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Daniel Dlab

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Rotationally Multiplicative Integers

Daniel Dlab, Sardinia, Italy

1 Introduction

On the eve of Christmas 1949, the polymath Jacob Bronowski posed an intriguing numerical puzzle: find the smallest positive integer such that moving its leading digit to the end yields a new number exactly one and a half times the original [1]:

$$\tfrac{3}{2} \times \mathbf{1}\,176\,470\,588\,235\,294 = 1\,764\,705\,882\,352\,94\mathbf{1}.$$

In the February 1955 issue of *The Mathematical Gazette*, D. E. Littlewood provided an elegant derivation of the solution [2]. In that same issue, E. J. F. Primrose took the problem further [3]. Rather than fixing the ratio at $\frac{3}{2}$, Primrose asked which pairs (p, q) admit a solution at all—that is, for which rational numbers $\frac{p}{q}$ does there exist an integer whose digit rotation produces a number exactly $\frac{p}{q}$ times the original, observing that many pairs render the puzzle insoluble. That question of realizability is not the one we settle here; we mention it because it shows how quickly the puzzle opens out once the ratio is allowed to vary.

A few years later, in *The Gentle Art of Mathematics*, Daniel Pedoe revisited Bronowski's puzzle, offering a concrete solution and systematic method [4]. Pedoe showed that the problem can be understood by analyzing how cyclic digit shifts correspond to multiplication by fractions whose decimal expansions are repeating. Yet Pedoe himself was dismissive of the puzzle's mathematical standing, remarking that "the problem itself is one which most mathematicians would call *trivial*, because it leads nowhere, and does not fit into a general theory" [4, p. 16]. It is precisely this qualification that we wish to gently contest: as the sections below demonstrate, the puzzle is in fact an instance of a general theory of digit rotations driven by repeating decimal expansions, one that admits a clean algebraic formulation and extends naturally to composite divisors and arbitrary offsets.

Variants of such rotational multiplication puzzles appear in earlier recreational works, notably Henry Ernest Dudeney's *Amusements in*

Mathematics (1917) [5], suggesting that the underlying idea predates Bronowski's formulation.

Although often introduced as recreational puzzles, these problems have a natural algebraic structure: rotating the digits of a repeating block is exactly what happens when the *Euclidean division* (that is, integer division with remainder) producing that block is restarted one step further along, so the whole question is governed by the cycle of remainders that the division runs through. In this paper, we examine the problem in greater depth and present several generalizations along with notable findings.

We believe the material is elementary enough—requiring nothing beyond division and careful bookkeeping of its remainders—to be a worthwhile addition to mathematics at primary and secondary school level.

2 From Remainders to Rotations

We begin by defining the function $\rho(x)$, which reorders the digits of a positive integer x by moving its leftmost digit to the rightmost position, effectively performing a left rotation (or cyclic shift) of the number's digits. Bronowski's puzzle is therefore asking for the smallest x such that $\rho(x) = \frac{3}{2}x$, or equivalently, $2\rho(x) = 3x$.

There is nothing special about the value $\frac{3}{2}$. Let us give a few examples of numbers x for which $\rho(x) = \frac{1}{k} \times x$ for $k = 2, 3, 4, 5, 6, 7, 8$ and 9.

Dividing the number 10^{19} by 19 gives

$$\frac{10^{19}}{19} = 526\,315\,789\,473\,684\,210.\overline{526\,315\,789\,473\,684\,210}$$

and the digits after the decimal point never settle: they repeat forever in blocks of eighteen. Such a repeating block is called the *repetend* (sometimes also the *period*) of the division, and it is the repetend itself that interests us here. Taking it as our number

$$x = 526\,315\,789\,473\,684\,210,$$

we can readily check that $\rho(x) = \frac{1}{2} \times x$. Every example below is obtained the same way, as the repetend of the division of a suitable dividend by a specified divisor; the process is examined in detail in the rest of this section.

The dividend decides only where we enter the repetition. Dividing 10^{18} by 19 instead gives

$$\frac{10^{18}}{19} = 52\,631\,578\,947\,368\,421.\overline{052\,631\,578\,947\,368\,421},$$

the very same eighteen digits in the very same order, but read from one step earlier. Notice that this block opens with a zero. A repetend may perfectly well begin with 0, and it is worth being alert to this from the outset; Remark 3 returns to the difference it makes when we wish to read such a block as an integer.

