ALGEBRA IDENTIFIED WITH GEOMETRY.

I. Euclid's Conception of Ratio and Proportion.

1. *Nature of the Conception.*—(i.) The Latin terms *ratiō* = calculation, and *prōportiō* = portioning forward, do not convey the force of the Greek λόγος and ἀναλογία, and have by their arithmetical character served to lead the mind astray. Of the second Greek term Cicero, to whom its Latinisation is due, says (*Timaeus, seu de Universo*, cap. iv.): "Omnia duo ad cohaerendum tertium aliquid requīrunt, et quasi nōdum vinculumque dēsīderant. Sed vinculōrum id est aptissimum atque pulcherrimum, quod ēx sē, atque dē hīs, quae astringit, quam maximē ūnum efficit. Id optimē assequitur quae Graecē ἀναλογία, Latīnē, (audendum est enim, quoniam haec prīmum ā nōbīs novantur) comparātiō prōportiōve dīcī potest." It is a pity that subsequent Latinists preferred Cicero's second proposal to his first. But Cicero was not thinking mathematically. The Greek term λόγος has its radical sense in col-*lec*-ting, or bringing together for the purpose of thought, and ἀναλογία was the comparison of such collections, by running them through from bottom to top (ἀνά). This general conception must necessarily have influenced any Greek in applying the terms. Euclid meagrely defines λόγος thus, in *two* separate definitions, of which the second has not been usually construed as a development of the first.

γ΄. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποιὰ σχέσις.

δ΄. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

(ii.) Now I first observe that Euclid does not define *homogeneity*, as he uses the term ὁμογενῶν without any explanation, as if well understood, and hence that it is an error to suppose that in def. 4. he intended to define it, although of course that definition is incomprehensible unless the magnitudes compared are homogeneous. In modern language we may I think render the meaning of these definitions thus:

"3. The term *logos* is used to express a certain standing towards one another in respect to size, of two homogeneous magnitudes.

"4. Two magnitudes will be said to have a *logos* towards each other, when a multiple of either can be formed so as to exceed a multiple of the other."

(iii.) The term *multiple*, of which much more in art. 3, is not, properly speaking, defined by Euclid. He first tells us that he intends to limit the ordinary word μέρος by using it as an *aliquot part*, which he defines by means of the unexplained term μέτρον, thus: α΄. μέρος ἐστὶ μέγεθος μεγέθους, τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον· that is: "The less magnitude will be termed a *meros* of the greater, when it measures the other without remainder (κατά)." And then he observes as an additional remark (shewn by δέ) meant to render this notion more complete, and also distinguish a multitude from an aggregate: β΄. πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος, "In this case (δέ) the greater magnitude will be a multiple of the less, when it is measured by the less without remainder (κατά)," which is only saying: "of course, then, any magnitude is a *meros* of any multiple of it."

(iv.) Returning to the definitions in (ii.) The term σχέσις πρὸς is exactly rendered by our "standing towards." The use of "mutual relation" seems to be tautological, on account of the popular use of the word *ratio*, which the Germans have even translated by the same word, *verhältniss*, that they use for *relation*, just as in French our distinction of *ratio* and *reason* is lost in the single word *raison*. The use of ποιά before σχέσις is precisely similar to our use of the word *certain*, meaning "undefined, of some kind or other," and hence requiring future limitation, and in Plato's Greek constantly it is joined to τὶς, as ποιός τις. In def. 3. the only limitation regards *size*, which is expressed by πηλικότης *compared*, as distinct from μέγεθος *un*compared magnitude. There is no notion of *measuring out* καταμετρεῖν, in πηλικότης, which is therefore not well rendered by quantu*plicity* or *manifoldness*, for which in literary Greek as in literary English there seems to have been no term. Now there are many ways in which two magnitudes may be compared in respect to size; 1) with regard to greater and less, the only method used in the previous books of Euclid, and by that very circumstance here excluded, 2) with regard to one *measuring out* the other, which was a particular case, already considered in def. 1. and 2.; 3) with regard to both being measurable out by a third magnitude, which Euclid wisely saw to be included in the next case; 4) with regard to successive multiples of one continually exceeding successive multiples of the other, and as a particular case one multiple of one being of the same size as the same or another multiple of the other. The object of def. 4., appears to me to have been the limitation of the ποιὰ σχέσις πρὸς ἄλληλα, or *certain standing towards one another*, to this last case, which is alone general and includes all the preceding. Observe that the article ἡ points out σχέσις as the subject of the sentence. Euclid proceeds then to examine this conception, namely, that *logos* is *the interdistribution of multiples*.

