b \in \mathbb{R}$, which can also be described as a *horizontal flip* of its vertices: $\mathsf{a} \leftrightarrow \mathsf{d}$ & $\mathsf{b} \leftrightarrow \mathsf{c}$.[15] Nevertheless, because of $\lambda(\mathsf{a}, \mathsf{b}) = 1 \neq \sqrt{2} = \lambda(\mathsf{d}, \mathsf{c})$, only the identity is an isometry of P; thus, the group of isometries of P (onto itself) has only one element, the identity.

The group of order isomorphisms of $\mathfrak{F}(\mathsf{P})$ is larger, it has eight elements. Specifically, besides the identity and the horizontal flip, it contains 4 other flips, namely, $\mathsf{a} \leftrightarrow \mathsf{c}$ (it means that b and d remain in their positions), $\mathsf{b} \leftrightarrow \mathsf{d}$, $\mathsf{a} \leftrightarrow \mathsf{c}$ & $\mathsf{b} \leftrightarrow \mathsf{d}$, and $\mathsf{a} \leftrightarrow \mathsf{b}$ & $\mathsf{c} \leftrightarrow \mathsf{d}$, and two cycles, namely $\mathsf{a} \to \mathsf{b} \to \mathsf{c} \to \mathsf{d} \to \mathsf{a}$ and $\mathsf{a} \to \mathsf{d} \to \mathsf{c} \to \mathsf{b} \to \mathsf{a}$.[16]

Another example of a polytope, whose automorphism groups differ is the rhombus $\mathsf{Q} \subseteq \mathbb{R}^N$ shown in the right picture of Figure 1. The group of its isometries has 4

[15] For this, one can apply Remark 3.2 and the observation from Section 2.3.1 about the form of affine mappings on $\mathbb{R}^N$ and use the fact that vertices must be mapped to vertices (implied by Lemma 3.3).

[16] This is because, besides the vertices, the only proper faces of P are its geometric edges; hence, the order automorphisms of P correspond to its graph automorphisms.

elements, besides the identity, it has 3 flips: $\mathsf{a} \leftrightarrow \mathsf{c}$, $\mathsf{b} \leftrightarrow \mathsf{d}$, and $\mathsf{a} \leftrightarrow \mathsf{c}$ & $\mathsf{b} \leftrightarrow \mathsf{d}$. The group of its affine automorphisms has 8 elements. For example, the flip $\mathsf{a} \leftrightarrow \mathsf{b}$ & $\mathsf{c} \leftrightarrow \mathsf{d}$ is realized by the linear transform $\Psi(x_a, x_b) := (2 \cdot x_b, \frac{1}{2} \cdot x_a)$ for $x_a, x_b \in \mathbb{R}$.

3.3. Order isomorphisms induce graph isomorphisms

Lemma 3.5. Given an order isomorphism $\iota : \mathfrak{F}(\mathsf{P}) \to \mathfrak{F}(\mathsf{Q})$ for polytopes $\mathsf{P}, \mathsf{Q} \subseteq \mathbb{R}^N$, the mapping Ψ which assigns to $x \in \mathrm{ext}(\mathsf{P})$ that $z \in \mathrm{ext}(\mathsf{Q})$ for which $\iota(\{x\}) = \{z\}$ is a graph isomorphism from $G(\mathsf{P})$ onto $G(\mathsf{Q})$.

P r o o f. This is more or less evident from the definitions because $\{x, y\}$ is an edge of $G(\mathsf{P})$ iff the segment $[x, y]$ is a 1-dimensional face of P. The order isomorphisms of face lattices preserve the dimension of faces, because this is their height function. □

Note that, given an affine isomorphism on a polytope P, the graph isomorphism assigned to the respective order isomorphism ι from $\mathfrak{F}(\mathsf{P})$ is what one expects: the restriction of Ψ on $\mathrm{ext}(\mathsf{P})$. To see this combine Lemma 3.3 and Lemma 3.5.

Thus, any order automorphism of $\mathfrak{F}(\mathsf{P})$ defines, by Lemma 3.5, a graph automorphism of $G(\mathsf{P})$, because $\mathsf{Q} = \mathsf{P}$ then. Clearly, if $\iota = \iota_1 \circ \iota_2$ then $\Psi = \Psi_1 \circ \Psi_2$ for the respective graph automorphisms of $G(\mathsf{P})$. Hence, the group of order automorphisms of $\mathfrak{F}(\mathsf{P})$ is embedded as a subgroup into the group of graph automorphisms of $G(\mathsf{P})$.

The following example, however, shows that the groups might have different sizes. The discussed polytope belongs to a special class of neighbourly polytopes [3, § 14] in which every two distinct vertices form a geometric edge.

Example 3.6. Consider the case $N := \{a, b, c, d\}$ and construct our polype $\mathsf{P} \subseteq \mathbb{R}^N$ as a *cyclic polytope* of the type $C(6, 4)$ as discussed in [3, § 13]. Specifically, we define P as the convex hull of the following six vectors (x_a, x_b, x_c, x_d) in $\mathbb{R}^N$:[17]

$$\begin{array}{lll} \mathsf{a} := (-2, 4, -8, 16) & \mathsf{b} := (-1, 1, -1, 1) & \mathsf{c} := (0, 0, 0, 0) \\ \mathsf{d} := (1, 1, 1, 1) & \mathsf{e} := (2, 4, 8, 16) & \mathsf{f} := (3, 9, 27, 81)\,. \end{array}$$

The reader can verify directly that $\mathrm{ext}(\mathsf{P}) = \mathsf{V} := \{\, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f} \,\}$ and every two distinct vertices of P form a geometric edge; see also [3, Theorem 13.4] and [3, Theorem 14.1]. In particular, $G(P)$ is a complete graph over V and any bijection $\Psi : \mathsf{V} \to \mathsf{V}$ is a graph automorphism of $G(P)$; so, the graph automorphism group has 720 elements.

To show that not all of them correspond to an order automorphism of $\mathfrak{F}(\mathsf{P})$ we first observe that $\mathsf{S} := \{\, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \,\}$ represents a face of P. Indeed, to this end consider the vector $y := (-2, -1, 2, 1)$ and observe that the inequality $\langle z, y \rangle \geq 0$ is tight for $z \in \mathsf{S}$ and strict for $z \in \mathsf{V} \setminus \mathsf{S}$. Hence, it is valid for any $z \in \mathsf{P}$ and $\mathrm{conv}\,(\mathsf{S})$ is a face of P.

Second, observe that $\mathsf{T} := \{\, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{e} \,\}$ does not represent a face of P. Indeed, to show that $\mathsf{C} := \mathrm{conv}\,(\mathsf{T})$ is not a face P realize that $\mathsf{g} := (\frac{13}{11}, \frac{19}{11}, \frac{37}{11}, \frac{91}{11}) = \frac{10}{11} \cdot \mathsf{d} + \frac{1}{11} \cdot \mathsf{f} \in \mathsf{P}$. On the other hand, $\mathsf{g} \notin \mathsf{C}$: to this end consider the vector $y' := (-4, 0, 1, 0)$ observe that the inequality $\langle z, y' \rangle \geq 0$ is valid for $z \in \mathsf{T}$, while $\langle \mathsf{g}, y' \rangle = -\frac{15}{11} < 0$. Then

[17] These are just the vectors obtained by the choice of values $\alpha \in \{-2, -1, 0, 1, 2, 3\}$ on the moment curve $\phi(\alpha) := (\alpha, \alpha^2, \alpha^3, \alpha^4)$, as described in [3, § 13].

$\mathsf{h} := (\frac{1}{2}, \frac{3}{2}, 2, 6) = \frac{1}{16} \cdot \mathsf{a} + \frac{10}{16} \cdot \mathsf{c} + \frac{5}{16} \cdot \mathsf{e} \in \mathsf{C}$. However, because $\mathsf{h} = \frac{5}{16} \cdot \mathsf{b} + \frac{11}{16} \cdot \mathsf{g}$, one has $\mathsf{h} \in]\,\mathsf{b}, \mathsf{g}\,[\subseteq \mathsf{P}$ implying that C is not a face of P.