Similarly, dividing 24×10^{28} by 29, we obtain the repetend

$$x = 8\,275\,862\,068\,965\,517\,241\,379\,310\,344,$$

satisfying $\rho(x) = \frac{1}{3} \times x$.

Dividing 12×10^6 by 13, 16×10^6 by 39, or 4×10^6 by 39, we obtain the repetends $x = 923\,076$, $x = 410\,256$ and $x = 102\,564$ respectively, all three satisfying $\rho(x) = \frac{1}{4} \times x$. The last of the three shows the same thing from the other side: its rotation $\rho(102\,564) = 025\,641$ is a block beginning with a zero. The repetend $x = 714\,285$, obtained from the division $\frac{5\times10^6}{7}$, satisfies $\rho(x) = \frac{1}{5} \times x$.

Dividing 54×10^{58} by 59, we obtain the repetend

$$x = 9\,152\,542\,372\,881\,355\,932\,203\,389\,830\,508\,474 \\ 576\,271\,186\,440\,677\,966\,101\,694,$$

satisfying $\rho(x) = \frac{1}{6} \times x$. The two repetends

$$x = 9\,130\,434\,782\,608\,695\,652\,173 \quad \text{and} \quad x = 7\,101\,449\,275\,362\,318\,840\,579,$$

resulting from the division of 21×10^{22} by 23 and of 49×10^{22} by 69 respectively, both satisfy $\rho(x) = \frac{1}{7} \times x$.

Finally, dividing 64×10^{13} by 79 results in the repetend

$$x = 8\,101\,265\,822\,784,$$

for which $\rho(x) = \frac{1}{8} \times x$, while dividing 81×10^{44} by 89 yields the repetend

$$x = 91\,011\,235\,955\,056\,179\,775\,280\,898\,876\,404\,494\,382\,022\,471,$$

satisfying $\rho(x) = \frac{1}{9} \times x$.

Let us point out that the examples above use five of the nine primes p smaller than 100 whose repetends have length $p-1$. The remaining four are $p=17$, $p=47$, $p=61$ and $p=97$.

Our focus will be on characterizing those integers x that exhibit a scaled rotational symmetry; that is, integers for which a digit rotation satisfies

$$B \times \rho(x) = A \times x,$$

where A and B are positive integers with $A \neq B$.

To understand the structure underlying such numbers, we begin by examining the division of natural numbers, paying particular attention to the appearance of repeating decimal blocks. As a first example, consider

$$\frac{1\,000\,000}{7} = 142\,857.\overline{142\,857}.$$

Again the quotient repeats, here with a repetend of six digits. Writing out the division explicitly

$$\begin{array}{r} 1000000 : 7 = 142\,857 \\ -7 \\ \hline \mathbf{3}0 \\ -28 \\ \hline \mathbf{2}0 \\ -14 \\ \hline \mathbf{6}0 \\ -56 \\ \hline \mathbf{4}0 \\ -35 \\ \hline \mathbf{5}0 \\ -49 \\ \hline \mathbf{1} \end{array}$$

We eventually obtain a remainder of 1, so the division process must repeat, producing the six-digit repetend 142 857. The intermediate remainders

$$3,\ 2,\ 6,\ 4,\ 5,\ 1$$

are not incidental; they encode the rotational structure. Writing each step of the division, we have

$$\begin{aligned} 10 &= 1 \times 7 + 3 & 3 \times 10 &= 4 \times 7 + 2, \\ 2 \times 10 &= 2 \times 7 + 6 & 6 \times 10 &= 8 \times 7 + 4, \\ 4 \times 10 &= 5 \times 7 + 5 & 5 \times 10 &= 7 \times 7 + 1. \end{aligned} \tag{1}$$

We can also take each remainder as being the start of a new number dividend, so we can generate a series of interesting equations

$$\begin{aligned} 3 \times 1\,000\,000 &= 428\,571 \times 7 + 3, \\ 2 \times 1\,000\,000 &= 285\,714 \times 7 + 2, \\ 6 \times 1\,000\,000 &= 857\,142 \times 7 + 6, \\ 4 \times 1\,000\,000 &= 571\,428 \times 7 + 4, \\ 5 \times 1\,000\,000 &= 714\,285 \times 7 + 5, \\ 1 \times 1\,000\,000 &= 142\,857 \times 7 + 1, \end{aligned}$$

hence, subtracting each remainder from both sides and dividing by 7,

$$\frac{428\,571}{3} = \frac{285\,714}{2} = \frac{857\,142}{6} = \frac{571\,428}{4} = \frac{714\,285}{5} = \frac{142\,857}{1}.$$

