III. The Laws of Tensor, or the Algebra of Proportions


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made less than any assignable, the last proportion may be written
\( C - D : C' - D' :: R : R' \); whence by Eucl. vi. 16,
\[ \text{rect.} (C, R') = \text{rect.} (D, R') = \text{rect.} (C', R) = \text{rect.} (D', R). \]

The second rectangles on each side may become less than any assignable, and hence, by "Carnot's principle," we have always, whether \( D, D' \) are large or small,
\[ \text{rect.} (C, R') = \text{rect.} (C', R), \text{ and rect.} (D, R') = \text{rect.} (D', R), \]
that is,
\[ b = c : c' : c' : c, \text{ and } d : d' : d' : d'. \]

The last result was not wanted, (although it is useful to draw attention to it when \( D \) is large,) and hence, as soon as we had stated the proportion as \( C - D : C' - D' :: R : R' \), we might have inferred
\[ C : C' :: R : R', \]
which was all of the truth we wanted, although not the whole truth.

(iii.) Apply the reduced process to find the ratio \( A : A' \) of the areas of the circles, \( Q : Q' \) being that of the areas of the similar polygons, which is the same as that of \( \text{sq. on } R : \text{sq. on } R' \). \( E \) and \( E' \) being the varying differences between the invariable areas of the circles and of the variable areas of the polygons, which differences may become less than any assignable areas, the constant proportion \( Q : Q' :: \text{sq. on } R : \text{sq. on } R' \) can be expressed as
\[ A - E : A' - E' :: \text{sq. on } R : \text{sq. on } R', \]
and hence by "Carnot's principle" we infer
\[ A : A' :: \text{sq. on } R : \text{sq. on } R', \]
and also, if required, \( E : E' :: A : A' \). This solves Art. 3. v.

(iv.) It is obvious that the application of this principle in higher algebraical geometry, the differential calculus, &c., is impossible unless we assume that we can deal with incommensurable expressions by the ordinary laws of commutative algebra. I have never seen any attempt to prove the justifiability of this condition, which seems to be taken as an axiom. Yet it is evident that if the limit of a convergent series is incommensurable, we cannot, from conclusions drawn from the (commensurable) sum of any finite, or ever increasing (infinite) number of its terms, conclude an exact relation which depends solely on its limit, until we know what are the laws by which we may calculate with incommensurables. The ordinary algebraical proof that \( \sqrt{2} \cdot \sqrt{3} = \sqrt{6} \), by "squaring each side," is absurd if we do not know the meaning of multiplying \( \sqrt{6} \) by \( \sqrt{6} \) or of multiplying \( \sqrt{2} \) by \( \sqrt{3} \). To this question, then, the next Tract is devoted.

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III. The Laws of Tensors, or the Algebra of Proportion.

12. Proportion expressed by Tensors.—(i.) Eucl. v. and vi. are assumed. It is also assumed that no ratio is known till two straight lines have been found having that ratio. And only known ratios are here dealt with.

(ii.) \( OI, \text{fig. } 3 \), is a straight line continued indefinitely beyond \( I \), on which
I is a fixed known point, OI being the standard length to which all others are referred. All other points mentioned, as A, B, C, &c., are supposed to lie on OI and on the I side of O. The length, position, and direction of OI are quite arbitrary. But for ulterior purposes I shall assume OI to be horizontal, and drawn from right to left.

(iii.) Given two points A and B, find a third point C, so that OI : OA :: OB : OC. This is a perfectly simple known geometrical operation (art. 5. iii.), not in the slightest degree involving, but also not excluding, commensurability. As this is an operation performed on OB, through the instrumentality of OA, (it is not necessary to consider the invariable OJ,) I designate it by the small letter a, of which A is the capital, (a relation between forms of letters constantly observed,) and call it a tensor, (the name is borrowed from Sir W. R. Hamilton,) and I write

\[ OC = a \cdot OB \]

(read a. OB as “a ante OB”),

to show that OC is the result of performing the operation called the tensor a, upon the operand OB. Hence this equation has no other meaning than the original proportion OI : OA :: OB : OC.

