(2) If c^2-ab is not $= o$, then, fig. 21, putting $2m = a+b$ and $2p = a-b$, on adding m^2 to each side we have

$$(x-m)^2 = m^2-ab+c^2 = p^2+c^2 = n^2 \text{ or } n'^2,$$

where N, N' are constructed as in art. 33. v., and $x = m+n = x'$, or $x = m-n = m+n' = x''$. The mean bisectors of $X'A, X'B$ and $X''A, X''B$ are $X'E$, $X'E'$ and $X''F$, $X''F'$, which are $= OC$, OD respectively.

(3) If $p^2+c^2 = o$, or OP, OC are of the same length and at right angles to each other, fig. 22; then $n = n' = o$, and the two positions X', X'' coalesce at M, so that there is only one position which satisfies the conditions. The mean bisectors of MA, MB are MG, MG'.

(x.) To determine the points where a line perpendicular to OA cuts a circle with radius OB.

As in iii. (2) we have in the line $S.xRa = i$, and as in (vi.) for the circle $T^2x = T^2b$. Then, by art. 26. vi., $xRa+K.xRa = 2SxRa = 2i$, and $xKx = T^2b$, whence eliminating Kx, we have

$$x^2 = 2ax - U^2a.T^2b,$$

and $$x = a \pm Ua\sqrt{(T^2a-T^2b)};$$

whence $$x.Ra = i \pm RTa.\sqrt{(T^2a-T^2b)}.$$

Unless then $Ta = < Tb$, $S.xRa$ will not $= i$, and this is therefore the condition of possibility. There are no "imaginary" intersections. No "imagination" can make $i = i \pm k$, where k is not $= o$, for this would lead to the impossibility of Appendix II. A circle and straight line have therefore *no* "imaginary" intersections. This term applies only to a derived case, considered in art. 49. v. The meaning of this distinction is assigned in art. 36. v.

When $Ta=Tb$, $x=a$, and there is only one point of intersection A. When $Ta<Tb$, $x = a \pm j.Ua.\sqrt{(T^2b-T^2a)}$, which gives the two points determined by drawing $X'AX''$ perpendicular to OA, and making len AX' = len AX'' = length of the perpendicular of a right-angled triangle, of which the lengths of base and hypothenuse are the lengths of OA and OB respectively.

V. Stigmatic Geometry, or the Correspondence of Points in a Plane.

35. *No previous complete representation of Algebra by Geometry.*—Some of the results hitherto adduced have been already obtained (although less directly, and always by a more or less implied use of limits) from various geometrical "explanations" of "imaginaries," advanced with some degree of hesitation, often on metaphysical grounds, and (except by Sir W. R. Hamilton) always by means of "complex numbers," or clinants of the form $Sa+jWa$, where Sa, Wa were considered as the limits of convergent "possible" (that is, scalar) series. The class of problems embraced under the theory of Stigmatics have also been attacked with immense acuteness and wide success, in particular instances, but the occurrence of imaginaries have constantly baffled

the very lions of mathematical science, towards whom I feel but as the mouse that gnaws their net asunder by my clinant teeth. My firm belief is that there is not known to exist any intelligible, workable general theory but my own, nay, even any tenable, hypothetical particular explanation of the geometry of those imaginaries which constantly occur in the algebraical plane geometries of Descartes and Plücker, or the higher plane geometry of Chasles; and that, until such a general theory has been furnished, there is no complete representation of geometry by algebra, or of algebra by geometry. The solution of this problem, the furnishing of one general theory which will embrace *all* cases of plane geometry from a single simple point of view, which shall never meet with any difficulties by the way from "imaginary" lines, "imaginary" angles, or "imaginary" figures; which shall make every step in every problem a pure piece of geometry (conceding the division of angles in any ratio and the interposition of any number of geometrical means between two extremes); *which shall, in fact, identify Algebra with Geometry*,—this has been the ideal of my mathematical life, and I believe that it has at length been realised to the letter by means of my clinants and stigmatic geometry.

Other labours have hitherto prevented me from sending it out in the form I have always wished to give it, with numerous illustrative and comparative diagrams; and I am now so far advanced in life that my power ever to do so becomes very problematical. The following brief notes, which contain my last unpublished notations and nomenclature, will enable any one of those distinguished mathematicians to whom they will be sent, if he finds time to scan them, to apply my theory far better than I could do it myself. Those who care to learn the history of the birth and growth of my conception of Stigmatic Geometry will find it in Appendix III. On the facts therein detailed, and on the citations from the works of eminent mathematicians in Appendix II., I distinctly claim originality for a conception, in forming which I have not obtained a scrap of help from the best writings of the best writers that I could consult. The mouse asserts her teeth.

36. *General Conception of Stigmatic Geometry.*—(i.) Let X and Y, fig. 23, be two points on a plane, connected by the clinant equation $f(x, y) = o$, which, so far as it can be solved, or so far as the properties of clinant equations are known, will enable us to construct the different positions of Y for every assumed position of X, (that is, with certainty so far as biquadratic equations extend,) and to deduce various relations between X and Y in all other cases. The continuous correspondence of the points X and Y, given by any such law, while X moves continuously over the plane, forms a *stigmat'ic*. The point X, which moves independently, is called the *in'dex*, and geometrically represents the independent variable x. The point Y, which is determined from X by the given law $f(x, y)$, is called the *stig'ma*, and geometrically represents the dependent variable y. The pair of corresponding points, index and stigma, is termed a *stig'mal*, (*stigm*-a + *al*; see an explanation of the origin of this nomenclature in Appendix III.,) and is written (XY), or (xy), or (x, y), according to convenience. The line OX is called the *abscis'sa*, the line XY the *or'dinate*, and the line OY the *ra'dius*

of the stigmal (xy), and x, $y-x$, y are their clinants respectively. These three lines form the sides of the *stigmal triangle* OXY. To each index there may correspond several stigmata, in the same or different stigmatics. Stigmals with a common index are called *co-stigmals*, and their stigmata are called *co-stigmata*.