Finally, take any auto-bijection of V mapping S onto T and get an example of graph automorphism of $G(\mathsf{P})$ which does not yield an order automorphism of $\mathfrak{F}(\mathsf{P})$.

4. MAIN RESULT AND ITS PROOF

The structure of this section is as follows. We start with the main result formulation in Section 4.1. Then, a number of auxiliary concepts needed in its proof is introduced in Sections 4.2 and 4.3. What follows is a series of observations on the automorphisms proved in Section 4.4. The formerly proved facts are put together in Section 4.5.

4.1. Main theorem: formulation

First, let us show some of the claims made in the Introduction.

Lemma 4.1. Any linear isometry from $\mathbb{R}^N$ onto itself is uniquely determined by its values on the set of rank vectors $\Re(N)$. The mappings Ψ_π and $\Psi_\mathfrak{r}$ defined by (1) and (2) are both such linear isometries and map the permutohedron $\Pi(N)$ onto itself. Moreover, they commute: $\Psi_\pi \circ \Psi_\mathfrak{r} = \Psi_\mathfrak{r} \circ \Psi_\pi$ for any permutation $\pi \in S_N$.

P r o o f . Every linear isometry is an affine mapping and is, thus, uniquely determined by its values on an affinely independent generator of $\mathbb{R}^N$ (see Section 2.3.1). There is such a generator $\mathsf{G} \subseteq \mathrm{ext}(\Pi(N)) = \Re(N)$ of $\mathrm{aff}(\Pi(N)) = \{\, y \in \mathbb{R}^N \,:\, \sum_{u \in N} y_u = \binom{n+1}{2} \,\}$, and adding the vector $\mathbf{0}$, for which $\Psi(\mathbf{0}) = \mathbf{0}$, yields an affine generator of $\mathbb{R}^N$.

The linearity of Ψ_π and $\Psi_\mathfrak{r}$ is immediate from (1) and (2). Given $x, y \in \mathbb{R}^N$, writing

$$\lambda(\Psi_\pi(x), \Psi_\pi(y))^2 \overset{(1)}{=} \sum_{u \in N} \{x_{\pi(u)} - y_{\pi(u)}\}^2 = \sum_{v \in N} \{x_v - y_v\}^2 = \lambda(x, y)^2$$

shows that Ψ_π are isometries. Put $z_u := x_u - y_u$ for $u \in N$, $\bar{z} := \sum_{w \in N} \frac{z_w}{n}$, and write

$$\begin{aligned}
\lambda(\Psi_\mathfrak{r}(x), \Psi_\mathfrak{r}(y))^2 &\overset{(2)}{=} \sum_{u \in N} \left\{ \frac{2}{n} \cdot \sum_w x_w - x_u - \frac{2}{n} \cdot \sum_w y_w + y_u \right\}^2 = \sum_{u \in N} \left\{ \frac{2}{n} \cdot \sum_w z_w - z_u \right\}^2 \\
&= \sum_{u \in N} \{2 \cdot \bar{z} - z_u\}^2 = \sum_{u \in N} \{\, 4 \cdot \bar{z}^2 - 4 \cdot \bar{z} \cdot z_u \,\} + \sum_{u \in N} z_u^2 \\
&= 4 \cdot \bar{z} \cdot \overbrace{\sum_{u \in N} \{\bar{z} - z_u\}}^{=0} + \sum_{u \in N} z_u^2 = \sum_{u \in N} \{x_u - y_u\}^2 = \lambda(x, y)^2
\end{aligned}$$

to see that $\Psi_\mathfrak{r}$ is an isometry.

If we view a rank vector $x = (x_u)_{u \in N} \in \Re(N)$ as a mapping $x : N \to [n]$ and $\pi : N \to N$ is a bijection then $\Psi_\pi(x) = x \circ \pi$. Similarly, $\Psi_\mathfrak{r}(x) = (n+1-x_u)_{u \in N} = \mathfrak{r} \circ x$, where $\mathfrak{r} : [n] \to [n]$ is the reverse (see p. 758). Thus, it is evident that both Ψ_π and $\Psi_\mathfrak{r}$ map $\Re(N)$ onto $\Re(N)$. Since both these mappings are affine, they map $\Pi(N)$ onto itself.

By the transitivity of composition, one has $\Psi_\pi \circ \Psi_\mathfrak{r}(x) = \mathfrak{r} \circ x \circ \pi = \Psi_\mathfrak{r} \circ \Psi_\pi(x)$ for any $x \in \Re(N)$. And again, because the rank vectors together with $\mathbf{0}$ generate affinely $\mathbb{R}^N$, this commutativity equality extends from $\Re(N) \cup \{\mathbf{0}\}$ to $\mathbb{R}^N$. □

This is the main result.

Theorem 4.2. For every $\Psi : \Re(N) \to \Re(N)$ the following conditions are equivalent:

(i) either $\Psi = \Psi_\pi$ or $\Psi = \Psi_\pi \circ \Psi_\mathfrak{r}$ for some $\pi \in S_N$, with $\Psi_\pi, \Psi_\mathfrak{r}$ restricted to $\Re(N)$,

(ii) Ψ is the restriction of a linear auto-isometry of $\mathbb{R}^N$ which maps $\Pi(N)$ onto itself,

(iii) Ψ is the restriction of an isometry of $\Pi(N)$ which maps it onto itself,

(iv) Ψ is the restriction of an affine automorphism of $\Pi(N)$,

(v) there is an order automorphism ι of the face lattice $\mathfrak{F}(\Pi(N))$ which determines the mapping Ψ by the relation $\iota(\{x\}) = \{\Psi(x)\}$ for any $x \in \Re(N)$,

(vi) Ψ is a graph automorphism of the graph $G(\Pi(N))$ of the permutohedron.

Note in this context that, if $n \geq 3$, then, by Lemma 4.1, the condition (i) in Theorem 4.2 means that Ψ belongs to the direct product of the permutational group $\{\, \Psi_\pi \,:\, \pi \in S_N \,\}$ and of the two-element group $\{\, \mathrm{id}, \Psi_\mathfrak{r} \,\}$ (see Section 2.4).

The proof of Theorem 4.2 is postponed to Section 4.5. The hardest part is the implication (vi)⇒(i), for which several auxiliary concepts are needed (see below).

4.2. Combinatorial view on graph automorphisms

Graph automorphisms of the permutohedron can perhaps be treated in geometric terms of rank vectors, but we found it to be more convenient to describe them in combinatorial terms of enumerations (see the beginning of Section 2). This is because faces of the permutohedron have their natural description in these terms (see Section 2.3.5).

Therefore, in the rest of Section 4, we interpret our graph automorphisms as the automorphisms of a graph, whose *nodes are enumerations* of N. Thus, we will deal with the *permutohedral graph* (over N), which is a (finite undirected) graph with node set $\Upsilon(N)$, where $\varepsilon, \eta \in \Upsilon(N)$ form an edge iff $[\rho_\varepsilon, \rho_\eta]$ is a geometric edge of $\Pi(N)$. Also, to avoid degeneracies we practically assume $n \geq 3$ is the rest of this section.

It follows from the results reported in Section 2.3.5 that geometric edges of $\Pi(N)$ correspond to ordered partitions $|A_1| \dots |A_{n-1}|$ of N, which means there is exactly one $i \in [n-1]$ such that $|A_i| = 2$, while $|A_\ell| = 1$ for $\ell \in [n-1] \setminus \{i\}$. By the *numeral label* of such an edge we will mean that particular number $i \in [n-1]$.

It follows from the description of enumerations consistent with $|A_1| \dots |A_{n-1}|$ that $\{\varepsilon, \eta\} \subseteq \Upsilon(N)$ form an edge iff they differ by an *adjacent transposition*, which means

$$\begin{aligned} \exists\, i \in [n-1] \quad &\text{such that} \;\; \varepsilon(i) = \eta(i+1),\; \varepsilon(i+1) = \eta(i), \\ &\text{and} \quad \varepsilon(k) = \eta(k) \qquad \text{for remaining } k \in [n] \setminus \{i, i+1\}\,. \end{aligned} \tag{3}$$

The set of *neighbours* of an enumeration $\varepsilon \in \Upsilon(N)$ in the permutohedral graph will be shortly denoted by $\mathrm{ne}(\varepsilon) := \{\, \eta \in \Upsilon(N) \,:\, (3) \text{ holds} \,\}$.