(iv.) Since OI : OA :: OB : 00 gives by Eucl. v. 16, OT : OB :: OA : 00; and hence, by precisely the same process as before, c = ba, and hence \( ab = ba \), or the multiplication of tensors is commutative.

13. Commutative Multiplication of Tensors. (i.)—Now let \( ab \) be a symbol which has the same resultant meaning as c, but shews that c has been reached by the two operations a, b, performed in the order of the equation in (art. 12. iv.), then \( c = ab \) (read ab as “a ante b”), represents the effect of that equation without the use of OI, and defines what may be termed the multiplication of tensors, the order of the symbols being observed. It must be remembered that no knowledge of arithmetical operations is assumed.

(ii.) But OI : OA :: OB : OC gives by Eucl. v. 16, OI : OB :: OA : OC; and hence, by precisely the same process as before, \( c = ba \), and hence \( ab = ba \), or the multiplication of tensors is commutative.

(iii.) A product of tensors represents a compound ratio; compare art. 19. iii.

14. Division of Tensors.—Given any two of the three points A, B, C, the third may be found by well-known tensor operations. Express them by writing \( a = \frac{c}{b}, \quad b = \frac{c}{a}, \) (read \( \frac{a}{b} \) as “c super b”).

Then

\[ \frac{c}{b} \cdot b = ab = c, \quad a = \frac{c}{b} = \frac{ab}{b}, \]

which are the two laws of division of tensors.

15. Associative Multiplication of Tensors.—(i.) Let

\[ OI : OM :: OA : OB \] \hspace{1cm} (1),

\[ OI : OM :: OC : OD \] \hspace{1cm} (2),

whence

\[ OA : OB :: OC : OD \] \hspace{1cm} (3),

and also

\[ OA : OC :: OB : OD \] \hspace{1cm} (4),

and

\[ OB : OA :: OD : OC \] \hspace{1cm} (5).
Then, by art. 14, \( m = \frac{b}{a} \), \( m = \frac{d}{c} \), and \( \frac{b}{a} = \frac{d}{c} \) ............(6); and hence equation (6) becomes the tensor expression of the proportion (3); and as this involves (4) and (5), which may be similarly expressed, we find that

if \( \frac{b}{a} = \frac{d}{c} \), then \( \frac{a}{b} = \frac{c}{d} \), \( \frac{c}{a} = \frac{d}{b} \), and \( \frac{a}{c} = \frac{b}{d} \) ............(7);

whence follow various results, and among others,

if \( \frac{b}{a} = \frac{d}{c} \), then \( \frac{b}{a} \cdot c = \frac{d}{c} \cdot c = d \), and \( \frac{c}{a} \cdot b = \frac{d}{b} \cdot b = d \),
or

\( \frac{b}{a} \cdot c = \frac{c}{a} \cdot b \) .................................. (8),

which is, in fact, the great law of association of tensors in multiplication, which may be put in the usual form thus:

(iii.) Given \( m = ab \), find \( n = bc \); then \( a = \frac{m}{b} \), \( c = \frac{n}{b} \), and

\( ab \cdot c = m \cdot c = \frac{n}{b} = n \cdot \frac{m}{b} = n \cdot a = bc \cdot a \),
or

\( ab \cdot c = a \cdot bc \),

the usual form of this law, the extreme importance of which, and its occasional independence of the law of commutation, is well shewn by the laws of quaternions.

16. Results for Multiplication and Division of Tensors.—(i.) The following are immediate results of these laws:

If \( p \) is any tensor,

\[ \frac{b}{p} \cdot \frac{a}{p} = \frac{b}{p} \cdot \frac{a}{p} = \frac{p}{a} \cdot a = a, \quad \text{or} \quad \frac{p}{a} = \frac{a}{b}, \]

and

\[ \frac{a}{b} \cdot \frac{c}{d} = \frac{a}{b} \cdot \frac{c}{d} = \frac{a}{d} \cdot \frac{d}{b} = \frac{a}{d} \cdot \frac{d}{b} \]

and

\[ \frac{a}{b} + \frac{c}{d} = (\frac{a}{b} \cdot bd) + (\frac{c}{d} \cdot bd) = \frac{ad}{bc} ; \]

and if \( \frac{a}{b} = \frac{c}{d} \), then \( \frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd \), or \( ad = bc \), &c.