This leads us to several equations of the desired format

$$\begin{aligned} 3 \times 285\,714 &= 2 \times 428\,571, \\ 2 \times 857\,142 &= 6 \times 285\,714, \\ 6 \times 571\,428 &= 4 \times 857\,142, \\ 4 \times 714\,285 &= 5 \times 571\,428, \\ 5 \times 142\,857 &= 1 \times 714\,285. \end{aligned}$$

We can now clearly see that the factors A and B are determined by consecutive remainders. An integer read off from one full period of a repeating block in this way—142 857, 428 571, 285 714 and the other three above—will be called a *repetend integer*.

3 Offsets and t-Rotations

As a further step, we can look at jumps of two or more, where each factor is determined not by immediately adjacent remainders but by those separated by a fixed *offset*.

We define the function $\rho_t(x)$ on an integer x as the operation that rotates its digits to the left by t positions. A single such step is the rotation we started with, so ρ_1 is the function ρ of Section 2, and we use the two names interchangeably from here on.

So by taking the remainders 3 and 6, for example, with a remainder offset of 2, we isolate these two ratios from above

$$\frac{428\,571}{3} = \frac{857\,142}{6},$$

to get

$$3 \times 857\,142 = 6 \times 428\,571 \quad \Rightarrow \quad 1 \times \rho_2(428\,571) = 2 \times 428\,571.$$

We can describe all possible offset-ratio combinations in Table 1 below. To use it, locate the column corresponding to your chosen starting remainder, then look down to the row for your desired offset. The cell gives the ratio $\frac{A}{B}$ such that $B \times \rho_t(x) = A \times x$, where x is the repetend integer associated with that remainder. The topmost row records, for each remainder r, the leading digit $\lfloor \frac{10r}{7} \rfloor$ of its repeating block—read left to right, these digits spell out the repetend 142 857 itself. For example, starting at remainder 3 with offset 2, we read off ratio 2, confirming the equation

$$1 \times \rho_2(428\,571) = 2 \times 428\,571$$

derived above.

Repetend Digit	1	4	2	8	5	7
Intermediate Remainder	1	3	2	6	4	5
Offset 1	**3**	2/3	**3**	2/3	5/4	**1/5**
Offset 2	**2**	**2**	**2**	5/6	**1/4**	3/5
Offset 3	**6**	4/3	5/2	**1/6**	3/4	2/5
Offset 4	**4**	5/3	**1/2**	**1/2**	**1/2**	6/5
Offset 5	**5**	**1/3**	3/2	**1/3**	3/2	4/5

Table 1: Offset-Ratio Table for Divisor 7

Entries with bold figures represent the special cases where either $A = 1$ or $B = 1$.

Remark 1. Note that $p - 1 = 6$ is exactly the number of nonzero remainders that division by 7 can leave, and that these six remainders—$1, 3, 2, 6, 4, 5$ in cycle order—are precisely the "Intermediate Remainder" row of Table 1, with none left over. That is, the single cycle traced out by repeatedly multiplying by 10 and keeping the remainder on division by 7 is as long as it could possibly be: it visits every nonzero remainder, leaving none outside it. Consequently any two remainders in the cycle can serve as the starting pair—each such pair already appears somewhere in the table, at some offset.

It is worth noting what happens when we begin with a different starting dividend $R \times 10^n$. Provided the remainder left by dividing R by 7 is nonzero, this simply enters the same remainder cycle seen in (1) at a different point; the cyclic structure is a robust property of the divisor itself. (For $d = 7$ specifically, this excludes $R = 7$, which instead gives an initial remainder of 0 and hence a terminating, rather than repeating, division.)

4 Algebraic Formulation

Now that we understand the basics we can describe this formally. Before giving a definition, we agree to write an integer

$$x = \sum_{i=0}^{k} c_i 10^i$$

simply as a sequence of $k + 1$ digits $x = c_k c_{k-1} \dots c_1 c_0$.