To distinguish the combinatorial interpretation from the geometric one, we will denote the automorphisms of the permutohedral graph by means of lowercase letters, that is, $\psi : \Upsilon(N) \to \Upsilon(N)$ contrary to $\Psi : \Re(N) \to \Re(N)$. The relation of these maps is clear: $\psi(\varepsilon)$ for $\varepsilon \in \Upsilon(N)$ is that $\eta \in \Upsilon(N)$ such that $\rho_\eta = \Psi(\rho_\varepsilon)$. Put it in another way, $\psi = \mathrm{inv}_{-1} \circ \Psi \circ \mathrm{inv}$, where inv denotes the 'inversion' operator from $\Upsilon(N)$ onto $\Re(N)$. In particular, given a bijection $\pi : N \to N$, one has $\psi_\pi(\varepsilon) = \pi_{-1} \circ \varepsilon$ for any $\varepsilon \in \Upsilon(N)$, while $\psi_{\mathfrak{r}}(\varepsilon) = \varepsilon \circ \mathfrak{r}$ for any $\varepsilon \in \Upsilon(N)$.[18]

Another particular notation applied in the rest of this section is the symbol $\mathbb{G}$ used to denote the *group of all graph automorphisms* $\psi : \Upsilon(N) \to \Upsilon(N)$ of the permutohedral graph and the symbol $\mathbb{H}$, used to denote its *subgroup generated by permutational automorphisms* ψ_π *and the reversing automorphism* $\psi_{\mathfrak{r}}$. These have the form ψ_π or $\psi_\pi \circ \psi_{\mathfrak{r}} = \psi_{\mathfrak{r}} \circ \psi_\pi$ for $\pi \in S_N$.

With little abuse of notation, the symbol $\mathrm{fac}(T)$ for $T \subseteq \Upsilon(N)$ will also be used to denote those enumerations, whose rank vectors belong to the *facial closure* of the set of rank vectors $\{\, \rho_\varepsilon \,:\, \varepsilon \in T \,\}$. The following is the consequence of the fact that the permutohedron is a simple polytope.

Lemma 4.3. If $\psi \in \mathbb{G}$ is a graph automorphism of the permutohedral graph then, for any $T \subseteq \Upsilon(N)$, one has $\psi(\mathrm{fac}(T)) = \mathrm{fac}(\psi(T))$.

P r o o f . It follows directly from (3) that, for any $\varepsilon \in \Upsilon(N)$, one has $|\mathrm{ne}(\varepsilon)| = n - 1$. In the geometric context this means that every rank vector has exactly $n - 1$ geometric neighbours. Thus, the permutohedron $\Pi(N)$ is a simple polytope, one can apply the results from the Appendix to it, in particular, Corollary A.3. If one reformulates this in our combinatorial terms, one gets the observation above. □

4.3. Squares, hexagons, and different types of numeral labels

By the results from Section 2.3.5, there are two kinds of two-dimensional faces of $\Pi(N)$.

- Faces described by ordered partitions $|A_1| \dots |A_{n-2}|$ of N with $|A_i| = 2 = |A_j|$ for distinct $i, j \in [n-2]$ are geometrically *squares*. These involve four vertices and, in graphical context, they correspond to induced 4-cycles.

- Faces characterized by $|A_i| = 3$ for some $i \in [n-2]$ are geometrically regular *hexagons*. These involve six vertices and the respective induced subgraphs of the permutohedral graph are special 6-cycles.

See Figure 2 for illustration of the graphical representation of squares and hexagons.

[18] This is because of $\Psi_\pi(\rho_\varepsilon)_{-1} = (\rho_\varepsilon \circ \pi)_{-1} = \pi_{-1} \circ (\rho_\varepsilon)_{-1} = \pi_{-1} \circ \varepsilon$ and $\Psi_{\mathfrak{r}}(\rho_\varepsilon)_{-1} = (\mathfrak{r} \circ \rho_\varepsilon)_{-1} = (\rho_\varepsilon)_{-1} \circ (\mathfrak{r}_{-1}) = \varepsilon \circ \mathfrak{r}$.

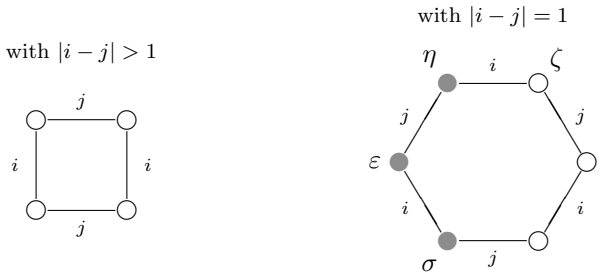


Fig. 2. A square and a hexagon in the permutohedral graph.

Special terminology concerning the numeral labels will be used in the sequel. The *opposite* label to $i \in [n-1]$ is the label $n-i$, also in $[n-1]$. The label which is its own opposite label will be named *central*. It is clear that it exists only in case $n = 2 \cdot k$ is even, in which case it is just k. We say that two distinct numeral labels $i, j \in [n-1]$, $i \neq j$, are *close* if $|i-j| = 1$, while in case $|i-j| > 1$ we say they are *distant*. Note in this context that the distinct labels on edges of a square are distant, while they are close in case of the hexagon; see Figure 2 for illustration.

We also say that a graph automorphism $\psi : \Upsilon(N) \to \Upsilon(N)$ *preserves labels* if, for any edge $\{\varepsilon, \eta\}$ of the permutohedral graph, the numeral labels on $\{\varepsilon, \eta\}$ and $\{\psi(\varepsilon), \psi(\eta)\}$ coincide. On the other hand, we say that $\psi : \Upsilon(N) \to \Upsilon(N)$ *converts labels* if, for every edge $\{\varepsilon, \eta\}$, the label on $\{\psi(\varepsilon), \psi(\eta)\}$ is the opposite one to the label on $\{\varepsilon, \eta\}$.

An example of an automorphism preserving labels is any permutational one, that is, any ψ_π for $\pi \in S_N$. The reversing automorphism $\psi_{\mathfrak{r}}$, however, evidently converts labels. One of substantial facts shown below is that any graph automorphism either preserves or converts labels, which leads to the equality $\mathbb{G} = \mathbb{H}$ proved below.

Example 4.4. In case $N := \{a, b, c, d\}$ there exists an edge $\{\varepsilon, \eta\}$ of the permutohedral graph and a <u>non-identical</u> graph automorphism $\psi : \Upsilon(N) \to \Upsilon(N)$ such that $\psi(\varepsilon) = \varepsilon$ and $\psi(\eta) = \eta$, which means that ψ is identical on that edge. The point is that the edge has a central numeral label. Take $\varepsilon := |a|b|c|d|$, $\eta := |a|c|b|d|$, and ψ which performs 8 (non-local) switches between enumerations, namely

$$\begin{array}{llll} |a|b|d|c| \leftrightarrow |b|a|c|d| & |a|c|d|b| \leftrightarrow |c|a|b|d| & |a|d|b|c| \leftrightarrow |b|c|a|d| & |a|d|c|b| \leftrightarrow |c|b|a|d| \\ |b|c|d|a| \leftrightarrow |d|a|b|c| & |b|d|c|a| \leftrightarrow |d|b|a|c| & |c|b|d|a| \leftrightarrow |d|a|c|b| & |c|d|b|a| \leftrightarrow |d|c|a|b| \end{array} ,$$

while the remaining 8 enumerations remain in their positions. The construction was $\psi := \psi_\pi \circ \psi_{\mathfrak{r}}$, where $\pi \in S_N$ is given by $\pi(a) = d$, $\pi(b) = c$, $\pi(c) = b$, and $\pi(d) = a$.

An analogous construction as in Example 4.4 can be done for any edge $\{\varepsilon, \eta\}$ with a *central numeral label*: take that $\pi \in S_N$ for which $\pi_{-1} \circ \varepsilon = \varepsilon \circ \mathfrak{r}$. In particular, for every such an edge, if two graph automorphisms ψ_1 and ψ_2 coincide on this edge, they still need not coincide everywhere. Therefore, one cannot propagate the uniqueness of automorphisms from this edge to the rest of the permutohedron. Nonetheless, we are going to show below that this cannot happen for an edge with a *non-central label*.