(v.) He begins by considering the possibilities that may occur. If we take two magnitudes A and B, and two others C and D, and compare any of their multiples mA, nB, and mC, nD with regard to greater, equal and less, we find that for each of the three cases of mA being greater than, or equal to, or less than nB, mC may be greater than, or equal to, or less than nD. There are therefore 9 cases to consider. Euclid already knew from the properties of parallel transversals cutting two intersecting straight lines, that it was *possible* that when $mA >=< nB$, then mC might be $>=<$ nD *respectively*, and therefore he begins by saying, def. 5., that in *that* case the *logos* of A to B is the *same* as the *logos* of C to D, ἐν τῷ αὐτῷ, not ἐν τῷ ἴσῳ λόγῳ, adding def. 6., "let then (δὲ) two pairs of magnitudes which have the same *logos*, as thus determined (included in δὲ), be called *analoga*," τὰ δὲ τὸν αὐτὸν ἔχοντα μεγέθη λόγον, ἀνάλογα καλείσθω. He does not think it necessary to shew from the first, that if one and one only of the two, A or B, be altered in any way however slight, the *logos* will be changed. He proceeds to the cases in which the interdistribution of multiples is *not* the same for each pair of magnitudes considered. These he reduces to one. Suppose that when $mA > nB$, mC is *not* $> nD$; "in that case (τότε) the first *logos* is said (λέγεται) to be *greater* than the second," the metaphorical use of greater and less as applied to *logos* is justified by the ordinary use of the term greater and less applied to the multiples considered, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον. This being settled, he is able to introduce the abstract term *analogía* for *sameness* of ratios. The word used is ὁμοιότης, usually rendered *similarity*. It is evident from the ἐν τῷ αὐτῷ λόγῳ in def. 5., that the Aristotelian ταὐτότης should have been used, but perhaps Euclid, if he was acquainted with the word (we know that he was no school-logician) possibly thought it barbarous. It remained for theologians to wrangle over ὁμοιοούσιος and ὁμοούσιος. Euclid at any rate did not invent ὁμότης (which was never Greek), but contented himself with using ὁμοιότης. Perhaps logically considered *two* thoughts, just because they are *two*, are not the *same*, although indistinguishable except in point of time of entertainment. But the use of *similarity* has led to the use of *equality* as applied to *logoi*, which Euclid did not contemplate, and this use of equality has led to bringing *analogía* under the axiom of "*things* which are *equal* to the same *thing* are equal to one another," which is a mere verbal quibble. What Euclid says is in English: "sameness of *lógoi* then (δὲ) is *analogía*," ἀναλογία δὲ ἐστιν ἡ τῶν λόγων ὁμοιότης, the use of the ἡ pointing out the subject of the sentence, and its absence the predicate, as before (iv.)

(vi.) This appears to me Euclid's real conception, and it is a conception which places its author in the very first rank of thinkers, that is, among those who have discovered the one simple key to an apparently insoluble difficulty—in this case the passage from discontinuity to continuity. It remains to shew how this conception can be imparted to learners, whose minds have been arithmetically cribbed, confined, and hence distorted from earliest childhood. Of course the Greek words *logos*, *analogía*, here used to prevent ambiguity, will henceforth be discontinued.

2. *Paedagogical Exposition of the Conception. First step.*—(i.) In the following pages a method is suggested for leading pupils up to the conception of ratios of magnitudes, independently of commensurability, and to the mode of comparing them. No child who has not been taught arithmetic has any general conception on these points. Every child who has been so taught has a more or less incorrect conception. We have to furnish him with progressive experience to make him familiar with the geometrical conception, and understand how far the arithmetical conception is useful and where it makes default. It is not till after the modes of comparing magnitudes are understood that

the term proportion should be introduced, as proportion is only one case of comparison. It will be understood that these are merely hints, and not even a detailed syllabus.

(ii.) Arrange boys (or, for convenience, straws or sticks) in order of height (or length). Shew how this can be done by marking their heights in any order against the same standard, because the terminal points of lengths which have the same origin arrange themselves in the order of the lengths of the lines. No statement is to be made of actual height or length in reference to a standard.

(iii.) Arrange boys (or, for convenience, stones) in order of weight. Shew that this may be done by scales, but more conveniently by taking the stones at hazard and weighing them by a balanced lever with arms of unequal length, a fixed scale being attached to the shorter arm, and a small weight (another stone) hitched by a string over the longer, a mark being made on the longer arm where the balance is attained. Shew that these marks naturally arrange themselves in order, the mark for the heaviest being furthest from the fulcrum. No statement is to be made of actual weight in reference to a standard. This is an extremely important reduction of order of weights to order of lengths. Practically it leads to a mechanical mode of finding two straight lines which bear to each other the same ratio as any two weights, without any considerations of commensurability. But this reduction requires some mechanical knowledge and is not to be attempted at first.