Here, all numbers c_i, $0 \leq i \leq k$ are digits, i.e., natural numbers $0 \leq c_i \leq 9$, with the requirement that $c_k \neq 0$ (ensuring the representation has exactly $k + 1$ digits).

This notation will make what follows much easier to read.

Definition 1. Let r and d be natural numbers with $0 < r < d$ and $\gcd(r, d) = 1$, where d is a multiple of neither 2 nor 5. A natural number $x = c_k c_{k-1} \dots c_1 c_0$ with $c_k \neq 0$ is called the *minimal repetend* of the fraction $\frac{r}{d}$ if:

1. The equation

$$r \times 10^{k+1} = d \times x + r \tag{2}$$

holds, and

2. The length $k+1$ is minimal, i.e., there is no smaller positive integer $m < k+1$ and no natural number x' with m digits such that $r \times 10^m = d \times x' + r$.

Remark 2. We use the term *rotation* throughout, though this can equivalently be described as moving the first digit to the end of the number.

Remark 3. Definition 1 is what makes the repetend integers of Section 2 precise: the requirement $c_k \neq 0$ is what allows a repeating block to be read as an integer of exactly $k+1$ digits, and the minimality condition pins it to a single period. The distinction is worth drawing, because the repeating block of $\frac{r}{d}$ can itself begin with 0—for instance $\frac{1}{13} = 0.\overline{076\,923}$. Such a block is a valid repeating *digit string*, but it is not a minimal repetend. Every cycle of remainders nonetheless contains at least one remainder whose block has a nonzero leading digit—for the $\frac{1}{13}$ cycle, $\frac{10}{13} = 0.\overline{769\,230}$—and it is such a remainder that supplies the repetend integer we multiply.

This fixes our notation for the rest of the paper. An integer is never written with a leading zero: the digit string 042 857 is not a notation for the number 42 857. A repeating block, by contrast, is a digit string of the fixed length $k+1$, and it may perfectly well begin with 0. Rotation acts on such a block as a string of $k+1$ digits, and reading the result as a number discards any leading zeros—so a rotation of a repetend integer need not itself be one. Wherever we multiply, then, the multiplicand is a repetend integer in the sense of Definition 1, its leading digit c_k nonzero, while the rotated result is allowed to carry leading zeros.

Now let's write out the division steps in (1) using similar notation. For convenience we number our 6 steps from 5 to 0 as our repetend digits come out left to right.

In general at each step i we have

$$r_i \times 10 = c_i \times d + r_{i-1}. \tag{3}$$

Here the indices of r (and, throughout, of c) are understood cyclically: they run $k, k-1, \ldots, 1, 0$ and then begin again at k. So when $i = 0$ the symbol r_{i-1} is to be read as r_k, the step wrapping back around to where it started.

Now let's formalize the results from our offset-ratio combinations table.

Theorem 4. *Using the notation in* (3) *for a repetend* $x = c_k c_{k-1} \dots c_1 c_0$ *as defined in* (2)*, then for any* $0 \le i \le k$*,*

$$c_{i-1}c_{i-2}\dots c_1c_0c_k\dots c_{i+1}c_i = \left(\frac{r_{i-1}}{r_i}\right)(c_ic_{i-1}\dots c_1c_0c_k\dots c_{i+1}). \quad (4)$$

Proof. By (2) we have $r_i \times 10^{k+1} = d \times (c_ic_{i-1}\dots c_1c_0c_k\dots c_{i+1}) + r_i$ and $r_{i-1} \times 10^{k+1} = d \times (c_{i-1}c_{i-2}\dots c_1c_0c_k\dots c_{i+1}c_i) + r_{i-1}$.

Hence

$$c_ic_{i-1}\dots c_1c_0c_k\dots c_{i+1} = \frac{r_i(10^{k+1}-1)}{d}$$

and

$$c_{i-1}c_{i-2}\dots c_1c_0c_k\dots c_{i+1}c_i = \frac{r_{i-1}(10^{k+1}-1)}{d}.$$

Therefore

$$c_{i-1}c_{i-2}\dots c_1c_0c_k\dots c_{i+1}c_i = \left(\frac{r_{i-1}}{r_i}\right)(c_ic_{i-1}\dots c_1c_0c_k\dots c_{i+1}),$$

as required. □

Theorem 4 can be extended by generalizing the offset-ratio combinations we discussed above, applying (4) repeatedly to obtain the following.