4.4. Observations on graph automorphisms

Given $\varepsilon \in \Upsilon(N)$, the symbol $\hat{\varepsilon}$ will be used as a shorthand for its *reverse* enumeration $\hat{\varepsilon} := \psi_{\mathfrak{r}}(\varepsilon) = \varepsilon \circ \mathfrak{r}$. Enumerations can be mapped to each other by the elements of $\mathbb{H}$.

Lemma 4.5. The following two facts are easy.

(i) For any $\varepsilon, \sigma \in \Upsilon(N)$ there exists $\pi \in S_N$ such that $\psi_\pi(\sigma) = \varepsilon$.

(ii) For any $\varepsilon \in \Upsilon(N)$ there exists $\phi \in \mathbb{H}$ such that $\phi(\varepsilon) = \varepsilon$ and ϕ converts labels.

Proof. For (i) we simply put $\pi := \sigma \circ \varepsilon_{-1}$. For (ii) we take $\sigma := \hat{\varepsilon} = \psi_{\mathfrak{r}}(\varepsilon)$ and apply (i) to it to get $\pi \in S_N$ such that $\varepsilon = \psi_\pi(\hat{\varepsilon}) = \psi_\pi \circ \psi_{\mathfrak{r}}(\varepsilon)$.

Thus, $\phi := \psi_\pi \circ \psi_{\mathfrak{r}} = \psi_{\mathfrak{r}} \circ \psi_\pi \in \mathbb{H}$ satisfies $\Phi(\varepsilon) = \varepsilon$. Being a composition of the automorphism ψ_π, which preserves labels, and of the automorphism $\psi_{\mathfrak{r}}$, which converts labels, ϕ converts numeral labels. □

Further lemma is about local quasi-uniqueness of graph automorphisms nearby any enumeration, being the crucial step to verify our relative uniqueness result.

Lemma 4.6. Assume $\varepsilon \in \Upsilon(N)$ and $\psi \in \mathbb{G}$ such that $\psi(\varepsilon) = \varepsilon$. Then ψ maps $\mathrm{ne}(\varepsilon)$ onto $\mathrm{ne}(\varepsilon)$ and one of the next two options occurs, which exclude each other if $n \geq 3$:

- ψ is identical on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$,
- ψ converts numeral labels within $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$, that is, for any $\eta \in \mathrm{ne}(\varepsilon)$, the edge $\{\varepsilon, \psi(\eta)\}$ has the opposite numeral label to the label on the edge $\{\varepsilon, \eta\}$.

Proof. To avoid the trivial case assume $n \geq 3$ in the rest of the proof. The fact that $\mathrm{ne}(\varepsilon)$ is mapped onto $\mathrm{ne}(\varepsilon)$ is immediate. The neighbours of ε can then be identified with the natural numbers from $[n-1]$. Specifically, we let ε^i denote that (unique) $\eta \in \mathrm{ne}(\varepsilon)$ for which the edge $\{\varepsilon, \eta\}$ has the numeral label i.

We base our arguments on observations about certain faces generated by neighbours of ε. The following facts easily follow from the results reported in Section 2.3.5:

- if $i, j \in [n-1]$ are close then $\mathrm{fac}(\{\varepsilon, \varepsilon^i, \varepsilon^j\})$ is a hexagon, with 6 enumerations,[19]
- if $i, j \in [n-1]$ are distant, then $\mathrm{fac}(\{\varepsilon, \varepsilon^i, \varepsilon^j\})$ is a square, with 4 enumerations,[20]
- if $i \in [n-1]$ then $\mathrm{fac}(\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^i\})$ is described by $|\varepsilon(1) \ldots \varepsilon(i)|\varepsilon(i+1) \ldots \varepsilon(n)|$ and involves precisely $i! \cdot (n-i)!$ enumerations.

We apply those facts to derive three auxiliary observations.

I. For every $\varepsilon^i \in \mathrm{ne}(\varepsilon)$, $i \in [n-1]$, either $\psi(\varepsilon^i) = \varepsilon^i$ or $\psi(\varepsilon^i) = \varepsilon^{n-i}$.

Indeed, assume $\psi(\varepsilon^i) = \varepsilon^j$ for $j \in [n-i]$. Thus, ψ maps $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^i\}$ onto $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^j\}$ and Lemma 4.3 gives

$$\psi(\,\mathrm{fac}(\,\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^i\}\,)\,) = \mathrm{fac}(\,\psi(\,\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^i\}\,)\,) = \mathrm{fac}(\,\{\varepsilon\} \cup \mathrm{ne}(\varepsilon) \setminus \{\varepsilon^j\}\,)\,.$$

[19] Specifically, if $j = i+1$ then the face is $|\varepsilon(1)| \ldots |\varepsilon(i-1)|\varepsilon(i)\varepsilon(i+1)\varepsilon(i+2)|\varepsilon(i+3)| \ldots |\varepsilon(n)|$.

[20] If $i+1 < j$ then the face is $|\varepsilon(1)| \ldots |\varepsilon(i-1)|\varepsilon(i)\varepsilon(i+1)| \ldots |\varepsilon(j)\varepsilon(j+1)|\varepsilon(j+2)| \ldots |\varepsilon(n)|$.

That means $|\varepsilon(1)\dots\varepsilon(i)|\varepsilon(i+1)\dots\varepsilon(n)|$ is mapped onto $|\varepsilon(1)\dots\varepsilon(j)|\varepsilon(j+1)\dots\varepsilon(n)|$. Since ψ is a bijection, this implies $i!\cdot(n-i)! = j!\cdot(n-j)!$ and necessitates that either $j=i$ or $j=n-i$, what is claimed.

II. If $i+1<\frac{n}{2}$ and $\psi(\varepsilon^i)=\varepsilon^i$ then $\psi(\varepsilon^{i+1})=\varepsilon^{i+1}$.

Indeed, by Step I., one has either $\psi(\varepsilon^{i+1})=\varepsilon^{i+1}$ or $\psi(\varepsilon^{i+1})=\varepsilon^{n-i-1}$. The latter option, however, means that ψ maps $\{\varepsilon,\varepsilon^i,\varepsilon^{i+1}\}$ onto $\{\varepsilon,\varepsilon^i,\varepsilon^{n-i-1}\}$. Using Lemma 4.3 one gets $\psi(\mathrm{fac}(\{\varepsilon,\varepsilon^i,\varepsilon^{i+1}\})) = \mathrm{fac}(\{\varepsilon,\varepsilon^i,\varepsilon^{n-i-1}\})$. Because i and $n-i-1$ are distant labels it would mean that a hexagon is mapped by a bijection ψ to a square, which is impossible. Hence, one has to have $\psi(\varepsilon^{i+1})=\varepsilon^{i+1}$.

III. If $i+1<\frac{n}{2}$ and $\psi(\varepsilon^i)=\varepsilon^{n-i}$ then $\psi(\varepsilon^{i+1})=\varepsilon^{n-i-1}$.

Analogously, by Step I., one has either $\psi(\varepsilon^{i+1})=\varepsilon^{n-i-1}$ or $\psi(\varepsilon^{i+1})=\varepsilon^{i+1}$. The latter option means that ψ maps $\{\varepsilon,\varepsilon^i,\varepsilon^{i+1}\}$ onto $\{\varepsilon,\varepsilon^{n-i},\varepsilon^{i+1}\}$ and Lemma 4.3 gives $\psi(\mathrm{fac}(\{\varepsilon,\varepsilon^i,\varepsilon^{i+1}\})) = \mathrm{fac}(\{\varepsilon,\varepsilon^{n-i},\varepsilon^{i+1}\})$. Because $n-i$ and $i+1$ are distant, a hexagon would be mapped to a square, giving a contradiction. Hence, $\psi(\varepsilon^{i+1})=\varepsilon^{n-i-1}$.