(ii.) The whole of the laws of commensurable fractions, so far as multiplication and division are concerned, therefore, hold for (indifferently, commensurable or incommensurable) tensors, and each equation represents a geometrical relation between points on the line \( OI \), that is, between pure lengths.

17. Addition and Subtraction of Tensors.—(i.) To find the point \( C \) from \( A \) and \( B \), by the addition of \( OA \) to \( OB \) in either order, as

\[ OA + OB = OC = OB + OA, \]

is a geometrical commutative operation. It must now be shewn (v.), that under these circumstances we can find \( B \) from \( A \) and \( C \), or \( A \) from
(i.) Expressed by means of tensors, these equations give
\[ a \cdot OI + b \cdot OI = c \cdot OI = b \cdot OI + a \cdot OI, \]
\[ a \cdot OI = c \cdot OI - b \cdot OI, \quad b \cdot OI = c \cdot OI - a \cdot OI, \]
\[ (c \cdot OI - b \cdot OI) + b \cdot OI = c \cdot OI, \quad (a \cdot OI + b \cdot OI) - b \cdot OI = a \cdot OI. \]

(iii.) Taking then the symbol \( a + b \) to mean \( c \), as derived from \( OC \),
thus related to \( OA \) and \( OB \), and similarly for \( c - b \), we have
\[ a + b = c = b + a \quad \text{(1)}, \]
\[ a = c - b, \quad b = c - a, \]
and
\[ (c - b) + b = c, \quad (a + b) - b = a. \]

(iv.) From these definitions it follows that in \( c - b \) the \( c \) is always
greater than \( b \), that is, derived from a line \( OC \) longer than \( OB \). No
meaning can be given to \( b - c \), because \( OB - OC \) has no meaning in the
geometry of magnitude. The algebra of tensors has therefore, as re­
gards subtraction, the same defect as the ordinary algebra of com­
mensurables. The resultant difficulty of negatives and consequent imagi­
naries is overcome in the next section.

(v.) To form \( OC - OB \), we set off a line \( CA = OB \) from \( C \) towards
\( O \), and find \( A \) such that \( OA = OC - OB \). Now although \( CA \) could not
project beyond \( O \), the point \( A \) might lie on \( O \) itself, in which case \( B \)
lies on \( O \), and we may shew this by writing \( OO = OC - OB \), wherever
\( C \) may lie. Now from the point \( OO \) we can form no tensor, because
\( OI, OO \) can have no ratio, as no multiple of \( OO \) has any length (art. 1.
ii.). But the operation indicated by \( OC - OC \) is that of reducing any
length to a point, forming one of its extremities; and if we represent
this by \( o \), we may inquire what are the laws of \( o \).

(vi.) First, by the nature of \( o \), whatever be \( C \), \( o \cdot OC = OC \), that is,
\( o \cdot (c \cdot OI) = o \cdot OI \), so that we may write \( o \cdot c = o \), whatever be \( c \).
This equation, wherever it arises, shews that \( c \) is indeterminate. It
also shews that if any conditions require \( o \cdot c = b \), where \( b \) is not \( = o \),
those conditions are impossible. That is, there is no point \( C \) at all
which will answer the condition that, when you measure from \( O \) to \( C \)
and then back to \( O \), you should stop at a point \( B \) short of \( O \). This is
the one geometrical impossibility of simultaneous coincidence and
separation (Appendix I.). \( C \) is not “at infinity,” as usually stated.
And with this the whole of the conceptions deprecated in art. 6. i. fall
to the ground, to be replaced by intelligibilities. See the observations
and citations in Appendix I.