(ii.) The points X, Y are said to be *co-ordinated* by the equation $f(x, y) = o$. If by simple geometrical constructions X', Y' can be determined from X, Y, so that X', Y' may be co-ordinated by a derived equation $f(x', y') = o$, then X, Y are said to be *trans-ordinated* to X', Y'; and the second stigmatic is said to be a *transordination* of the first. Such transordinations are frequently convenient for the purpose of simplifying the discovery of the points X, Y by means of the points X', Y'. The general theory is given in art. 47. Thus we may form *subsidiary stigmatics* having the same index X, but different stigmata U, V, by putting, as in fig. 23, 24, $y-x=v$, $ju=v$, $y=x+v=x+ju$, whereby the stigmatic equations become

$$f(x, x+v) = o, \quad f(x, x+ju) = o,$$

forming the connected *ordinar* and *orthar* stigmatics, which are related to the original stigmatic, stigmal for stigmal, as particular cases of transordinated stigmatics. If from the orthar stigmatics we *select* those particular stigmals for which both x and u are *scalars* (fig. 24), the stigmata of the corresponding stigmals form the *real* points of Cartesian plane geometry referred to *rectangular* co-ordinates, the Cartesian axes of the abscissae and ordinates being OI OJ; and all stigmata for which the one or the other or both of the points X, U do not lie on OI, or V does not lie on OJ, form the *imaginary* points of Cartesian plane geometry so referred. If (no figure) we make $v = hu'$, where h is any unit radius, $y = x+hu'$, and the new stigmatic is $f(x, x+hu') = o$, from which those stigmals (xy) for which x, u' are scalar, have as their stigmata the *real* points of Cartesian plane geometry referred to the *oblique* co-ordinates of which OI, OH are the axes. For comparing stigmatic and Cartesian geometry it is convenient to have special names for these cases, which may be provided by the prefixes *Cartesian* (abbreviated to *car*-,) and *non-Cartesian*, more briefly *incar*- (*in* = negative + *Car*-tesian). Thus *carstig'mal*, *carstig'ma*, *carin'dex*, and so forth. *Carstig'mata*, are "real points;" not simply geometrical points, but points referred by ordinates to other points in the axis of the abscissae; *incarstigmata* are "imaginary" points, that is, points which the former algebra indicated should be similarly referred, but which no one had been able to refer on the old theory, and hence merely "imagined" to be so referred, in order to preserve the old terminology. Rectangular co-ordinates will be assumed unless otherwise expressed, but the prefixes *rec*-, *ob*-, will distinguish the two cases. A carstigmatic is that part (if any) of a stigmatic for which the stigmals are carstigmals. A Cartesian stigmatic contains a carstigmatic, that is, some carstigmals, but also contains incarstigmals.

(iii.) As any plane geometric curve whose properties are known may be treated as a carstigmatic, and expressed by $f(x, x+ju) = o$, with the condition that x, u are scalar; and as this can be immediately thrown into the general form $f(x, y) = o$, which will agree with the former as long as x, u are scalar, and which will *also* give all the relative positions of Y, when x is still scalar, but u not scalar, (that is, "imaginary,") or even when x is also not scalar,—it is evident that every result from any Cartesian form can be immediately included in its proper general clinant stigmatic, in which shape it is usually much easier to treat. "Imaginary" points can only thus arise in Cartesian Geometry; compare art. 34. x. If we further proceed to make the constants clinants, that is, refer them to any point on the plane, instead of those from which the scalar case was deduced, any such particular carstigmatic will suggest a still more general stigmatic, which is equally easy to treat, and is the only form which fully shews the geometrical relations.

(iv.) Stigmatics are said to *intersect* in their *common* stigmals or *stin'nals* (*sti*-gmals of *in*-tersection + *al*), of which the stigmata and indices are called *stig'mins* (*stigm*-ata + *in*-tersection) and *indins* (*ind*-ices + *in*-tersection) respectively. The laws of such intersection are now *precisely* those in Plücker's *Theorie der algebraischen Curven* (Bonn, 1839), the *whole* of which, transferred to stigmatic geometry, after the following theory of primals and quadrals is understood, may be interpreted as *strictly* geometrical.

(v.) When the index moves on any path, the stigma moves on another path, corresponding point by point; these are the *in'dit* (*ind*-icis *it*-er) and *stig'mod* (*στίγμ-ατος ὀδ-ός*). All indits which intersect in the index of a stinnal, have stigmods which intersect in its stigma. In carstigmatics the indit is a straight line, part or all of the Cartesian axis of abscissae, and the stigmod is that curve which was alone considered when Descartes founded his algebraical geometry, by referring any curve, point for point, to the axis of the abscissae by ordinates parallel to the ordinate axis. This reference was the egg from which the present stigmatic geometry was hatched. It was an addition to the ancient geometry, invented as a mere expedient for reducing it to algebraical computation, without any perception of the principle involved. It is evident from the preface to Chasles's *Géométrie Supérieure* that he had not recognised this principle as identical with that of his own homographic geometry. But the fact of the identity of principle is shewn by the present inclusion of *both* as particular cases under Stigmatic Geometry, so that the method of working the two becomes indistinguishable. It will be seen, also, that the clinant stigmatic view is the only one which perfectly explains the principles of "signs" and "continuity." A carstigmod differs from a simple curve of the same form, by its *implying* a carindit, to which it is referred. The distinction is important. Thus when a simple straight line does not cut a simple circle, the line and circle have only to be considered as carstigmods, and Cartesian stigmatics are generated, which *do* intersect, although only in two in-carstin'nals. Compare art. 34. x. with art. 49. v.

(vi.) From the theory of intersection, the analogous theories of contact (of any order) and asymptoticity may be immediately deduced. If $f(x,y) = f_1(x,y) \,.\, f_2(x,y) + c = o$, then $f_1(x,y) = o$, and $f_2(x,y) = o$ give stigmatics which have no stigmal in common with $f(x,y) = o$, but, as X recedes, have stigmata continually approaching to the co-stigmata in the original stigmatic, and are hence called its *asymp'tals* (*asympt*-otes + *al*).