We can now summarize. By Step I., either $\psi(\varepsilon^1)=\varepsilon^1$ or $\psi(\varepsilon^1)=\varepsilon^{n-1}$. In the former case, Step II. allows one to show by induction that $\psi(\varepsilon^j)=\varepsilon^j$ for $j<\frac{n}{2}$. This allows one to apply (again) Step I. to $j>\frac{n}{2}$ and show that $\psi(\varepsilon^j)=\varepsilon^j$ also for these j. It together means that ψ is identical on $\{\varepsilon\}\cup\mathrm{ne}(\varepsilon)$. In the latter case $\psi(\varepsilon^1)=\varepsilon^{n-1}$, Step III. allows one to show by induction that $\psi(\varepsilon^j)=\varepsilon^{n-j}$ for $j<\frac{n}{2}$. Application of Step I. to $j>\frac{n}{2}$ similarly gives $\psi(\varepsilon^j)=\varepsilon^{n-j}$ for $j>\frac{n}{2}$. Hence, $\psi(\varepsilon^j)=\varepsilon^{n-j}$ for $j\in[n-1]$, meaning that ψ converts numeral labels within $\{\varepsilon\}\cup\mathrm{ne}(\varepsilon)$. □

The next lemma concerns the uniqueness of automorphisms on hexagons.

Lemma 4.7. If $|A_1|\dots|A_{n-2}|$ is a hexagon and $\psi\in\mathbb{G}$ a graph automorphism for which $\psi(\xi)=\xi$ for three consecutive nodes ξ of the hexagon, then ψ is identical on the hexagon.

P r o o f. Let η,ε,σ be the three consecutive nodes of the hexagon; see the right picture in Figure 2 for illustration of this situation. Because $\mathrm{fac}(\{\eta,\varepsilon,\sigma\})$ is the hexagon and ψ maps $\{\eta,\varepsilon,\sigma\}$ onto itself, Lemma 4.3 implies that ψ maps the hexagon onto itself. Let ζ be the other neighbour of η in the hexagon, besides ε. Since $\psi(\zeta)$ must be a neighbour of $\eta=\psi(\eta)$ and belonging to the hexagon, one has $\psi(\zeta)=\zeta$. The same consideration works with the other neighbour of σ. The fact that ψ is identical on five nodes of the hexagon implies that it is identical on the whole hexagon. □

The next lemma brings our uniqueness extension result.

Lemma 4.8. Let $\{\varepsilon,\eta\}$ be an edge of the permutohedral graph which has a non-central label. If $\psi\in\mathbb{G}$ is such that $\psi(\varepsilon)=\varepsilon$ and $\psi(\eta)=\eta$ then $\psi=\mathrm{id}$ is identical everywhere.

P r o o f. Assume $n\geq 3$ as otherwise the claim is trivial. Observe that the permutohedral graph can be seen as a finite metric space, in which the distance $\mathrm{dist}(\sigma,\zeta)$ of two enumerations $\sigma,\zeta\in\Upsilon(N)$ is the least possible length of a path in this graph from σ to ζ. This also allows one to define the distance of σ from a set od enumerations $\Upsilon\subseteq\Upsilon(N)$: $\mathrm{dist}(\sigma,\Upsilon) := \min\{\,\mathrm{dist}(\sigma,\zeta)\,:\,\zeta\in\Upsilon\,\}$.

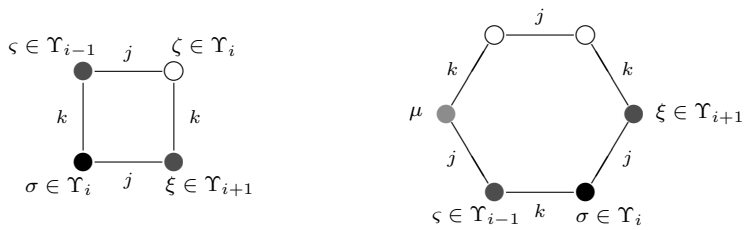


Fig. 3. The illustration of the proof of Lemma 4.8.

Put $\Upsilon_0 := \{\varepsilon, \eta\}$ and $\Upsilon_i := \{\, \sigma \in \Upsilon(N) \,:\, \mathrm{dist}(\sigma, \Upsilon_0) = i \,\}$ for $i = -1, \ldots, \binom{n}{2} - 1$. The idea is to prove, by induction on i, that $\psi = \mathrm{id}$ on $\bigcup_{\ell \leq i} \Upsilon_\ell$. We introduce auxiliary classification of nodes (in every Υ_i) for this purpose.

- We say that a node $\sigma \in \Upsilon_i$, $i \geq 0$, is *good* if there exists $\zeta \in \mathrm{ne}(\sigma) \cap (\Upsilon_0 \cup \Upsilon_{i-1})$ such that $\{\sigma, \zeta\}$ has a non-central label.

It follows from the assumption of Lemma 4.8 that both nodes $\varepsilon, \eta \in \Upsilon_0$ are good. Also, once $\sigma \in \Upsilon_i$, $i \geq 1$, has more neighbours in Υ_{i-1}, the node σ is necessarily good.

- Contrarily, we say that a node $\sigma \in \Upsilon_i$, $i \geq 1$, is *bad* if $\mathrm{ne}(\sigma) \cap \Upsilon_{i-1} = \{\varsigma\}$ is a singleton and the edge $\{\sigma, \varsigma\}$ has the central label.

To verify the induction step we assume $\psi = \mathrm{id}$ on $\bigcup_{0 \leq \ell \leq i} \Upsilon_\ell$, $i \geq 0$, and consider $\xi \in \Upsilon_{i+1}$. Note that it follows from the definition of Υ_{i+1}, that $\mathrm{ne}(\xi) \cap \Upsilon_i \neq \emptyset$. The aim is to show $\psi(\xi) = \xi$. We distinguish two cases for ξ.

- If *there is a good* $\sigma \in \mathrm{ne}(\xi) \cap \Upsilon_i$ then we apply Lemma 4.6 to σ and ψ. Recall that σ has a neighbour ζ in $\Upsilon_0 \cup \Upsilon_{i-1}$ such that $\{\sigma, \zeta\}$ has a non-central label. By the induction premise, both $\psi(\sigma) = \sigma$ and $\psi(\zeta) = \zeta$, which implies that ψ cannot convert labels on $\{\sigma\} \cup \mathrm{ne}(\sigma)$. Thus, ψ is identical on this set and $\psi(\xi) = \xi$.

- If *every* $\sigma \in \mathrm{ne}(\xi) \cap \Upsilon_i$ *is bad* then we choose and fix one such bad node $\sigma \in \Upsilon_i$. Recall that the edge $\{\sigma, \varsigma\}$ connecting it to the respective unique $\varsigma \in \Upsilon_{i-1}$ has the central label $\frac{n}{2}$. The arguments for this case are more complicated, requiring the application of Lemma 4.7; see two steps below.

This is the first basic observation in this more complicated case.

I. The label j on $\{\xi, \sigma\}$ is close to the central label $k := \frac{n}{2}$.

Indeed, because σ is bad, there exists $\varsigma \in \Upsilon_{i-1}$ such that the edge $\{\sigma, \varsigma\}$ has the central numeral label $k := \frac{n}{2}$. Assume for a contradiction that j is not close to k, which means that ξ, σ, ς are nodes of a square determined by σ and distant labels j and k;[21] see the left picture of Figure 3 for illustration of this situation. Thus, the fourth enumeration

[21] It is either the square $|\sigma(1)| \ldots |\sigma(j-1)|\sigma(j)\sigma(j+1)| \ldots |\sigma(k)\sigma(k+1)|\sigma(k+2)| \ldots |\sigma(n)|$ or the square $|\sigma(1)| \ldots |\sigma(k-1)|\sigma(k)\sigma(k+1)| \ldots |\sigma(j)\sigma(j+1)|\sigma(j+2)| \ldots |\sigma(n)|$.

$\zeta \in \Upsilon(N)$ of this square is adjacent both to $\varsigma \in \Upsilon_{i-1}$ and to $\xi \in \Upsilon_{i+1}$. This necessitates $\zeta \in \Upsilon_i$, by the definition of sets Υ_i. The edge $\{\zeta, \varsigma\}$ has the non-central label j, which implies the node ζ is good. However, $\zeta \in \Upsilon_i$ is also a neighbour of $\xi \in \Upsilon_{i+1}$, which contradicts the assumption on ξ that every neighbour of ξ in Υ_i is bad.