(vii.) Then from the proved associative character of geometrical ad­
dition and subtraction, namely, that
\[ (OA \pm OB) \pm OC = OA \pm (OB + OC), \]
taking all the upper or all the lower signs, we find for tensors
\[ (a \pm b) \pm c = a \pm (b \pm c), \]
provided always that the subtractions be possible. And thence we
obtain the further laws of \( o \) : \( a \pm o = a \), whatever be \( a \).

(viii.) Since \( i, o \) answer to the arithmetical \( 1, 0 \), the laws of addition
give \( i + i \) as \( 2, 2 + i \) as \( 3 \), and so on. But in these Tracts the geometrical
III. THE LAWS OF TENSORS. [ART. 17. viii.—19. iii.

operations i, o will be kept distinct from 1, 0. Thus geometry and arithmetic will be completely separated. The figures 2, 3, &c. represent certain arithmetical operations so that \(2\mathbf{a} = \mathbf{OA} + \mathbf{OA},\) \(2a = a + a,\)
\[2i = i + i,\]
\[2\mathbf{i} = \mathbf{i} + \mathbf{i}\]
&c. Again, \(a^2 = a \cdot a,\) \(i^2 = i \cdot i,\) and so on. While \(\mathbf{o} \cdot \mathbf{OA} = \mathbf{OA} - \mathbf{OA},\)
\[\mathbf{o} \cdot a = a - a,\]
\[\mathbf{0} \cdot \mathbf{i} = \mathbf{i} - \mathbf{i}.

18. Distributive Character of Tensors.—(i.) Let A, B, C, D, P be such points that \(pa = b,\) \(pc = d,\) whence \(\frac{a}{c} = \frac{b}{d},\)
or \(\mathbf{OA} : \mathbf{OC} :: \mathbf{OB} : \mathbf{OD},\) by Art. 15. i. (6.). Then Euclid v. 17 and 18, supposing \(\mathbf{OA} > \mathbf{OC}\) for the lower sign,
\[(\mathbf{OA} \pm \mathbf{OC}) : \mathbf{OC} :: (\mathbf{OB} \pm \mathbf{OD}) : \mathbf{OD},\]
or \(\frac{a \pm c}{c} = \frac{b \pm d}{d},\)
whence \(b \pm d = \frac{d}{c}(a \pm c),\) or \(pa \pm pc = p \cdot (a \pm c),\)
that is, tensors are distributive.

(ii.) And taking, therefore, \(p \cdot (a - a)\) to mean \(pa - pa,\) we have \(p \cdot o = o,\) as well as \(o \cdot p = o\) (art. 17. vi.), or \(o\) is commutative with any tensor.

19. Applications of Tensors.—(i.) This completes all the laws that need be adduced: powers with integral coefficients follow immediately, roots become intelligible as expressed either geometrically (for square roots, and for any roots by "Peaucellier's cell"), or as tensor limits to converging sums of tensors or fractions (art. 11. iv.). The whole of arithmetical algebra has been shown to hold for tensor algebra, which also includes incommensurable algebra.

(ii.) The application to all the numerous cases for which merely quantitative geometry is used, is too evident to need explanation. And if, when \(\mathbf{OM} = \mathbf{AB},\) we agree to represent the tensor \(m\) by \(\mathbf{AB}\) in calculations, the whole of the usual algebra of quantitative geometry becomes exact, without any trace of limits. Moreover it would be possible to take proportion immediately after the theory of parallels, and prove by tensors numerous propositions which, although usually (and very properly) otherwise proved, it will be a useful exercise to prove by tensors.

(iii.) The whole of Euclid ii. admits of this treatment. The algebraical proofs of these propositions, so much eschewed, now become strictly geometrical. Thus \((AC + CB)^2 = AC^2 + CB^2 + 2AC \cdot CB,\)
with the conventional notation of (ii.), is a tensor relation.