(iv.) Arrange stones by volume. Shew that this may be done by placing a large enough vessel *full* of water within a larger one which drains into a glass cylinder outside of which a slip of paper is pasted vertically. On immersing any stone in any order carefully in the first vessel, the overflow is conducted through the second into the cylinder, and the height to which the water rises is to be marked on the paper. Empty the cylinder and fill the first vessel again. Immerse a second stone and proceed as before, and so on. The marks on the slip of paper arrange themselves naturally in the ascending order of the size of the stones. This will subsequently reduce ratios of any volumes to ratios of lengths without regard to commensurability. No reference to any standard volume is to be made.

(v.) Arrange any number (4 or 5 are enough) of rectilinear areas (mixed, triangles and polygons) in order of magnitude. Shew that they may be all reduced to rectangles of the same altitude, and then that the bases may be arranged as in (ii.)

(vi.) Arrange curvilinear, or amorphous, or mixed rectilinear and other areas, plane or other, in order of magnitude. Cut them out in "lead paper," which is sufficiently homogeneous, flexible and heavy, to convey the required notion, and treat the slips as weights (iii.).

(vii.) Arrange curves or broken lines in order of length. Pass threads round them, and straighten them by tension, and apply (ii.).

(viii.) The processes in (ii., v.) are strictly geometrical. The other processes require "idealising," and suggest geometrical problems, which the teacher should carefully explain have not been completely solved, but that in general we can by refined geometrical methods approach more nearly to the truth than by the rough physical methods here employed, when some of the magnitudes to be compared are very nearly the same; a difficulty which should be introduced in a second or third trial in every case. But shew also that the idealisation of those rough methods conclusively proves that we can always *conceive* a series of straight lines arranged in the order of magnitude of any series of magnitudes such as those already experimented on.

(ix.) Then draw attention to the fact that we first compared straight lengths with one another, then weights with one another, then volumes, then areas, and then general lengths, but that we did not compare lengths with weights, &c., for we could not say of a length that it was either greater or less than a weight, although we were able to arrange lengths in the same order as weights. Hence lead to a conception of *kinds*, and to the order of arrangements of things of the *same kind* independently of the *particular kind*. These are difficult abstractions, and must be treated cautiously. Terrible mistakes are made by children who have to grub them out unguided. But merely to *tell* them is pouring water on a duck's back—neither tale nor water is ever taken in.

(x.) This completes the first step in the way of preparation, and the absence of all approach to arithmetic or commensurability is of the utmost importance for what follows.

3. *Second step.*—(i.) The next step includes the formation of multiples, and the point to be borne in mind by the teacher is that the child, through arithmetic, has been trained to consider "bags of stones,"—that is, separate discontinuous magnitudes artificially aggregated without losing their discontinuity,—and that he has to be led to comprehend an addition which results in absolute continuity, without a trace of the original individuality. This is best done by grouping quantities of liquids. Take a small glass vessel, with an external band marked on it, but *not* all round it (a short slip of paper is best); pour coloured water in till the top of the water is seen to coincide with the top of the band. Have ready a series of larger glass vessels of the same shape, cylinders of the same radius, which, to avoid arithmetical conceptions, are marked by the letters *A*, *B*, *C*, &c. Empty the small vessel into *A*. Fill it again and empty into *B*; fill it again and empty into *B* again. Fill it three more times and empty each time into *C*, and so on. Then place the vessels in the order of the height of the water. This will be also in order of the volumes and also of the weights of the water. Draw attention to the fact that the water in each vessel shews no trace of having been poured in by instalments, so that it is absolutely impossible to say in what manner it was poured in. But as the operation was *witnessed*, it is *known* that this continuity resulted from the *discontinuous* operation of adding equal instalments. These *discontinuous* instalments can be counted like anything else. *A* had 1, *B* had 2, *C* had 3, and so on. Hence the volumes of the water are called the first, second, third &c. *multiple* of the volume of water in the original smaller vessel, and the order of arrangement of these *multiples* of volume is consequently the order of the arrangement of the scale of whole numbers, and *this order must be the same* whatever be the size of the original small vessel, *although the multiple volumes themselves are different.* Moreover if any volumes are arranged in order of mag-

nitude it is easy to see,—not whether they *have* been formed by instalments, but—whether they *can* be formed by instalments, by simply emptying A into a new vessel, marking the height of the water, and throwing it away. Then pouring from B into this new vessel up to the line, emptying, seeing if the remainder will fill the new vessel up to the same line, and so on.

(ii.) The points which should be gained are: 1) that multiples of magnitudes are simple continuous magnitudes; 2) that these can be arranged in order of magnitude; 3) that this order is constant, and is that of the numerical scale by which they are named; 4) that any magnitudes being arranged in order, it can be ascertained whether they are or are not multiples of the same magnitude, whenever subtraction is possible.

(iii.) Next make the learner *construct* multiples of straight lines in the form of straight lines with no mark of division; multiples of rectilinear areas not being parallelograms, in the form of parallelograms of the same height with no mark of division; multiples of circular arcs in the form of circular arcs, also with no mark of division, but with a rough internal or external spiral which by the number of its coils shews the amount of revolution when exceeding a semi-revolution; and finally multiples of angles in the form of angles in the same way.