(vii.) There is nothing in the form of the stigmatic equation $f(x, y) = o$ to distinguish the index from the stigma. Either may be assumed as either, but the two stigmatics thus formed necessarily differ, unless the equation is symmetrical with regard to x and y, as in $(s-x)(s-y) = (s-e)^2$, see art. 44. Given the *direct* stigmatic, with X as index, and Y as stigma, the *inverse* stigmatic, with Y as index and X as stigma, is the geometrical representative of the inversion of functions, which can be here only indicated. In this case one stigma may have many indices, giving *con-in'dices* and *con-indic'ial* stigmals.

(viii.) From the general conception of functions the meaning of clinant differential and clinant integral calculus, &c., is given. These are the only points which I have not yet worked out in detail. But the indications in Sir. W. R. Hamilton's *Elements of Quaternions*, Book III. chap. ii., in Martin Ohm's *Geist der Differential- und Integral-Rechnung* (Erlangen, 1846), in Casorati's *Teorica delle Funzioni di Variabili Complessi* (Pavia, 1868), in Hankel's *Vorlesungen über die Complexen Zahlen und ihre Functionen* (Part I., Leipzig, 1867, Part II. will be the especial part when published), will suffice, with the present indications, to work out this part of the complete reconstruction of plane geometry. For the differential calculus, Taylor's theorem holds, and processes analogous to those for maxima and minima, and for tangents, immediately follow.

37. *Integral Stigmatics*—(i.) Henceforth attention will be confined to the *integral stigmatic* equations of the form

$$x^m . (ay^n + a'y^{n-1} + \ldots) + x^{m-1}(by^n + b'y^{n-1} + \ldots) + \ldots = o,$$

where m and n are integers and the other letters clinants. This is the fundamental form of equation assumed by Chasles in his Theory of Characteristics, (*Comptes Rendus*, 27 June, 1864, vol. 58, p. 1175), the whole of which theory (after primals are understood) may be incorporated in stigmatics, and applied to any points on a plane.

(ii.) Dividing by y^n, the sum of the terms not containing powers of y in the denominator is $ax^m + bx^{m-1} + \ldots$, and if we put this $= o$, we shall obtain m values of x, which, when substituted for x in the original equation, have no corresponding values of y. These point out m *solitary indices*, having *no* corresponding stigmata. Similarly $ay^n + a'y^{n-1} + \ldots = o$ gives n *solitary stigmata*, which have *no* corresponding indices. If we put $x = y = z$, we find an equation of $m+n$ dimensions in z; these give $m+n$ *double points* Z, in which the index *coincides* with the stigmata. When any one point is *at once* a solitary index and a solitary stigma, it is termed simply a *solitary point*. The above are called the *peculiar points* in a stigmatic.

(iii.) Of this general form I shall give only the fundamental cases of *primal* (arts. 38. to 42.), *uniquadral* (arts. 43. to 46.), and *duoquadral* (arts. 48. to 51.) stigmatics, but none will be treated with even a distant approach to detail. My second memoir on Plane Stigmatics, when the nomenclature is properly changed in accordance with that here used, and the notation altered by putting the present $b-a$ and $(b-a)(d-c)$ for the ab and $ab . cd$ there used, gives sufficient details to shew the power of the method; but it is impossible to abstract, much less to reproduce in the present improved form, the whole even of that memoir (itself a mere sketch) within the time and space at my command.

38. *Primals, or Cartesian Straight Lines generalised.*—(i.) The simple stigmatic equation $a'x+b'y+c'=o$, can, when b' is not $=o$, be reduced to the form $y+(a-i).x=b=ac$,

which is the standard form of a primal stigmatic. There is no solitary index or stigma. C is the double point, B is the *original* point, that is, the stigma when the origin is taken as index. A is called the *direction point*, the triangle $IOA \bigtriangleup CXY$ (fig. 25) being the *direction triangle.* As it is necessary to become familiar with the geometrical relations of the primal, the reader should construct many figures with different positions of A, B, and hence C, beginning with cases where A and B lie on OJ, and C on OI, for which CB is the ordinary Cartesian line, as in fig. 34, and if X is chosen on OI, XY is parallel to OJ. But positions of X_1 not on OI should also be chosen, and the abscissa OX_1 and ordinate X_1Y_1 then give the imaginary Cartesian abscissa and ordinate of the imaginary point Y_1. Fig. 25 gives a general case, and will indicate the method to be pursued.

(ii.) Any two stigmals (xy), $(x'y')$, or (xy), (cc), or (xy), (ob), or (cc), (ob), will determine a primal, which may be written pri $(xy, x'y')$, &c. The direction point A and any stigmal (xy) or (cc) will also determine a primal, which may then be written pri (A, xy) or pri (A, cc) &c., the capital letter distinguishing the point. A primal is said to be *drawn* when a quadrilateral $XYY'X'X$ has been constructed by joining the extremities of the ordinates XY, $X'Y'$. In drawing stigmatics generally it is convenient to guide the eye to the correspondences by making the stigmod YY' an unbroken line ————, the indit curve XX' a broken line — — —, and the ordinates XY, $X'Y'$ dotted lines .... This will make the constant directional similarity, $CXY \bigtriangleup IOA$, very evident in the primal.

(iii.) The general form does not hold when $b'=o$, in (i.) In this case $x=o$, or $x=m$, and there is no direction point. The following eight peculiar cases occur so frequently that I have found it convenient to give them special names; they are here given in terms of both y and $v=y-x$, see art. 36. ii., for which the general equation becomes $v+ax=b$. Assume $m+m'=o$.