Extension 5. *For any* $1 \le t \le k$*,*

$$c_{i-t}c_{i-t-1}\dots c_1c_0c_k\dots c_{i-t+1} = \left(\frac{r_{i-t}}{r_i}\right)(c_ic_{i-1}\dots c_1c_0c_k\dots c_{i+1}),$$

where the indices of c *and* r *are again understood cyclically, as in* (3)*.*

Taking $t = 1$ recovers (4) exactly. We are now in a position to say something about the question Primrose raised [3], mentioned in the Introduction: which ratios can be obtained at all?

Corollary 6. *Suppose there exist arbitrarily large primes* p *whose repetends have length* $p - 1$*. Then*

(i) *every ratio* $\frac{u}{v}$ *of two natural numbers* u *and* v *with* $u \ne v$ *and* $\frac{u}{v} < 10$ *can be obtained by the procedure described in* Extension 5*, the multiplicand being a repetend integer as in* Definition 1*; and*

(ii) *if the multiplicand is allowed to be a repeating block with leading zeros, then every ratio* $\frac{u}{v}$ *with* $u \ne v$ *whatsoever can be obtained.*

The bound $\frac{u}{v} < 10$ in part (i) cannot be dropped, and the convention of Remark 3 is exactly what makes it true. A repetend integer x of $k+1$ digits satisfies $x \geq 10^k$, while any rotation of it is at most $10^{k+1} - 1$, so the ratio of a rotation to x itself is always smaller than 10. If the leading digit is permitted to be 0, the argument fails at its very first step, and part (ii) is what results instead.

This is also why the tables that follow carry entries such as 16 and 19. An entry is $\frac{r_{i-t}}{r_i}$, and it reaches 10 or more only if $r_{i-t} \geq 10\, r_i$; since $r_{i-t} \leq d - 1$, this forces $10\, r_i < d$, which says precisely that the leading digit $\lfloor 10 r_i / d \rfloor$ of the block at position i is 0. Conversely, when that leading digit is 0 we have $r_{i-1} = 10\, r_i$ exactly, so the offset-1 entry of that column is 10. Such a remainder is starred in the tables that follow: its column is the one whose block begins with 0, so by Remark 3 the identities read off there hold between digit strings of fixed length rather than between integers, and it is exactly there that an entry of 10 or more can appear.

The supposition, on the other hand, is not something we can discharge. Whether there are arbitrarily large primes whose repetends attain the full length $p - 1$ is a long-standing open question, a special case of a conjecture of Artin; see [6] for a survey.

5 Composite Divisors

Neither the Definition of a minimal repetend nor the Theorem in Section 4 assumed that d was prime; both require only that $\gcd(r, d) = 1$ and that d be a multiple of neither 2 nor 5. The example built around $d = 7$ was simple in a particular way: all six nonzero remainders $1, \ldots, 6$ are automatically coprime to a prime d, and they happened to form a single cycle under repeated multiplication by 10, so every one of them was a rotation of the same repetend, 142 857. For composite d this need not happen.

Consider $d = 21$. There are 12 remainders coprime to 21, and under repeated multiplication by 10, keeping the remainder on division by 21 at each step, they split into two disjoint cycles of length 6:

$$\{4, 19, 1, 10, 16, 13\} \quad \text{and} \quad \{5, 8, 17, 2, 20, 11\}.$$

These give two genuinely different repetend integers, neither a rotation of the other:

$$\frac{4}{21} = 0.\overline{190\,476} \quad \text{and} \quad \frac{5}{21} = 0.\overline{238\,095}.$$

Let us point out that these two cycles do not exhaust the twenty nonzero remainders. Setting aside for a moment the requirement that r be coprime to d, the division $\frac{3}{21} = 0.\overline{142\,857}$ gives yet another repetend of length six, namely 142 857, with remainders $\{3, 9, 6, 18, 12, 15\}$—and one we have already met, since $\frac{3}{21} = \frac{1}{7}$. Thus $d = 21$ has three repetends of length six; the remaining two repetends, $\frac{7}{21} = 0.\overline{3}$ and $\frac{14}{21} = 0.\overline{6}$, are of length one.

Table 2 works out the offset-ratio combinations for the first cycle exactly as Table 1 did for $d = 7$; the second cycle has its own, independent table of the same shape, built from 238 095 instead. For example, starting at remainder 19 with offset 1, we read off ratio $\frac{1}{19}$, confirming $19 \times \rho_1(904\,761) = 904\,761$.