(iv.) De Morgan said that Euc. vi. 33 fairly gave up Euclid's conception of angle, Euc. i., def. 8 to 12. But really this was given up in Euc. i. 13 and i. 32, especially in its corollaries. I think it advisable to retain the term angle for sums of angles not exceeding two right angles, and to use the term *ro'tate* for larger amounts. An extension of the term angle to any sums of less than four right angles does not meet the case of Euc. i. 32, cor. And it will be seen that for directional angles the limitation here proposed is important (art. 20. x.). Also it is clear that only in the case of such limitation can we dispense with the use of the subsidiary spirals. Angles themselves will then become rotates of less than a certain amount. The sums of angles (*i. e.* rotates) are always rotates, and may (exceptionally) be angles. Great trouble is at present experienced by learners from the sum of several angles exceeding even four right angles, and hence not being an angle at all, even when its meaning is extended as above.

(v.) The next point is to shew that, knowing Euc. i. to iv., we cannot take multiples of curvilinear magnitudes; for example, we cannot draw a circle which shall be double the area of a given circle. Shew, however, that we can easily describe one much more or much less than double, and hence that it is only our want of geometry that prevents us from hitting the exact radius required. State that for this particular case we shall find a solution (art. 11. iii.); but that in general we are at present able only to form such multiples hypothetically in conception, and approximatively in practice. Thus we cannot with geometrical accuracy compare the length of a circular arc with its chord. The teacher should, however, shew why we know the arc to be greater, as this is a very important result.

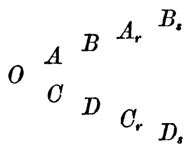
4. *Third step.*—Take two series of multiples of two original magnitudes of the same kind, and, as they are all magnitudes, arrange them

in order of magnitude. Our preparation (art. 2. ii.—ix.) enables us to reduce this case to that of comparing multiples of length. Make the learner mark off lengths, as OA_2, OA_3, &c., and OB_2, OB_3, &c., which are multiples of OA and OB, by marking their terminations A_2, &c., B_2, &c. with short ticks on *opposite* sides of the same straight line. The point to be established is that, if one of the original lines be ever so slightly altered, some multiple of the altered line can be found which will exceed or fall short of the same multiple of the unaltered line by more than the other original line, and that consequently the order of the multiples of the *two* original lines, if enough of them are taken, will *differ* from the order of the multiples of *one* of the original lines, *and* of a line differing from the other. Hence, when two magnitudes are known, the order of their multiples is fixed and known. And it must be also seen that, conversely, if by any means the order of multiples is known, and also one of the original lines, the other is of fixed length, although we are not yet in a condition to find it.

5. *Fourth step.*—(i.) Shew that it is possible to alter the lengths of *both* the original lines of art. 4. in such a way that the order of the multiples of the two altered lines will be the same as that of the two original lines.

(ii.) The first case is that of commensurability. If $m \,.\, OA = n \,.\, OB$, (notation to be thoroughly explained as representing multiples, *not* aggregates,) then the multiples divide into groups of m multiples of OA and n multiples of OB, and the order in which the multiples interlie will be the same in each group. This should be exemplified by a figure. The consequence is that when we know the order for the first group we know the order *for ever*—without any veiled application of the principle of limits. This conclusion is extremely important.

(iii.) The second case is that of parallels. Let OAB, OCD be straight lines (the unconnected letters in the margin will show how any figures are to be constructed) drawn from a common origin O; and AC, BD parallel lines. Take $OA_r = r.OA$, $OB_s = s.OB$, and draw the lines A_rC_r, B_sD_s parallel to AC, BD; then $OC_r = r.OC$, and $OD_s = s.OD$. And *as parallels do not intersect*, the order of the multiples of OA, OB, determined by the terminal points A_r, B_s, will be the same as the order of the multiples of OC, OD, determined by the terminal points C_r, D_s. Here the geometrical property of parallels enables us to know with certainty that the order of multiples of OC, OD is the same as that of those of OA, OB, *independently of the number of multiples compared*, without any veiled application of limits, and also *independently of commensurability*. This conclusion therefore holds for all those cases which have been shewn to be reducible to straight lines. It should be verified by examples of triangles and rectilinear areas generally. The converse must also be proved. Four arrangements are possible: 1) the lines are parallel, and the order the same; this we have seen to be the case, and it excludes 2), the lines are parallel, and the order is *not* the same; 3) but the lines may *not* be parallel, and yet, for all we know, the order may



be the same; but this is excluded by 4), when the lines are not parallel, the order not the same; because we know that when the order is not the same, the lines joining the extremities of the multiples must cross, which is impossible for parallels. There remain therefore only the first and fourth cases, and this proves the correctness of the conversion.