Referring each term to \(\mathbf{OI},\) it becomes a relation of lengths only, and as such should be drawn by the learner, that he may fully feel its meaning. Thus take \(\mathbf{OM}, \mathbf{ON}, \mathbf{OP}, \mathbf{OQ}\) so that in lengths \(\mathbf{OI} : \mathbf{AC} :: \mathbf{OM},\)
\(\mathbf{OI} : \mathbf{CB} :: \mathbf{CB} : \mathbf{ON},\)
\(\mathbf{OI} : \mathbf{AC} :: \mathbf{CB} : \mathbf{OP},\)
and \(\mathbf{OI} : \mathbf{AB} :: \mathbf{AB} : \mathbf{OQ}.

The meaning of the equation is that \(\mathbf{OQ} = \mathbf{OM} + \mathbf{ON} + 2\mathbf{OP}.\)

Referred to the square on \(\mathbf{OI},\) it becomes the relation of rectangular areas, which is figured in Euclid ii. 4; for (fig. 4.)
since \(\mathbf{OI} : \mathbf{OM} :: \text{rect.} (\mathbf{OI}, \mathbf{OP}) : \text{rect.} (\mathbf{OM}, \mathbf{OP}),\)
we have \(\text{rect.} (\mathbf{OM}, \mathbf{OP}) = m \cdot \text{rect.} (\mathbf{OI}, \mathbf{OP}).\)
Similarly, on taking $OJ$ on $OP$ of the length $OI$, so that
\[
\text{rect. } (OI, OJ) = \text{square on } OI,
\]
we have
\[
OJ : OP :: \text{rect. } (OI, OJ) : \text{rect. } (OI, OP),
\]
so that, on using $p$ for the tensor of $OP$,
\[
\text{rect. } (OI, OP) = p \cdot \text{square on } OI,
\]
whence
\[
\text{rect. } (OM, OP) = mp \cdot \text{square on } OI,
\]
on using the ordinary notation. The proof is here conducted by proportion only, and quite independently of commensurability, so that the objections to the "algebraical proof"—really, commensurable proof—of Euclid ii. no longer hold.

Referred to cube on $OI$, it becomes a relation of rectangular solids, having one constant dimension $OI$.

(iv.) The demonstration of all these relations flows at once from the laws of tensors. But there is no room for negatives or imaginaries in an algebra derived from the geometry of magnitude only. The laws of both are obtained at once from the geometry of direction, as follows.

IV. THE LAWS OF CLINANTS, OR THE ALGEBRA OF SIMILAR TRIANGLES LYING ON THE SAME PLANE.

20. Data from the Geometry of Direction.—(i.) The following propositions are borrowed from the geometry of direction, as opposed to that of ratio, or magnitude only. See fig. 5.

(ii.) $AB$, without further limitation, always represents the line $AB$, as respects both magnitude and direction, considered as the trace of the motion of a point along a straight line from $A$ to $B$, $A$ being its initial and $B$ its final point. When length is considered without direction, write $\text{len } AB$, and read "length of $AB".$

(iii.) The equation $AB = CD$ implies that $AB$ and $CD$ are the opposite sides of the parallelogram $ABDCA$, the points lying in this order.

(iv.) Directional Addition is defined by the equation
\[
AB + AC = AB + BD = AD,
\]
or the sum of two adjacent sides of a parallelogram measured from their point of intersection is the included diagonal. And since in this case
\[
AC + AB = AC + CD = AD,
\]
directional addition (which is quite different in its results from quantitative or rational addition) is commutative. It is also easily shewn to be associative.

(v.) Directional Subtraction is defined by the equation
\[
AB - AC = AB + CA = AB + BE = AE = CB,
\]
or the directional difference of two adjacent sides of a parallelogram is the transverse (non-included) diagonal, measured from the final extremity of the subtrahend to that of the minuend.

(vi.) $OO_1$, $OO_1'$ (fig. 6 and 8) are diameters of a standard (or unit) circle, drawn at right angles. The position of centre $O$ (taken as origin) and radius $OI$, (taken as the standard of length and of original direction,) and the direction of right angle $IOJ$, (taken as standard of angular rotation,) are arbitrary, but once fixed remain throughout the problem, and deter-