Thus, one can consider the hexagon determined by σ and this pair of close labels j and k, where k is central. The hexagon contains the unique node ς in $\mathrm{ne}(\sigma) \cap \Upsilon_{i-1}$; see the right picture of Figure 3 for illustration.

II. The automorphism ψ is identical on the hexagon, which yields $\psi(\xi) = \xi$.
Indeed, let $\mu \in \Upsilon(N)$ be the unique neighbour of ς such that the edge $\{\varsigma, \mu\}$ has the label j. By its definition, the node μ also belongs to the considered hexagon. Because $\varsigma \in \Upsilon_{i-1}$, one has $\mu \in \Upsilon_{i-2} \cup \Upsilon_{i-1} \cup \Upsilon_i$. Hence, by the induction premise, one has $\psi(\mu) = \mu$, $\psi(\varsigma) = \varsigma$, and $\psi(\sigma) = \sigma$. The automorphism is thus identical on three consecutive nodes of the hexagon, and the application of Lemma 4.7 implies that it is identical on the whole hexagon. Clearly, ξ also belongs to the hexagon.

Hence, in both cases for ξ we got $\psi(\xi) = \xi$ and verified the induction step. □

Finally, we can verify the non-trivial inclusion of the considered groups.

Lemma 4.9. One has $\mathbb{G} \subseteq \mathbb{H}$.

Proof. Given $\psi \in \mathbb{G}$, our aim is to show $\psi \in \mathbb{H}$. Assume $n \geq 3$ without loss of generality. Then there exists an edge $\{\varepsilon, \eta\}$ of the permutohedral graph with a non-central numeral label. Put $\sigma := \psi(\varepsilon)$. By Lemma 4.5(i), there exists $\pi \in S_N$ such that $\psi_\pi(\sigma) = \varepsilon$. The definition of $\mathbb{H}$ gives $\psi_\pi \in \mathbb{H}$. We put $\psi' := \psi_\pi \circ \psi \in \mathbb{G}$ and observe $\psi'(\varepsilon) = \varepsilon$.

According to Lemma 4.6 applied to ε and ψ'

- either ψ' is identical on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$,
- or ψ' converts labels on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$.

In both cases there exists $\phi \in \mathbb{H}$ such that $\phi(\varepsilon) = \varepsilon$ and $\psi'' := \phi \circ \psi'$ preserves labels on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$. Indeed, in the former case we simply put $\phi := \mathrm{id}$. In the latter case we apply Lemma 4.5(ii) to ε and get $\phi \in \mathbb{H}$ which converts labels and satisfies $\phi(\varepsilon) = \varepsilon$. Clearly, the composition $\psi'' := \phi \circ \psi'$ must then preserve labels on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$.

Thus, we have $\psi'' \in \mathbb{G}$, $\psi''(\varepsilon) = \varepsilon$, and, because ψ'' preserves labels on $\{\varepsilon\} \cup \mathrm{ne}(\varepsilon)$ and the edge $\{\varepsilon, \eta\}$ was chosen with a non-central label, also $\psi''(\eta) = \eta$. The next step is to apply Lemma 4.8 to ψ'' and observe that ψ'' is identical everywhere. Hence, $\mathrm{id} = \psi'' = \phi \circ \psi' = \phi \circ \psi_\pi \circ \psi$ gives $\psi = (\phi \circ \psi_\pi)_{-1} \in \mathbb{H}$, because $\mathbb{H}$ is a group, closed under composition and inversion. So, we showed $\psi \in \mathbb{H}$, which was the goal. □

4.5. Proof of the main theorem

Here we summarize our formerly proved observations and give the proof of Theorem 4.2.

P r o o f . The implication (i)⇒(ii) follows from observations made in Lemma 4.1. While (ii)⇒(iii) is evident, (iii)⇒(iv) follows from Lemma 3.1 applied to $\mathsf{P} := \Pi(N)$. Lemma 3.3 applied to $\Pi(N)$ then yields (iv)⇒(v), while Lemma 3.5 gives (v)⇒(vi). The crucial implication (vi)⇒(i) is that the subgroup $\mathbb{H}$ of graph automorphisms generated by the permutational automorphisms ψ_π, $\pi \in S_N$, and the reversing automorhism $\psi_{\mathfrak{r}}$ is already the whole group $\mathbb{G}$ of graph automorphisms. This is what is claimed in Lemma 4.9. □

Remark 4.10. The reader may ask whether one can possibly offer simpler arguments for the validity of Theorem 4.2. That was a question the author asked himself as well. Three options were considered by him and none of them was found to lead to a desired simplification of the proof. It is clear that the proof in the case of odd number n of variables is easier because, then, there are no edges with central numeral labels.

1. The first natural question was, therefore, whether the result for an even number of variables could be derived as a consequence of the result for an odd number of variables. The idea was to embed $\Upsilon(N)$ with $|N| = n$ using a graph isomorphism into $\Upsilon(M)$ with $|M| = n + 1$, show that any (transformed) graph automorphism from $\Upsilon(N)$ can be extended to a graph automorphism on $\Upsilon(M)$, apply the result for $\Upsilon(M)$, and derive the result for $\Upsilon(N)$ by 'backward' projection.

   Although such graph-isomorphic embeddings exist, it seems that not every such transformed automorphism ψ from $\Upsilon(N)$ can be extended on $\Upsilon(M)$ and, thus, this idea does not work. Specifically, if one considers $M := N \cup \{v\}$ then the embedding $|\varepsilon(1)| \dots |\varepsilon(n)| \mapsto |\varepsilon(1)| \dots |\varepsilon(n)|v|$ has $T_v := \{\, \eta \in \Upsilon(M) : \eta_{-1}(v) = n + 1 \,\}$ as its image. The point is that the number of graph automorphisms ϕ mapping T_v onto itself is $n!$,[22] while one has $2 \cdot n!$ automorphisms on $\Upsilon(N)$.

2. A different idea was to consider, in case of even n, a subgraph obtained from the permutohedral graph by the removal of all edges with central labels and prove crucial Lemma 4.8 by induction on the distance in this subgraph. This idea also fails because the considered subgraph is not connected, for any even n.

   Indeed, if $A \subseteq N$, $|A| = \frac{n}{2}$, then every edge from the set of enumerations described by $|A|N \setminus A|$ has the central label.

3. The reader may also wonder whether, for any even $n \geq 4$, the more complicated case in the proof of Lemma 4.8 necessarily occurs.

   It is indeed the case: given a bad $\sigma \in \Upsilon_1$ and consider $\xi \in \Upsilon_2$ such that the edge $\{\xi, \sigma\}$ has a non-central label which is close to the central label.

[22] Theorem 4.2 implies that these are permutational automorphisms ψ_π for $\pi \in S_M$ with $\pi(v) = v$.

5. CONCLUDING REMARKS

As indicated already in the Introduction one can distinguish four different equivalences for (non-empty) polytopes $\mathsf{P}, \mathsf{Q} \subseteq \mathbb{R}^N$. More specifically,

- the *isometry* equivalence means that there is an (Euclidean) isometry mapping the polytope P onto the polytope Q,
- the *affine* equivalence means that there is a bijective affine mapping transforming the polytope P onto the polytope Q,
- the *combinatorial* equivalence [3, § 10] means that the face lattices $\mathfrak{F}(\mathsf{P})$ and $\mathfrak{F}(\mathsf{Q})$ of the considered polytopes are order isomorphic, and
- the *graphical* equivalence is that their graphs $G(\mathsf{P})$ and $G(\mathsf{Q})$ are isomorphic.

The presented results indicate what is the relation of those equivalences. The isometric equivalence is the strongest one and the second strongest equivalence is the affine one. Then combinatorial equivalence follows, and the weakest equivalence is the graphical one. The reader can certainly find examples that the inclusions are strict in general. The hierarchy of polytopal equivalences then induces the hierarchy of the respective automorphism groups for any (non-empty) polytope $\mathsf{P} \subseteq \mathbb{R}^N$ (see Section 3).

In this paper, we have characterized all four conceivable groups of automorphisms of the permutohedron $\Pi(N)$ and proved that these are identical from the viewpoint of group theory (Theorem 4.2). Thus, there is only one symmetry group of the permutohedron!