(iv.) The third case is that of angles or rotates and their subtending arcs, which presents no difficulty.

6. *Parallels, a Parenthesis.*—(i.) Here I interpose some parenthetical remarks suggested by the assumption that parallel lines never meet, in order to shew that the theory of parallels is not one of veiled limits, for, if it were, then indeed Euclid's conception would be one also, except in the case of commensurables. Now in modern geometry any system of parallels is said to have *one* and *only one* point in common, which is conveniently placed out of sight, at infinity. *Townsend* (Modern Geometry, 1863, p. 11, see also the citations in Appendix I.) says that the truth of this conclusion has been "long placed beyond all question by the simplest considerations of projection and perspective." I believe that it has been much longer rendered *impossible* by the elementary consideration that two straight lines cannot inclose a space. The conclusion (*ibid.* p. 12) that "the two *opposite* directions of every [straight] line, not itself at infinity, are to be regarded, not as *reaching infinity* at two different and opposite points, but as *running into each other* and *meeting* at a single point at infinity," amounts to saying that diametrically opposite directions are the same. Again (*ibid.*), "every [straight] line not at infinity may be regarded as a circle of infinite radius whose centre is the point at infinity in the direction orthogonal to the line," *i.e.*, the single point common to a system of parallel straight lines is the common centre of the concentric circles with which they coincide circumferentially, and which have no common circumferential point. The assumption that such circles have two imaginary points at infinity where they are touched by the two imaginary non-touchers (*asymptotes*) common to all concentric circles, is in the mere field of imaginaries, and will be disposed of hereafter (art 48. v.) The touching of curves by real non-touchers has more to be said in favour of it than the intersections of parallels, because asymptotes do constantly approach the curve, but no two points in two parallel lines are ever nearer each other than their common normal which itself never diminishes in length, so that the assumption that parallel lines intersect requires that an unchangeable length should discontinuously shrink into nothingness "*at* infinity" (which has no "at"). To my mind these are mere contradictions in terms which must lead, and I believe have led, to serious error. They are however nothing more than terminology, invented to make discontinuity continuous, and thus hide the real state of the case. Hence I hold that unless we assert that two straight lines will under certain circumstances inclose a space,—and thus give up the proof of the fundamental proposition, Euc. i. 4, in which case plane geometry will itself drift off to infinity,—we must consider that there are no veiled limits in the proof from parallels in art. 5. iii.

(ii.) This leads me to consider the question of parallels as a subject to be taught to children. No one would dream of teaching them the bewilderments just mentioned; but some men whose opinions I respect, are inclined to make parallels a "reserved question," whereas it seems to me that no geometrical teaching, and especially none on proportion, is possible without making it elementarily exoteric. I must crave indulgence for briefly stating my own views on this subject, which, however crotchety in appearance, are the result of years of reflection tested by other years of application.

(iii.) Bring the edges of two surfaces (pieces of paper may be used for illustration, but any surfaces will do) to touch in two points; observe whether there is any *intermediate* point, at which they are also in contact; turn the surfaces about the first two points like a door on its hinges, and observe if they still touch in that third point, throughout the movement. If they do, for all such third points observable, the edges intermediate to the points are *straight* lines. This is our only test of straightness. Some writers gain the second line by cutting off a bit of the first, which disguises without altering the principle.

(iv.) Straight edges can *slide* one on the other, that is, can move so as always to have *two* points of the one coincident with two of the other, and hence coincide intermediately.

(v.) But straight edges can *also* move one on the other so as to have *one* fixed point of one coincident with *one* fixed point of the other, and at least one fixed point of the one *not* coincident with any point of the other. In this case they can have only the one first mentioned point in each coincident. They then *rotate*. Here explain the generation of planes, plane rotation, circularity, angularity.

(vi.) When straight edges have thus rotated they can be *clamped*, by a transversal having fixed points, one in common with each straight edge. They then form a *biradial* (Sir W. R. Hamilton's word, see also art 34. v.), of which the original straight lines are the *arms*, the transversal not being further regarded. In this case the motion of one arm entails the motion of the other, and *neither can rotate unless the other rotates also.*

(vii.) Now let one arm of a biradial *slide* on a given straight edge, the *trace* of the other arm having been marked in its original position. Then in every new position of this second arm there will exist a new straight line having at least one point *not* in common with the original trace, while, as it has not rotated, it can have *no other* point in common with the original trace, quite independently of length. None of these positions of the second arm therefore ever meet the original trace. The existence of parallels is therefore demonstrated without any veiled reference to limits. The experiment is best shewn to a single pupil by lines on tracing paper moved over lines on other paper, and to a class by lines drawn with gum and whiting on glass, and moved over the chalk lines on the black-board. It leads to the best practical method of drawing parallels by sliding one "set square" along another. A straight line thus moved is said to be *translated.* The advantage of early familiarity with the notions of rotation and translation is obvious.