Name and Equation.	direction point.	original point.	double point.
I. *Ax'als and Parax'als.*			
or'dinal, $x=o$	none	O	O
paror'dinal, $x=m$	none	none	M
abscis'sal, $y=x$, $v=o$	O	O	all
parabscis'sal, $y=x+m$, $v=m$ ...	O	M	none
II. *As'sals and Paras'sals.*			
u'nal, $y=o$, $v+x=o$..............	I	O	O
paru'nal, $y=m$, $v+x=m$........	I	M	M
du'al, $y=2x$, $v-x=o$	I'	O	O
paradu'al, $y=2x+m$, $v-x=m$	I'	M	M'

The name *axal* (*ax*-is + *al*) is given from the relation of these primals to the Cartesian axes, and the name *assal* (*as*-ymptote + *al*), because these primals are the asymptals of a cyclal (art. 48. v.), the so-called "imaginary asymptotes" of a circle. The prefix *par-*, or *para-*, denotes the sameness of the direction points, or *para*-llelism of the primals. If in the quadrilateral $XYY'X'$ of (ii.) the two indices X, X' coalesce in X_1, then pri (x_1y, x_1y') is a parordinal with constant index; but if the two stigmata Y, Y' coalesce in Y_2, then pri $(xy_2, x'y_2)$ is a parunal, with constant stigma. If the ordinate $XY =$ ordinate $X'Y'$, then pri $(xy, x'y')$ is a parabscissal with constant ordinate. If the line YY' joining two stigmata is always equal to *double* the line XX' joining the two corresponding indices, then the pri $(xy, x'y')$ is a paradual. In fig. 33 pri $(pe, \epsilon e)$ is a parunal, and pri $(pe, \phi f)$ a paradual; and in fig. 26, pri (mm, m_1i) is a parunal, and pri $(t't, ot)$ a paradual; in fig. 34, pri $(oo, x'o)$ is a unal, and pri $(oo, x'y_0)$ a dual, and these two are there the asymptals of the cy'clal; see art. 48. v.

(iv.) Given two stigmals (pq), $(p'q')$ to find, fig. 25, the direction point A, original point B, and double point C. Make $p-r = q-q'$, then $\frac{p-r}{p-p'} = i-a$, or $P'PR \bigtriangleup OIA$ giving A, and $\frac{q-p}{c-p} = a = \frac{r-p'}{p-p'}$, or $CPQ \bigtriangleup IOA \bigtriangleup PP'R$ giving C from A or from (pq), $(p'q')$, direct, and $CPQ \bigtriangleup COB$ giving B.

(v.) If two stigmals (pq), $(p'q')$ are given, any other stigmal (xy) can be found without previously constructing A, B, or C, by putting the equation to the primal into the form $\frac{x-p}{x-p'} = \frac{y-q}{y-q'}$, or $XPP' \bigtriangleup YQQ'$, which also shews that *every stigmod of a primal is similar to its own indit* (compare the stigmod $CQQ'YC$ with indit $CPP'XC$, fig. 25), and is the condition that three stigmals (xy), (pq), $(p'q')$ should be *coprimal*, or lie on one primal. As this equation is satisfied by $m = \frac{1}{2}(p+p')$ and $n = \frac{1}{2}(q+q')$, (mn) will be a stigmal on the pri $(pq, p'q')$. This stigmal (mn) is called the *middle* stigmal between the stigmals (pq), $(p'q')$, and is said to *bisect* the *chordal* $(pq, p'q')$, bounded by the stigmals (pq), $(p'q')$, or to be its *bisec'tional.*

(vi.) It is evident that if we take *any* set of points in a plane, and, considering them as stigmata, refer *two* of them to *any* other two points as indices, we can by (v.) construct indices to all the other points so that they should lie on a primal. All points in a plane may therefore be considered as stigmata of a primal, of which two indices are determined arbitrarily, and may be chosen so as to satisfy certain conditions. In particular, the points thus regarded as stigmata may be themselves indices and stigmata of any stigmatic. In this way is formed the homma-primal, from the stigmatic called a hommal, in fig. 33; see art. 46. iv. Generally the new primal thus formed may be called a *stigmato-primal.* The stigmals on these primals, which have former indices as their stigmata, may be distinguished as *indi-stigmals* (*indi*-cis + *stigmal*), and the others as *stigmo-stigmals* (*stigm*-at-*o*-s + *stigmal*). These terms save long periphrases in cases of frequent occurrence.

39. *Intersections of Primals.*—(i.) Let

$$y + (a-i)\,x = b = ac, \quad y + (a'-i)\,x = b' = a'c'$$

be two primals (for which a Cartesian case has been taken in fig. 26), it is easy to determine their stinnal (hk) from $(a-a')\,h = b-b'$, or from $\frac{h-c}{h-c'} = \frac{a'}{a}$, that is, $CHC' \bigtriangleup A'OA$. When merely two stigmals are given in each, it is generally most convenient to find A and C as in art. 38. iv., and apply this form.

(ii.) If two pairs of co-stigmals are given, forming the primals (xp, $x'p'$), (xq, $x'q'$), and (hk) be their stinnal, then $\frac{p-p'}{q-q'} = \frac{p-k}{q-k}$, which shews that the stigmin K is the double point of the pri (pq, $p'q'$), from which property it may be immediately constructed as before, and then the indin H can be found from either primal.

(iii.) A parordinal $x=m$ has a constant index M, and hence (mn) its stinnal with pri (cc, xp) is the stigmal of that primal for the index M, and is immediately found. A parabscissal $y = x+l$ has a constant ordinate $= OL$, so that the index R of its stinnal (rs) with pri (A, ob) is found from $ar = b - l = l'$, whence $IOR \bigtriangleup AOL'$, or, from $l=a\,(c-r)$, whence, on putting $l=c-l''$, we have $L''CR \bigtriangleup AOI$; and then the stigma S is constructed from R as an index in the primal. A parunal $y=m$ has a constant stigma, which will therefore be that of the stinnal (m_1m), the index of which M_1 in the primal is immediately constructed from $CM_1M \bigtriangleup IOA$. A paradual $y = 2x+t$, of which T is the original and T' the double point, where $t+t' = o$, intersects pri (A, ob) in (uv) where $(a+i)\,u = b-t$, or, (putting $d = a+i$, $e = b-t$,) where $du = e$, that is, $IOU \bigtriangleup DOE$, and then $T'V = 2T'U$. Observe that $CUV \bigtriangleup IOA$. The geometrical operation of finding the stinnal of two primals, especially in the four last named cases, must become extremely familiar to those who wish to construct figures in illustration of general stigmatics. The process is entirely disguised in ordinary Cartesian geometry.