Let us conclude with remarks on the permutohedron $\Pi(N) \subseteq \mathbb{R}^N$ and its graph, we call the *permutohedral graph* (see Section 4.2). It was already mentioned that $\Pi(N)$ is a simple polytope of the dimension $n-1$. Another interesting fact is that $\Pi(N)$ is a zonotope, that is, a polytope obtained by projecting a hypercube [12, Example 7.15].

As concerns the permutohedral graph, the basic fact about it is that it is *bipartite*. Nonetheless, this graph of our interest exhibits other remarkable properties the reader may appreciate. It follows directly from its definition that it is a *regular graph*, because every its node has $n-1$ adjacent nodes. It also follows from the properties of graphs defined by polytopes [3, § 15, Theorem 15.6] that it is $(n-1)$-*connected*, which means that it is connected and remains connected even after the removal of at most $n-2$ nodes. Additionally, it is a *Hamiltonian graph*, meaning that there exists a cycle in this graph which visits every node exactly once. This is related to a celebrated enumeration algorithm for generating all permutations of an n-element set (named after Steinhaus, Johnson, and Trotter).

The reader may be interested in the fact that, besides the numeral labels, we utilized here, two other types of labels on the edges of the permutohedral graph can be utilized, namely the combinatorial and independence labels; for details see [11, § 2.6]. Squares and hexagons can be characterized in terms of these alternative labels too.

Because any numeral label $i \in [n-1]$ can be associated with a transposition $i \leftrightarrow (i+1)$ in $[n]$ and these transpositions form a distinguished generator of the symmetric group $S_{[n]}$, the numeral labels can be interpreted as edge-colors on the permutohedral graph, when it is interpreted as the *Cayley graph* of the group $S_{[n]}$.

Another note is that the numeral equivalence of edges (= the shared numeral label) is a special case of certain equivalence of faces of the permutohedron $\Pi(N)$ (see Section 2.3.5): faces described by $|A_1|\dots|A_m|$ and $|B_1|\dots|B_\ell|$ have the same *ordinal type* if $m=\ell$ and $|A_i|=|B_i|$ for $i\in[m]$.

A. ON GRAPHICAL DESCRIPTION OF FACES OF A SIMPLE POLYTOPE

The aim of this appendix is two-fold: both to reformulate the results from [5] in the way used in context of our paper and to provide the reader with detailed arguments for their validity, fully based on well founded facts from a textbook [3]. This is because the original note [5] was too sketchy (skipping details) and directed to narrow audience of profound experts in polytopes, and, thus, not accessible to a general reader.

Throughout this supplement we assume that $\mathsf{P}\subseteq\mathbb{R}^N$ is a non-empty *simple polytope* of the dimension d, $0\leq d\leq n$. Moreover, shortened notation G will be used to denote the graph $G(\mathsf{P})$ of P, defined in Section 2.3.3, throughout the appendix. The set of its nodes, that is, the set of vertices of P, will briefly be denoted by $\mathsf{V}:=\mathrm{ext}(\mathsf{P})$.

Recall from Section 2.3.2 that a set $F\subseteq\mathsf{V}$ of vertices *represents a face* (of P) iff there exists a face $\mathsf{F}\subseteq\mathsf{P}$ of P such that $F=\mathrm{ext}(\mathsf{F})$ and that by the *facial closure* of a set $S\subseteq\mathsf{V}$, denoted by $\mathrm{fac}(S)$, is meant the set of vertices of the face of P generated by S.

Given a total order o of V, formalized by a bijection $o:\mathsf{V}\to\{1,\dots,|\mathsf{V}|\}$, the *directed version* of G induced by o will be denoted by G^o; that is, $y\to x$ in G^o iff $\{x,y\}$ is an edge of G and $o(y)<o(x)$. The induced subgraph of G^o for a non-empty $S\subseteq\mathsf{V}$ will be denoted by G^o_S. The set of *previous neighbours* of $x\in\mathsf{V}$ in G relative to o will be denoted as follows:[23]

$$\mathrm{ne}(x|G,o)\ :=\ \{\,y\in\mathsf{V}:\ o(y)<o(x)\ \&\ \{x,y\}\text{ is an edge in }G\,\}\,.$$

This allows one to define a particular function of orders, namely

$$f(o)\ :=\ \sum_{x\in\mathsf{V}}2^{|\mathrm{ne}(x|G,o)|},\qquad\text{where } o \text{ is a total order of } \mathsf{V},\,^{24} \tag{4}$$

and say that a total order o of V is *optimal* if it minimizes this function over all orders. Clearly, this is a fully graphical concept, depending solely on the graph G.

The point is that the optimality can be characterized in terms of faces of P.

Lemma A.1. The minimal value of $f(o)$ is f, the number of non-empty faces of P. An order o is optimal iff, for every $\emptyset\neq F\subseteq\mathsf{V}$ representing a face of P, there is a unique sink node in G^o_F, that is, unique $x\in F$ such that no $z\in F$ with $x\to z$ in G^o exists.

P r o o f . In the first four steps of the proof we fix a total order o of V.[25] Observe that the value $f(o)$ is just the number of such pairs (x,S), where $x\in\mathsf{V}$ and $S\subseteq\mathrm{ne}(x|G,o)$.

[23] It is nothing but the set of $y\in\mathsf{V}$ with $y\to x$ in G^o, that is, the set of *parents* of x in G^o.

[24] Note that $f(o)$ is an alternative re-writing of the expression f^O from [5] or [12, § 3.4].

[25] The assumption of simplicity of P is not needed in the first two steps.

Define a mapping from non-empty faces of P into the set of these pairs: $\mathsf{F} \in \mathfrak{F}(\mathsf{P}) \setminus \{\emptyset\}$ is assigned the last vertex x of $\mathrm{ext}(\mathsf{F})$ in the order o and the set $S := \mathrm{ne}_{\mathsf{F}}(x)$.

I. Given $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$, $F := \mathrm{ext}(\mathsf{F})$, if $x \in F$ is a sink node in G^o_F and $S = \mathrm{ne}_{\mathsf{F}}(x)$ then F is the least face of P containing $\{x\} \cup S$, that is, the face generated by $\{x\} \cup S$.

Indeed, let $\mathsf{F}' \in \mathfrak{F}(\mathsf{P})$ denote the least face of P containing $\{x\} \cup S$. Since $\{x\} \cup S \subseteq \mathsf{F}$, one has $\mathsf{F}' \subseteq \mathsf{F}$ and, by applying [3, Theorem 5.2], $S \subseteq \mathrm{ne}_{\mathsf{F}'}(x) \subseteq \mathrm{ne}_{\mathsf{F}}(x) = S$. Hence, $x \in \mathrm{ext}(\mathsf{F}')$ and $S = \mathrm{ne}_{\mathsf{F}'}(x)$. Then [3, Corollary 11.7] says $\dim(\mathsf{F}) = \dim(\{x\} \cup \mathrm{ne}_{\mathsf{F}}(x))$, and, applied both to F and F', yields $\dim(\mathsf{F}) = \dim(\{x\} \cup S) = \dim(\mathsf{F}')$. Since these faces are in inclusion, it implies $\mathsf{F} = \mathsf{F}'$ [3, Corollary 5.5].

II. One has $f(o) \geq f$.

It follows from Step I. that the considered mapping is injective, because, for any $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$, the last vertex x of $\mathrm{ext}(\mathsf{F})$ in o is a sink in G^o_F. Hence, $f(o) \geq f$.

III. A pair (x, S) has a pre-image in the considered mapping iff x is the last node of $\mathrm{ext}(\mathsf{F})$ in o, where F is the face (of P) generated by $\{x\} \cup S$.[26]

The necessity follows easily from Step I. For the sufficiency we take the face F generated by $\{x\} \cup S$ and apply [3, Theorem 12.17] to it to observe $S = \mathrm{ne}_{\mathsf{F}}(x)$ (the simplicity assumption on P is used here!). This implies that F is assigned the pair (x, S).

IV. One has $f(o) = f$ iff, for any $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$, there is at most one sink node in G^o_F.