(viii.) The usual propositions as to equality of external and internal angles, &c., in the case of parallels are now to be proved, but *not* their converse (Euc. i. 29).

(ix.) The addition of angles which have not a common vertex is now to be shewn, by first sliding and then rotating, the sum of any number of rotations being independent of interposed slides or translations. In this way Euc. i. 32 may be immediately proved *without* using Euc. i. 29, for which purpose this proposition is mainly required : *ABC* (fig. 1) being a triangle, the rotation of a line *A'D'* originally lying over *AD*, by turning it about *A* as a pivot until it falls on *AB*, is the same as if this line were first slid till *A'* fell on *C*, and then rotated to fall on *CB*; were then slid along *CB* till *A'* pass from *C* to *B*, and *A'D'* falls on *BF*; were then rotated about *B* to *BE*, (the angle *FBE* being shewn to be equal to *CBA* by merely continuing the line *A'D'* backwards to *C'* over *C*, and seeing that on rotation this *A'C'* comes to fall on *BF*,) and were then slid till *A'* falls on *A*, so that *A'D'* has rotated from *AD* to *AB* by the help of two rotations separated by intermediate slides. The exterior angle *DAB* is therefore equal to the two interior and opposite angles *ACB*, *CBA*, whenever two intersecting straight lines *AB*, *CB* are crossed by a transversal *DAC*.

(x.) To prove Euc. i. 29, we have however still to prove Ax. 12, which may be made to depend on this principle : if a straight line *BE* (fig. 2) pass through a given point *B* and be translated in any manner till it again pass through *B*, it will wholly coincide with its former trace. For if it did not, it would have rotated, which is against the hypothesis.

(xi.) Let *AC*, *BD* (fig. 2) be parallel lines, and angle *ABE* be less than angle *ABD*; to prove that *AC*, *BE* will meet. Take *A'B'E'* as a biradial over *ABE*; slide *B'A'* to fall on *AH*, so that *B'E'* falls on *AF*, and continue *AF* indefinitely both ways. *AF* necessarily cuts *AC*. Clamp *B'E'*, now falling over *AF*, with *A'G'*, falling over *AG*; slide *A'G'* along *GAC*. Then there is no point in the plane *ABE* over which *B'E'*, which is attached to *A'G'*, when sufficiently produced, will not pass. Hence it will pass over *B*. And then *B'E'*, having been only translated, coincides with *BE* again. And as *B'E'*, during the last translation, has never ceased to cut *AC*, *BE* also cuts *AC*. This seems to me a complete proof of this axiom on the data assumed, and the assumption of these data also appears to me more directly connected with the subject, and to make the point of this Axiom 12 more evident than any other.

7. *Paedagogical Exposition resumed.—Fifth step.* (i.) The paedagogical introduction to proportion is now resumed. Having shewn that the order of multiples is constant when the originals are constant, and may be constant when the originals are *both* altered in certain ways, it becomes convenient to have a name for this order. Let the magnitudes be *A* and *B*, then the order in which the multiples of *A* are distributed among the multiples of *B*, (so that, given any multiple of *A*, we know the two nearest multiples of *B* between which it lies,) is called the *ratio* of *A* to *B*, and is written *A* : *B*. Similarly *B* : *A*, or the ratio of *B* to *A*, means the order in which the multiples of *B* are distributed among those of *A*, (so that, given any multiple of *B*, we know the two nearest multiples of *A* between which it lies).

(ii.) If then the multiples of C are distributed among those of D in the *same order* as those of A among those of B, the ratio of C to D is the *same* as that of A to B. This is written $A : B :: C : D$, which I prefer reading "A to B same as C to D," omitting the word ratio, and using *same as* instead of *equal to* for the reasons in art. 1. v.; and I also prefer, at least paedagogically, not to use the old formula "*as* A is to B *so* is C to D," because of the marvellous ambiguity of the *as* and *so*, and because of the old false associations produced by the Rule of Three as usually taught. Of course in this case also $B : A :: D : C$.

(iii.) The idealised elementary processes (art. 2. ii.—viii.) now lead us to infer that, given any two magnitudes of the same kind, we might always find (if our processes were accurate enough) two straight lines which would have the same ratio—*i. e.*, whose multiples would have the same order of magnitude. Hence a ratio is always (conceptionally) expressible as that of two straight lines.

(iv.) And this leads us to consider the case where $A : B$ not $:: C : D$, that is, where the multiples of A are not distributed in the same order among those of B, as those of C are among those of D. Two cases will arise:

1) *Either* some multiple of A is *greater* than some multiple of B, while the multiple of C corresponding to that of A is *not* greater than that of D corresponding to that of B. In this case, for brevity, the term *greater* is transferred from corresponding multiples to the orders of distribution of the multiples of A among those of B, and of C among those of D; and we say *laconically*, $A : B > C : D$, reading $>$ as "greater than" (compare art. 1. v.). Stress should be laid on this abbreviation, because in the ratios there is no real greater or less. Numerous examples *must* be formed.