(iv.) If in (ii.) the directionpoints A, A' have been determined, we have $\frac{p-p'}{q-q'} = \frac{a-i}{a'-i}$, which is the *an'nal* of AIA', art. 34. v., and may be spoken of as the annal between the two primals, but continue to be written an AA', where A, A' are their direction points. Similarly tal AA' may be spoken of as the tannal of the annal between the two primals. Here $w = \text{tal}\,AA' = \frac{a-a'}{i-aa'} = \frac{R\,(a-i)-R\,(a'-i)}{i+R\,(a-i)+R\,(a'-i)}$. When the primals are given by two stigmals each, as pri (xp, $x'p'$) and pri (xq, $x'q'$), then, since $(p-p')+(a-i)(x-x')=o$, and $(q-q')+(a'-i)(x-x')=o$, the second expression allows $\tan AA'$ to be expressed immediately in terms of the respective abscissae and ordinates and is often useful; see art. 48. x. It is seldom necessary actually to construct $w = \tan AA'$. In the Cartesian case of fig. 26, $\angle\, WIO = \angle\, AIA'$, and W lies on OJ; the same construction holds for all primals representing Cartesian straight lines. But generally put $a-a' = a_1$, $aa' = a_2$, $i-a_2 = a_3$, and $w = a_1 \,.\, Ra_3$. The points A_1, A_2, A_3 are omitted in the figure. By these expressions all cases where the sines and cosines and tangents of imaginary angles between real and imaginary lines, or two imaginary lines, occur, they may be treated with the greatest ease.

(v.) Also, $\frac{p-k}{q-k}=\frac{a-i}{a'-i}=\frac{b-k}{b'-k}$, or $PKQ \bigtriangleup AIA' \bigtriangleup BKB'$, which is a very useful property.

(vi.) *Para*-primals, or parallel primals, have $a = a'$, or $\text{an}\,AA' = i$, $\text{tal}\,AA' = o$. *Orthal* primals (art. 34. v.) have $aa' = i$, $\text{an}\,AA' = i'.a = i'.Ra'$, tal AA' = none, or $AO1 \bigtriangleup IOA'$. These generalise the conditions of parallelism and perpendicularity. Any parabscissal with direction point O is also said to be orthal to a parordinal which has no direction point, for the reason in (ix.) The direction point is the stigmin of the ordinal with a paraprimal through (ii).

(vii.) The condition that three primals, having the direction points A, A', A'' and original points B, B', B'', should be *co-stinnal*, or have a common stinnal, is $\frac{b-b'}{b-b''}=\frac{a-a'}{a-a''}$, or $BB'B'' \bigtriangleup AA'A''$.

(viii.) If in (vii.) we consider A as an index and B a stigma, and A', A'' and B', B'' as fixed points in the last equation, a primal results such that any other $y+(a-i)\,x = b$ having any such pair of points A, B as direction point and original point, will have the same stinnal. Hence this is the equation to a pencil of *rayals* (*ray* + *al*) or system of primals with a common stinnal, or to their common stinnal itself. The primal of their direction points is then called a *ray-primal*, with *ray-indices* and *ray-stigmata*. The direction points of any system of lines are the stigmins of pencils of rayals drawn through (ii) parallel to the primals in the system, to cut the ordinal; compare (vi.). For many purposes this is an important view of them to take.

(ix.) If from the common stinnal (hk) a pair of rayals be drawn having the direction points X', Y', and we substitute x', y' for x, y in the fundamental function $f(x, y) = o$, we determine relations, termed *direction-* or *ray-stigmatics*, between pairs of rayals by means of those between pairs of direction points which act as index and stigma. Stigmals, of which index and stigma are direction points, may be called *ray*-stigmals, with *ray*-indices and *ray*-stigmata, and the corresponding rayals may be termed *indi-rayals* and *stigmo-rayals*, and the pair composed of an indi-rayal and stigmo-rayal referred to each other may be termed simply a *rayar*. If we apply this transformation to the fundamental equation of art. 37. i., we shall have the results of Chasles's second lemma of Characteristics (*Comptes Rendus*, 27 June, 1864, vol. 58, p. 1175), so that the whole of that theory becomes perfectly generalised in stigmatic geometry, and its imaginaries become geometrically intelligible. Observe that when the ray-index X' is solitary, that is, has no ray-stigma Y', the stigmo-rayal, having no direction point, is a parordinal through (hk), and hence still exists, so that a rayar pair is always complete. Similarly for the case of a solitary ray-stigma Y', in which case the indi-rayal, having no direction point, is also a parordinal through (hk). The double rayals are coincident, corresponding to coincident ray-index and ray-stigma.

(x.) Thus, if we take $aa' = i$ as a direction-stigmatic, the corresponding rayals will be all *orthal* as long as either A or A' does not fall on O, in which case the other does not exist, (vi.). If $a = a' = i$, or $= i'$, (in which case the primals are parassals art. 38. iii.), and we continue to use the term orthal to express the relation of the rayals, we shall find that any parassal is orthal to itself (explaining the anomaly that either imaginary asymptote to a circle is perpendicular to itself). If $a=0$, or one rayal is parabscissal, A becomes solitary, and the corresponding rayal is parordinal; that is, retaining the term orthal, parabscissals and parordinals are mutually orthal (vi.), as in the usual Cartesian case of rectangular coordinates.

40. *Dis'tals, or Plücker's Coordinates generalised.*—(i.) Let (xy) be any stigmal and (xp') its co-stigmal on the primal $p'+(a-i)\,x = b$, (fig. 27 gives a Cartesian case,) then

$$y-p' = y+(a-i)\,x-b,$$

and $y-p'$ is called the ordinar *distal* (*dist*-ance + *al*), or simply the distal of the stigmal (xy) from pri (A, ob). It is evident that $y-p'=o$ may be used as the equation to that primal.