Repetend Digit	1	9	0	4	7	6
Intermediate Remainder	4	19	1*	10	16	13
Offset 1	19/4	**1/19**	**10**	8/5	13/16	4/13
Offset 2	**1/4**	10/19	**16**	13/10	**1/4**	19/13
Offset 3	5/2	16/19	**13**	2/5	19/16	**1/13**
Offset 4	**4**	13/19	**4**	19/10	**1/16**	10/13
Offset 5	13/4	4/19	**19**	**1/10**	5/8	16/13

Table 2: Offset-Ratio Table for Divisor 21 (Cycle Containing Remainder 4). The starred column's repeating block, 047 619, begins with 0; see Remark 3.

This splitting is not tied to primality: $d = 13$ is prime and still splits into two cycles of length 6 (repetends 769 230 and 153 846), while $d = 17$ is prime and stays as a single cycle of length 16—what matters is whether the remainders coprime to d happen to fall into one cycle or into several, as they fall into one for $d = 7$ and $d = 17$ but not for $d = 13$ or $d = 21$. A prime p whose remainders form a single cycle—necessarily of the greatest length available, $p - 1$—is called a *full repetend prime* (or *long prime*).

6 Further Divisors

We work out two more divisors in full, chosen to sit at opposite ends of the range just described: $d = 13$, a prime that still splits into multiple cycles, and $d = 17$, a prime whose single cycle is the longest we have seen yet.

$d = 13$

There are 12 remainders coprime to 13, splitting into two cycles of length 6:

$$\{10, 9, 12, 3, 4, 1\} \quad \text{and} \quad \{2, 7, 5, 11, 6, 8\}.$$

$$\frac{10}{13} = 0.\overline{769\,230} \quad \text{and} \quad \frac{2}{13} = 0.\overline{153\,846}.$$

Table 3 works out the first cycle. For example, starting at remainder 9 with offset 2, we read off ratio $\frac{1}{3}$, confirming $3 \times \rho_2(692\,307) = 692\,307$.

Repetend Digit	7	6	9	2	3	0
Intermediate Remainder	10	9	12	3	4	1*
Offset 1	9/10	4/3	**1/4**	4/3	**1/4**	**10**
Offset 2	6/5	**1/3**	**1/3**	**1/3**	5/2	**9**
Offset 3	3/10	4/9	**1/12**	10/3	9/4	**12**
Offset 4	2/5	**1/9**	5/6	**3**	**3**	**3**
Offset 5	**1/10**	10/9	3/4	**4**	3/4	**4**

Table 3: Offset-Ratio Table for Divisor 13 (Cycle Containing Remainder 10). The starred column's repeating block, 076 923, begins with 0; see Remark 3.

$d = 17$

Here 17 is a full repetend prime, so all sixteen remainders coprime to 17 form a single cycle:

$$\{2, 3, 13, 11, 8, 12, 1, 10, 15, 14, 4, 6, 9, 5, 16, 7\},$$

$$\frac{2}{17} = 0.\overline{1\,176\,470\,588\,235\,294}.$$

Table 4 gives the full offset-ratio grid—all 15 nonzero offsets against all 16 remainders. For example, starting at remainder 2 with offset 10, we read off ratio 2, confirming

$$1 \times \rho_{10}(1\,176\,470\,588\,235\,294) = 2 \times 1\,176\,470\,588\,235\,294.$$

This divisor also settles the puzzle we began with. Bronowski asked for an integer whose single left rotation is exactly $\frac{3}{2}$ times itself, which

in the notation of Section 3 is an entry of $\frac{3}{2}$ in the offset-1 row. Table 4 has one immediately, at remainder 2:

$$2 \times \rho_1(1\,176\,470\,588\,235\,294) = 3 \times 1\,176\,470\,588\,235\,294,$$

that is, $\rho_1(x) = 1\,764\,705\,882\,352\,941 = \frac{3}{2}x$.