2) *Or else* there will be some multiple of A which is *less* than a multiple of B, while the multiple of C corresponding to that of A is *not* less than that of D corresponding to that of C. Here, in the same way, we write $A : B < C : D$, reading $<$ as "less than," with the same warning as before. Numerous examples required.

8. *Sixth step.*—(i.) Up to this point there has not been a word of proportion. The word is used in common speech so ungeometrically, and has been so much perverted arithmetically, that I prefer reserving it for the Sixth step.

(ii.) Stand before a mirror. Hold a book parallel to its surface. Advance and withdraw it, keeping the head steady. Observe the great apparent change of *size* in the image of the book, shewn by the amount of the surface of the mirror covered by it (easily marked off), whereas the *shape* remains unaltered. When this occurs, we say that all the dimensions in any one image are *proportionate to* those in the other, or that they all alter *proportionably* or in *proportion*. Observe that the example is chosen so as to exclude commensurability, which would necessarily intrude in drawings made to different *scales* in the usual way. The shadow of a book cast on the wall from a single point of light (a candle) will serve as well; and better for a class. Examine what is meant by this.

(iii.) The simplest figure to deal with is a triangle. Draw one connecting three points on the book, or cast the shadow of a set square on the wall. Observe that the sameness of *shape* depends on the *sameness of angles* between corresponding sides, and that the difference of *size* depends on the *alteration of lengths*. What is the law by which the lengths alter? This should enable us, when we know the length of one line in the original figure and that of the corresponding line in the altered figure, from the length of any line in the original figure to construct the length of the corresponding line in the other figure.

Take two corresponding triangles. The sameness of angles allows of their superimposition so that any pair of corresponding vertices being brought together, the adjacent sides will lie on one another, and the opposite sides be *parallel*. We have the case of art. 5. iii. Hence if ABC, $A'B'C'$ be corresponding triangles of which AB, $A'B'$ are the parallel sides, we have, by art. 7. ii., $CA : CB :: C'A' : C'B'$. That is, *proportion* (or the law of alteration of length in figures of the same shape and different size) *consists in sameness of ratio between corresponding lengths.*

(iv.) Having thus arrived at an essentially geometrical view of proportion, exclusive of arithmetic and commensurability, it only remains to explain that figures which in popular language are said to be in proportion, are in geometry called *similar;* that their properties of size evidently depend on the sameness of the ratios of corresponding lengths; that the examination of the properties thus discoverable forms the principal part of geometry, and that it hence becomes important to discover *all* cases where this relation exists originally, and also what new relations of the same kind can be inferred from knowing one or more such relations. This then is the object of Euc. v. and vi., which would be made mutually illustrative if fused. In the elementary explanations, the main propositions, Euc. vi. 1. 2. 33, have already been proved. It would be of advantage to interpose Euc. vi. 3—17 between Euc. v. 16 and 17. The mode of treating the necessary propositions presents no difficulty whatever when this stage is reached, and I pass it over, to abridge this already too lengthy exposition, without which I felt that it was impossible to make my own views intelligible.

9. *Paedagogical Appendix to Proportion.*—(i.) After the general propositions on proportion in Euc. v., interspersed with some of their easiest and fundamental applications in Euc vi., have been thoroughly taught and understood in their real geometrical, as opposed to their arithmetical, which is also the usual algebraical sense, the question arises: how can we proceed, when it is not geometrically possible to find, as suggested in art. 2. ii.—viii., two straight lines which bear to each other the *same* ratio as any two given homogeneous quantities, but it is at the same time important to deal with that ratio?

(ii.) This leads to the consideration of *approximate* ratios. Of two ratios $X : Y$ and $X : Z$ where X, Y, Z are straight lines, that is *nearer* to the ratio $A : B$, (where A and B are any homogeneous magnitudes,) for which the order of the multiples is the *same* for the *greater* number of multiples of the greater term. When the number of multiples is *very* great in both cases, and the ratio $X : Y < A : B$, but $X : Z > A : B$, there can be but a small difference between Y and Z, and the required line V, for which $X : V :: A : B$, will be $< Y$ and $> Z$. If then we can find successive values of Y and Z, nearer and nearer to each other, we shall obtain ratios which more and more nearly approximate to $X : V$.

(iii.) When we require to find V for practical use, we may previously determine the amount of error, E, deemed sensible, and we lay down the principle that *for such practical ends*, if we can find Y and Z such that $Y-Z$, shall be $< E$, we shall have *practically* solved the problem, because the error will be insensible.