(ii.) Draw pri (T, xy) cutting pri (A, xp') in $(x_1 p_1)$; then, as $(x_1 p_1)$ is the stinnal of these two primals, we have (by art. 39. iv.)

$$\frac{p_1-y}{p_1-p'} = \frac{t-i}{a-i},$$

whence $$y-p_1 = \frac{t-i}{t-a}\cdot(y-p') = \frac{t-i}{t-a}\cdot[y+(a-i)\,x-b];$$

and $y-p_1$ is called the general or T-distal of Y from the primal (A, ob), because T is the direction point of the primal which determines it. The usual or ordinar distal $y-p'$ is determined by the intersection of the parordinal through (xy) with pri (A, ob).

(iii.) It is evident that either $y-p'=o$ or $y-p_1=o$ may be taken as equations to the primal, and that the relations of the clinants $y-p'$ or $y-p_1$ determine relations between $P'Y$ or P_1Y which are real distances measured directionally towards the arbitrary stigma Y from its co-stigma P' on the primal, or from the stigma P_1 of the stinnal of a known pri (T, xy) with the original pri (A, ob), and these relations of distances, directionally measured, determine and generalise a multitude of relations, hitherto most imperfectly noted even by Plücker, who first drew attention to their value. The equations thus deduced are called *distal* equations.

(iv.) Taking another primal (A', ob') intersecting the former, and determining the distals $y-q'$ or $y-q_1$ as before, we may determine x and y from the corresponding values,

$$y-p_1 = \frac{t-i}{t-a}\cdot(y-p') = \frac{t-i}{t-a}\cdot[y+(a-i)\,x-b] = p,$$

$$y-q_1 = \frac{t'-i}{t'-a}\cdot(y-q') = \frac{t'-i}{t'-a'}\cdot[y+(a'-i)x-b'] = q.$$

Finding from these equations the values of x, y in terms of p, q, and substituting them for x and y in $f(x, y) = o$, obtain first the distal equation $\phi\,(y-p_1, y-q_1) = o$ to the original stigmatic, and next $\phi\,(p, q) = o$ as the equation to a subsidiary (or *bi-primal*) stigmatic, in which the relations of the original points X, Y, are determined by means of the subsidiary points P, Q, where OP, OQ represent the directional distances P_1Y, Q_1Y of the corresponding stigmata P_1, Q_1 in two fixed primals (A, ob), (A', ob') from a movable stigma Y. The indices X_1, X_2 to the stigmata P_1, Q_1 are found from the two known primals, and the index X to the stigma Y is known, because (xy) is the stinnal of the primals (x_1p_1, T), (x_2q_1, T'). This may be called the *bi-primal* stigmatic, and is the basis of Plücker's *Punct-Coordinaten.*

(v.) The equation to a ray-primal (art. 39. viii.) allows of establishing precisely similar transformations answering to Plücker's *Coordinaten gerader Linien*, giving *bi-stigmal* stigmatics, in which the index and stigma relate to subsidiary points derived from the distals of two fixed stigmals from a movable primal, instead of the distals of a movable stigmal from two fixed primals.

41. *Trilat'erals, or Triangular Relations generalised.*—(Fig. 28 represents a Cartesian case.)—(i.) Let the three stigmals $(u'u)$, $(v'v)$, $(w'w)$ be connected two and two by the primals $(v'v, w'w)$, $(u'u, w'w)$, $(u'u, v'v)$, having the direction points T, T', T'' respectively. These three primals form a *trilateral* of which the three stigmals in the above order are the *apicals* (apical stigmals) *opposite* to the *laterals* (*lateral* or side primals) in the above order. This is written tri (uu, vv', ww').

(ii.) Let $(u'z)$ be a stigmal on the lateral opposite $(u'u)$, then (art.39.iv.) $\frac{u-v}{u-z}=\frac{t''-i}{t''-t}$ and $\frac{u-w}{u-z}=\frac{t'-i}{t'-t}$, whence $\frac{u-v}{u-w}=\frac{(t-t')(t''-i)}{(t-t'')(t'-i)}$, and generally $\frac{u-v}{(t-t')(t''-i)}=\frac{v-w}{(t'-t'')(t-i)}=\frac{w-u}{(t''-t)(t'-i)}$, the symmetry of which is evident. These equations give all the relations of all "triangles real or imaginary."

(iii.) The following particular cases for which the above assume inadmissible forms, with o in the denominator, are easily investigated independently.

The three stigmals lie on one primal $(u'u)$, $(v_1'v_1)$, $(v'v)$, so that $t=t'=t''$; the relation art. 38. v. must be used.

The tri $(u'v_2, u'u, w'w)$ has the parordinal lateral $(u'v_2, u'u)$ which has no direction point; but then $(u'u_2)$, $(u'u)$ are co-stigmals and $(w'w)$ the stinnal of primals $(u'v_2, w'w)$, $(u'u, w'w)$, having the direction points T_1, T' respectively, so that, by art. 39. iv., $\frac{u-w}{v_2-w}=\frac{t-i}{t_1-i}$. If further, as in fig. 28, pri $(u'v_2, w'w)$ is parabscissal,

$$t_1=o, \quad \text{and} \quad \frac{u-w}{v_2-w}=i-t', \quad \text{and} \quad \frac{v_2-u}{v_2-w}=t'.$$

(iv.) When the two last conditions are satisfied, we have an *orthal trilateral.* We may call its parabscissal lateral the *ba'sal*, and its parordinal lateral the *perpendic'ulal*, and the third lateral the *hypothenu'sal.* As we have shewn that tal $T'O=t'=\frac{v_2-u}{v_2-a}$, we might invent a *si'nal* (*sin*-e + *al*), *cosi'nal* (*cosin*-e + *al*) and *cotan'nal* (*cotan*-gent + *al*) of $T'O$, written sal $T'O$, cosal $T'O$, cotal $T'O$, defined thus, sal $T'O=\frac{v_2-u}{w-u}=\frac{t'}{t'-i}$, cosal $T'O=\frac{v_2-w}{w-u}=\frac{i}{t'-i}$, cotal $T'O=\frac{v_2-w}{v_2-u}=\frac{i}{t'}$, from which, in the Cartesian case, by taking tensors, the usual formulae of trigonometry, as derived from the triangle only, in this case the triangle *IOT'*, readily follow. For if $t' = pj$, where p is scalar, $T.\,\text{sal}\,T'IO = T.\,\frac{pj}{pj-i} = \frac{p}{\sqrt{(p^2+i)}}$, and $T.\,\text{cosal}\,T'IO = T.\,\frac{i}{t-i} = \frac{i}{\sqrt{(p^2+i)}}$. The former expressions, however, give what corresponds to the sines, cosines, tangents, and cotangents of imaginary angles. Thus the *direction triangle IOT'* gives rise to a *direction trilateral* tri (oo, ii, ot') which is clearly orthal. The imaginary trigonometrical functions in Cartesian and Plückerian and hence also in Chaslesian geometry arose from applying the terminology of the simple *triangle* to this *trilateral*, and the difficulties which hence arose are to be attributed to the omission to notice the directions of the sides of the triangle, that is, the direction points of the laterals of this trilateral.