Repetend Digit	1	1	7	6	4	7	0	5	8	8	2	3	5	2	9	4
Intermediate Remainder	2	3	13	11	8	12	1*	10	15	14	4	6	9	5	16	7
Offset 1	3/2	13/3	11/13	8/11	3/2	**1/12**	**10**	3/2	14/15	2/7	3/2	3/2	5/9	16/5	7/16	2/7
Offset 2	13/2	11/3	8/13	12/11	**1/8**	5/6	**15**	7/5	4/15	3/7	9/4	5/6	16/9	7/5	**1/8**	3/7
Offset 3	11/2	8/3	12/13	**1/11**	5/4	5/4	**14**	2/5	2/5	9/14	5/4	8/3	7/9	2/5	3/16	13/7
Offset 4	**4**	**4**	**1/13**	10/11	15/8	7/6	**4**	3/5	3/5	5/14	**4**	7/6	2/9	3/5	13/16	11/7
Offset 5	**6**	**1/3**	10/13	15/11	7/4	**1/3**	**6**	9/10	**1/3**	8/7	7/4	**1/3**	**1/3**	13/5	11/16	8/7
Offset 6	**1/2**	10/3	15/13	14/11	**1/2**	**1/2**	**9**	**1/2**	16/15	**1/2**	**1/2**	**1/2**	13/9	11/5	**1/2**	12/7
Offset 7	**5**	**5**	14/13	4/11	3/4	3/4	**5**	8/5	7/15	**1/7**	3/4	13/6	11/9	8/5	3/4	**1/7**
Offset 8	15/2	14/3	4/13	6/11	9/8	5/12	**16**	7/10	2/15	3/14	13/4	11/6	8/9	12/5	**1/16**	10/7
Offset 9	**7**	4/3	6/13	9/11	5/8	4/3	**7**	**1/5**	**1/5**	13/14	11/4	4/3	4/3	**1/5**	5/8	15/7
Offset 10	**2**	**2**	9/13	5/11	**2**	7/12	**2**	3/10	13/15	11/14	**2**	**2**	**1/9**	**2**	15/16	**2**
Offset 11	**3**	**3**	5/13	16/11	7/8	**1/6**	**3**	13/10	11/15	4/7	**3**	**1/6**	10/9	**3**	7/8	4/7
Offset 12	9/2	5/3	16/13	7/11	**1/4**	**1/4**	**13**	11/10	8/15	6/7	**1/4**	5/3	5/3	14/5	**1/4**	6/7
Offset 13	5/2	16/3	7/13	2/11	3/8	13/12	**11**	4/5	4/5	**1/14**	5/2	5/2	14/9	4/5	3/8	9/7
Offset 14	**8**	7/3	2/13	3/11	13/8	11/12	**8**	6/5	**1/15**	5/7	15/4	7/3	4/9	6/5	9/16	5/7
Offset 15	7/2	2/3	3/13	13/11	11/8	2/3	**12**	**1/10**	2/3	15/14	7/2	2/3	2/3	9/5	5/16	16/7

Table 4: Offset-Ratio Table for Divisor 17. The starred column's repeating block, 0 588 235 294 117 647, begins with 0; see Remark 3.

7 Where This Leads

The general theory the puzzle belongs to rests on three things: a single division, the cycle of remainders it runs through, and the observation that rotating a repetend is only that division restarted one step further along. Bronowski's number is then one entry in one row of one table, and the same table names every other ratio that divisor will support. What lies past the edge of these tables is the question of which ratios can be obtained at all, taking every divisor together. That is what Primrose asked in 1955, and Corollary 6 answers it only as far as an unproved conjecture allows.

Appendix: Generating Offset Tables

Every offset-ratio table in this paper (Tables 1–4) can be produced by tedious calculation from the sequence of remainders left by a suitable division. We produced ours with a short program instead, which will

do the same for any divisor d coprime to 10. Its source code is publicly available [7], and this appendix describes how it works.

Given d, the program partitions the remainders coprime to d into cycles, each step carrying a remainder r to the remainder left by dividing $10r$ by d—the relation behind (3)—reading off each step's repetend digit as it goes, exactly as Section 5 does by hand for $d = 21$. Nothing in this process assumes d is prime, so composite divisors fall out of the same code path as prime ones, splitting into as many independent cycles as the remainders require. For each cycle, the program builds its offset-ratio table from the relation in Theorem 4 and expands every entry into the corresponding equation $B \times \rho_t(x) = A \times x$ by rotating the cycle's actual digit string, so every table and equation in this paper can be checked mechanically.

As the output grows past successive size thresholds, the program writes the table to a file instead of the console, then eventually reports only each cycle's repetend digits, avoiding a table that would run to gigabytes.

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