(iv.) Now, conceptionally, we may suppose that by actual formation of multiples we obtain $mA = nB + D$, where D is homogeneous with and $< B$, in which case, if $mX = nY$, and $mX = (n+1) Z$, (whence, when X is given Y and Z can be found by parallels,) we shall have $X : Y < A : B$, and $X : Z > A : B$, while $Y-Z = \frac{m}{n(n+1)} X$, and hence may be made less than any line E, by simply increasing n. And this *conceptionally* solves the *practical* problem.

(v.) Of course the idea of discovering m and n by actually forming multiples, when m and n are very large indeed, is practically illusory. Hence the usual process pursued for finding $m \div n$ is to throw it into a continued fraction, and I particularly urge teachers to approximate to the values of $\sqrt{2}$, $\sqrt{3}$, &c. from two given lines in each case, (diagonal and side of the corresponding rectangles,) first by actually forming multiples, and secondly by actually forming continued fractions; and especially to shew that the diagonal and side of a square are *in*commensurable, both geometrically and arithmetically, to force on the learner the sensation, impossible to acquire without such actual trials, of the meaning, first, of approximation (with its practical uncertainty), and secondly, of incommensurability. An attempt to approximate to the ratio of the circumference to diameter of a circle by using strings of the length of both, is also very instructive. A gallipot, or tub head, or, better, a circular table, will give one or two places of decimals. Taking the best approximate commensurable ratio to be expressed by 355 feet : 113 feet, and observing that $\frac{355}{113} = 3 + \frac{1}{7+\frac{1}{16}}$, it will be found extremely interesting to watch the hesitation about the 7, and see how it will wander from 5 to 8 or 9, according to circumstances, in different trials. To reach the Archimedean 7 is a triumph or a "fluke." Nothing is better adapted to make pupils feel the practical difficulty in the way of "squaring the circle," by such a simple process as "rolling a circle on a straight line and marking off the length." In a London draper's shop I learned that 11 metres are 12 yards, and I think the man who told me would have been puzzled had he been told that a yard is eleven-twelfths of a metre. In all comparisons of length we really use multiples. If we say that 1 yard is 0·9144 metres, we scarcely convey a notion to most people who would quite understand 10000 yards being 9144 metres. Similarly, for general intelligibility, I would back against any fractional statement such approximations as 8 kilometres are 5 miles, 2 hectares are 5 acres, 5 kilogrammes are

11 pounds avoirdupois, 200 grammes are 7 ounces av., 4 litres are 7 pints, which the draper's information led me to calculate. I may state, by the way, that we come within one unit of the truth up to 1000 times the French units of measure (for metres up to 11000) by adding 1 part in 400 to the yards and pounds, subtracting 6 parts in 1000 from the miles, and adding the same to pints, and subtracting 12 parts in 1000 from the acres. The calculation is much easier than for decimals, and the results furnish admirable materials for exercising pupils in approximating to ratios of magnitudes arithmetically.

(vii.) Observe that if $mA = nB + D$, and we do not know the limit of the value of D, we can tell by the mere division $m'n \div m = n' +$ proper fraction, that $m'A$ lies between $n'B$ and $(n' + 2)\,B$, but that we cannot tell whether it lies between $n'B$ and $(n'+1)\,B$, or between $(n'+1)\,B$ and $(n'+2)\,B$, however great m may be. If, then, we want to find, not V, but $m'V$ within the limit E, we must find $m'mA = n''B + D'$, where $D' < B$. This is important in settling the limits of error, or "the number of decimal places required."

(viii.) But the processes of finding multiples, or throwing into a continued fraction, are alike illusory when certainty is required, as the suggested trials shew. Then arises the great problem of higher geometry: to find a series of terms (taken as geometrical magnitudes) continually diminishing, and *connected by a law* such that when a few are known any required number can be found, and such also that their (geometrical) sum continually approaches to the required limit, and may be made to differ from that limit by less than any assigned amount. The *practical* problem is then perfectly solved, but that practical problem gives birth to a *theoretical* problem. Suppose V to be the fixed limit toward which the series S converges, then $V-S$ will be a magnitude (a straight line, see ii.) of continually diminishing size, which can be made less than any assignable magnitude, while at every moment $V-(V-S) = S$. Can we then *neglect* $V-S$, and deal with S as if it were V, not merely for a practical approximation, but for theoretical exactness?

II. "Carnot's Principle" for Limits.

10. "*Carnot's Principle.*"—(i.) The only satisfactory answer which I have been able to find to the question just propounded, (and I have paid minute attention to the subject at various times for nearly 40 years,) is contained in *Réflexions sur la Métaphysique du Calcul Infinitésimal par* Carnot (3rd ed., Paris, 1839, pp. 254), which the name of the writer is enough to recommend to the careful study of all teachers. I wish here to state the principle in connection with the author's name, in that simple geometrical form which is suitable for learners, without any anticipation of the infinitesimal calculus.