(v.) The condition that the primals given by the distal equations $y-p' = y-q' = y-r' = o$, (art. 40. iii.) and having the direction points t, t', t'' respectively, should be the laterals of this trilateral, and hence have *no* common stinnal, is

$$(y-p').(t''-t')+(y-q').(t-t'')+(y-r').(t'-t) = e,$$

where
$$e = \frac{p'-q'}{p'-w}.(t-t'').(v-w) = \frac{(t'-t)(t-t'')}{i-t}.(v-w)$$
$$= \frac{(t'-t)(t''-t')}{i-t'}.(w-u) = \frac{(t''-t')(t-t'')}{i-t''}.(u-v).$$

(vi.) A multitude of propositions on the properties of the trilateral, deducible from these fundamental properties, are necessarily omitted.

42. *Pencil of Four Rayals, or the Anharmonic Properties of Rays generalised.*—(i.) Let there be *five* rayals, having the common stinnal (*he*) and the direction points T, T_1, T_2, T_3, T_4 respectively, (a Cartesian case is shewn in fig. 31). Let a transversal primal be drawn parallel to the first primal, and intersecting the four last in the stinnals (x_1y_1), (x_2y_2), (x_3y_3) and (x_4y_4) respectively.

(ii.) Then from tri (he, x_1y_1, x_2y_2) and tri (he, x_2y_2, x_3y_3) we find

$$\frac{y_1-y_2}{y_3-y_2} = \frac{t_1-t_2}{t_1-t}\cdot\frac{t_3-t}{t_3-t_2} = (t_1t_2t_3t), \text{ art. 34. iv.}$$

That is, the anral of the direction points is expressed by the simple quotient of the differences of the clinants of the stigmins.

(iii.) Similarly $\frac{y_1-y_4}{y_3-y_4} = \frac{t_1-t_4}{t_1-t}\cdot\frac{t_3-t}{t_3-t_4} = (t_1t_4t_3t)$,

and dividing the first of these results by the second,

$$(y_1y_2y_3y_4) = (t_1t_2t_3t_4),$$

that is, whatever be the direction point of the transversal, the anral of the four stigmins, when they exist, is constant and equal to the anral of the direction points. And if there be only *three* stinnals, from the coincidence of T with T_4, we see by (ii) that the anral, reducing to $(y_1y_2y_3\,..)$, remains $= (t_1t_2t_3t_4)$. This constant anral of the direction points is called the anral of their four rayals.

(iv.) This is a perfect generalisation of the fundamental property whence Chasles deduces the whole of his theory of anharmonic ratios, homography and involution (*Géom. Sup.*, art. 13.; see also below, art. 45. vii.). But this generalisation has the advantage of including every case of "imaginary" rays, angles, and points of intersection. The deductions in this general case may be made in a manner precisely similar to his, using the same arguments, *mutatis mutandis.* But the stigmatic calculus much facilitates the operation, as I have found by actually working out every proposition in the clinant form.

(v.) The whole of homography &c. has also been worked out with distals, on the method of Plücker, taking (hk, xp), (hk, xq) to be two fixed rayals, and (hk, xy), (hk, xy') two variable rayals determined by the equations $(y-p)-e\,.\,(y-q)=o$, $g\,.\,(y'-p)-e\,.\,(y'-q)=o$, where g is constant and e variable, which give

$$\frac{y-p}{y-q}\cdot\frac{y'-q}{y'-p}=g, \quad \text{or} \quad (ypy'q)=g,$$

which now becomes perfectly simple, because unperplexed by the "imaginaries" which are so plentifully strewn among Plücker's demonstrations.

43. *U'niqua'drals, or the Relations of Involution and Homography generalised.*—(i.) The general equation to quadrals is

$$ax^2+2\beta xy+\gamma y^2+2\delta x+2\epsilon y+\phi=o,$$

of which it is first convenient to consider the forms not involving x^2 and y^2, because they never give more than *one* value of y for each value of x, and conversely, whence the name *u'niqua'drals.* These are

(ii.) $$2\beta xy+2\delta x+2\delta y+\phi=o,$$

in which x and y are symmetrically involved, giving an *in'val* (*inv*-olution$+$*al*), and

(iii.) $$2\beta xy+2\delta x+2\epsilon y+\phi=o,$$

in which x and y are unsymmetrically involved, giving a *hom'mal* (*hom*-ography$+$*al*).

44. *In'vals, or Chasle'sian Involution of Points generalised.*—(i). From the general equation, art. 43. ii., determine the solitary index and solitary stigma, as in art. 37. ii. By dividing out first by y and then by x, and putting $=o$ the sum of the terms not containing y and x respectively in the denominator, we obtain $2\beta x+2\delta=o$, $2\beta y+2\delta=o$, so that there is merely one solitary point S, where $2\beta s+2\delta=o$. If e and f be the roots of the equation $2\beta z^2+4\delta z+\phi=o$, then $s=\frac{1}{2}(e+f)$, and E, F are the double points of the inval. These results give

$$(s-x)(s-y)=(s-e)^2=(s-f)^2,$$

to which is adapted fig. 29, where AA', BB', CC', DD', GG', HH', &c., are various ordinates.