For the necessity, assume for a contradiction that some face F has at least two sink nodes. Take such a sink node $x \in \mathrm{ext}(\mathsf{F})$ which is <u>not</u> the last one in o and put $S := \mathrm{ne}_{\mathsf{F}}(x)$. By Step I., F is the face generated by $\{x\} \cup S$. As x is not the last node of $\mathrm{ext}(\mathsf{F})$ in o, (x, S) has no pre-image, by Step III. Hence, the upper bound for f established in Step II. cannot be attained, which contradicts the assumption.

For the sufficiency it is enough to show that any pair (x, S) has a pre-image. Take the face F of P generated by $\{x\} \cup S$ and observe, by [3, Theorem 12.17], that $S = \mathrm{ne}_{\mathsf{F}}(x)$. As $S \subseteq \mathrm{ne}(x|G, o)$, the node x is a sink node in G^o_F and the assumption implies that x has to be the last node of F in o. Hence, F is assigned the pair (x, S).

It remains to show that the minimal possible value of $f(o)$ is indeed f.

V. There exists a total order o of V such that $f(o) = f$.

Indeed, by [3, Theorem 15.1] there exists an *admissible* vector $w \in \mathbb{R}^N$ for P, which means $\langle x, w\rangle \neq \langle y, w\rangle$ whenever $x, y \in \mathsf{V}$ are distinct. Fix (arbitrary) such w and define a total order o of V by $o(y) < o(x) \ := \ \langle y, w\rangle < \langle x, w\rangle$. Step III. can be used to show that any pair (x, S) has a pre-image. Specifically, if F is the face generated by $\{x\} \cup S$, then [3, Theorem 12.17] claims $\mathrm{ne}_{\mathsf{F}}(x) = S$. As $S \subseteq \mathrm{ne}(x|G, o)$, by the construction of o on basis of w, one has $\langle y, w\rangle < \beta$ for $y \in S$, where $\beta := \langle x, w\rangle$. Further, [3, Theorem 11.8] implies $\langle z, w\rangle < \beta$ for any $z \in \mathrm{ext}(\mathsf{F}) \setminus \{x\}$, implying $\mathrm{ext}(\mathsf{F}) \subseteq \{\, y \in \mathsf{V} \,:\, o(y) \leq o(x) \,\}$. Therefore, x is the last vertex of $\mathrm{ext}(\mathsf{F})$ in o. □

This is our re-formulation of the main result from [5], which characterizes faces of P solely in graphical terms.

[26] Since now, one does need the assumption of simplicity of P; the polytope from Example 3.6 can be used to illustrate that the claims are not true for a general polytope P.

Lemma A.2. A non-empty set $F \subseteq \mathsf{V}$ represents a face of P iff two conditions hold:

(i) the induced subgraph G_F is a connected regular graph,

(ii) there is an optimal order o of V such that $F = \{\, y \in \mathsf{V} : \; o(y) \leq |F| \,\}$.

Proof. For the necessity of (i) note that $G_F = G(\mathsf{F})$ for the represented face F of P, using [3, Thm 5.2]. Additionally, [3, Thm 12.15] says that F is a simple polytope, $G(\mathsf{F})$ is connected by [3, Thm 15.6], and regular by [3, Thm 12.12]. The necessity of (ii) follows from [3, Thm 15.3] saying that there exists an *admissible* vector $w \in \mathbb{R}^N$ for P and $\beta \in \mathbb{R}$ such that $\langle x, w\rangle \leq \beta$ for $x \in F$ and $\langle z, w\rangle > \beta$ for $z \in \mathsf{V} \setminus F$. The respective total order o of V is optimal, by the arguments from Step V. in the proof of Lemma A.1.

For the sufficiency fix an optimal order o of V from (ii), take the last node $x \in F$ in o, and define S as the set of $y \in \mathsf{V}$ such that $y \to x$ in G^o_F. By (i), every node of G_F has exactly k neighbours in G_F, where $k := |S|$.

Introduce $\mathsf{F} \in \mathfrak{F}(\mathsf{P})$ as the face generated by $\{x\} \cup S$. By [3, Thm 12.15], F is a simple polytope, by [3, Thm 12.17], $\dim(\mathsf{F}) = k$ and $\mathrm{ne}_{\mathsf{F}}(x) = S$. Hence, by [3, Thm 12.12], any node of $G(\mathsf{F})$ has exactly k neighbours in $G(\mathsf{F})$.

Let $F' := \mathrm{ext}(\mathsf{F})$ denote the set of vertices of $G(\mathsf{F})$. Using [3, Thm 5.2] realize that $G(\mathsf{F}) = G_{F'}$. As $\mathrm{ne}_{\mathsf{F}}(x) = S$, x is a sink node of $G^o_{F'}$. Since o is optimal, by Lemma A.1, there is no other sink node in $G^o_{F'}$ besides x. Therefore, x is the last node of F' in o, which implies that $F' \subseteq \{\, y \in \mathsf{V} : \; o(y) \leq o(x) \,\} = F$. Then $F' \subseteq F$ and $G(\mathsf{F}) = G_{F'}$ imply that, for every $y \in F'$, any of its k neighbours in $G(\mathsf{F})$ is also adjacent to y in G_F.

Assume for a contraction that $z \in F \setminus F'$ exists. By (i), there exists a path in G_F from z to $x \in F'$. Let y be the first node on this path which belongs to F'. Then y has k neighbours in $G(\mathsf{F})$, being also neighbours in G_F, and at least one additional neighbour $y' \in F \setminus F'$ in the graph G_F. So, it has at least $k+1$ neighbours in G_F, contradicting the assumption above. Hence, $F = F'$ and F represents a face of P. □

Recall that a graph automorphism of G is a bijection $\Psi : \mathsf{V} \to \mathsf{V}$ preserving edges: $\{x, y\}$ is an edge in G iff $\{\Psi(x), \Psi(y)\}$ is an edge in G, for any $x, y \in \mathsf{V}$, $x \neq y$.

Corollary A.3. Let $\Psi : \mathsf{V} \to \mathsf{V}$ be a graph automorphism of G. If $F \subseteq \mathsf{V}$ represents a face of P then so does $\Psi(F)$. Moreover, for any $S \subseteq \mathsf{V}$, one has $\Psi(\mathrm{fac}(S)) = \mathrm{fac}(\Psi(S))$.

Proof. Use Lemma A.2: the condition (i) is clearly transformed by Ψ. Additionally, if $o : \mathsf{V} \to \{\, 1, \ldots, |\mathsf{V}| \,\}$ is an optimal order with $F = \{\, y \in \mathsf{V} : \; o(y) \leq |F| \,\}$ then $\hat{o} := o \circ \Psi_{-1}$ is a total order with $\Psi(F) = \{\, z \in \mathsf{V} : \; \hat{o}(z) \leq |F| = |\Psi(F)| \,\}$. Since $\Psi(\mathrm{ne}(x|G, o)) = \mathrm{ne}(\Psi(x)|G, \hat{o})$ for any $x \in \mathsf{V}$, (4) gives $f(o) = f(\hat{o})$, and $\hat{o}$ is also optimal. Thus, if (ii) holds for F then it holds for $\Psi(F)$, and $\Psi(F)$ represents a face.

Let us put $R := \Psi(S)$, which means $S = \Psi_{-1}(R)$. Because $\mathrm{fac}(S)$ represents a face, so does $\Psi(\mathrm{fac}(S))$, and $S \subseteq \mathrm{fac}(S) \;\Rightarrow\; R \subseteq \Psi(\mathrm{fac}(S)) \;\Rightarrow\; \mathrm{fac}(R) \subseteq \Psi(\mathrm{fac}(S))$. Nonetheless, Ψ_{-1} is also a graph automorphism. Hence, $\Psi_{-1}(\mathrm{fac}(R))$ represents a face, which gives $R \subseteq \mathrm{fac}(R) \;\Rightarrow\; S \subseteq \Psi_{-1}(\mathrm{fac}(R)) \;\Rightarrow\; \mathrm{fac}(S) \subseteq \Psi_{-1}(\mathrm{fac}(R))$. Finally, $\Psi(\mathrm{fac}(S)) \subseteq \Psi(\Psi_{-1}(\mathrm{fac}(R))) = \mathrm{fac}(R)$. □

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