(ii.) To construct the stigmals, draw the *characteristic circle*, with centre S and radius SE or SF. A being any index, to find the stigma A', draw $ASE \bigtriangleup ESA'$, by making $\angle ESA'=\angle ASE$, and (B, B' being the intersections of SA, SA' with the char. cir.) BA' parallel to AB'. The lengths of the corresponding SX, SY are thus always found, and it is then easy to separate SX, SY by any angles from SE.

(iii.) From (i.) we find, on putting (aa'), (bb'), &c., for (xy),

$\dfrac{s-a}{s-b}=\dfrac{s-b'}{s-a'}=\dfrac{a-b'}{b-a'}=\dfrac{a-b'}{a-n}$, if $BA'=AN$, so that if two stigmals

(aa'), (bb') are known, the solitary point S is found by making $ASB \ \Delta \ B'AN$, and then the double points E, F are found from SA, SA', as in art. 33. iii. Two stigmals being then sufficient to determine an inval, we may write it as inv (aa', bb'), which for the solitary point may be inv (aa', S). The true nature of the equations $ab = i = i^2$ and $(y-i)^2 = (a-i)(b-i)$, art. 34. v., p. 37, is now evident.

(iv.) From equations similar to those in (iii.) it is easy to shew that all the properties of Chasles's Involution hold strictly, of which the following need only be cited.

First, from $\dfrac{s-a}{s-x} = \dfrac{s-y}{s-a'}$, $\dfrac{s-a}{s-b} = \dfrac{s-b'}{s-a'}$, $\dfrac{s-c}{s-x} = \dfrac{s-y}{s-c'}$, $\dfrac{s-c}{s-b} = \dfrac{s-b'}{s-c'}$.

we find $\dfrac{s-a}{a-x} = \dfrac{s-y}{y-a'}$, $\dfrac{s-a}{a-b} = \dfrac{s-b'}{b'-a'}$, $\dfrac{s-c}{c-x} = \dfrac{s-y}{y-c'}$, $\dfrac{s-c}{c-b} = \dfrac{s-b'}{c'-b'}$;

whence, eliminating $s-a$, $s-c$, $s-y$, $s-b'$, we find $(abcx) = (a'b'c'y)$, or any four indices have the same anral as their stigmata; and this would of course remain true if the former were drawn on a separate plane or different portion of the same plane from the latter. But this result is not characteristic of invals.

Second, $(abxy) = a'b'yx)$, or in any stigmal the index and stigma may be reversed. This result is characteristic, for on multiplying out we obtain the characteristic equation of invals, for which the planes cannot be separated.

Third, $(abs..) = (a'b'..s)$, as in (iii.) See art. 34. iv.

Fourth, $(efxy) = (efyx)$, whence $(eyfx) = i'$, or any index and stigma form a harmal with the double points, and hence these four points will lie either on the same straight line or the same circle, as shewn in the figure. Hence also the construction: draw any circle of which EF is a chord, take any points A, A' upon it, so that $\angle ESA = \angle A'SE$, then (aa') is a stigmal in the inval. In this case A and A' lie harmonically with respect to E, art. 34. iv. In the figure G is the centre of the circle containing $A'EAF$, which however is not drawn; but see fig. 14. If inv (ee, ff) and inv $(e'e', f'f')$, have the common stigmal (xy), then $(yexf) = (ye'xf')$, and hence (yy, xx) are the double points of inv $(ef, e'f')$, whence (xy) may be constructed. This fails when the invals have a common solitary point, and in that case only they can have no stinnal.

(v.) The equations of angles resulting from the above anrals also shew how the stigmod varies for different straight lines or circles assumed as indits; thus the indit circle ABC has the sigmod circle $A'B'C'$, but the indit circle $SHDL$, passing through S, has the stigmod straight line $H'D'L'$, S having no stigma. Möbius, in the papers cited in Appendix II., seems to have first treated the involution of points in a plane, but it will be found that his treatment is much more complicated, and that the present theory brings out all his results and many others with the greatest simplicity.

(vi.) It may be observed that, in the old theories of involution of points on a straight line, when X, Y lay as at D, D' on the same line as E, F, these last double points were called *real*, but when X, Y lay on a perpendicular to EF through S, as at G, G', these double points, though remaining unchanged, were called "imaginary." By forming two inva-primals (art. 38. vi.), so taken that the carstigmod gives the line ESF in the first case, and the perpendicular to ESF in the second, it will be seen that E, F are carstigmata in the first, and incarstigmata in the second case. This is the meaning of the above confusing distinction, which could not be previously avoided. Again, until a Cartesian inva-primal had been formed, since the ordinates XY lay on the same straight line, and not perpendicular to it, as in Cartesian geometry, the two cases were kept entirely separate. In uniquadrals XY was termed a segment, and in Cartesian geometry an ordinate. Until the stigmatic conception had been formed, it was impossible to perceive the real identity of the segments and the ordinates, as simply the straight lines connecting the indices with the stigmata, that is, shewing the pairs of corresponding points. The immense facilitation produced in the application of the homographic theories by the fusion of the Cartesian and Chaslesian geometries, will be strongly felt by every one who works out the cases in detail.

45. *Hom'mals, or Chaslesian Homography of Points generalised.*—(i.) To determine the solitary index S and solitary stigma Z' in the hommal, fig. 30, we find from art. 43. iii., first $2\beta s+2\epsilon = o$, and then $2\beta z'+2\delta = o$, and for the double points E, F we have

$$2\beta e^2+(2\delta+2\epsilon)\,e+\phi = o.$$

These values easily reduce the general form of equation to

$$(s-x)(z'-y) = (s-e)(z'-e).$$

(ii.) From this, by a process like that in art. 44. iv., we find $(abcx) = (a'b'c'y)$, which relation remains when the plane containing the indices is separated from that containing the stigmata. This enables us to determine the solitary index and stigma when three stigmals (aa'), (bb'), (cc') are known, because $(abcs) = (a'b'c'..)$, and $(abc..) = (a'b'c'z')$, that is to say,

$$\frac{a-b}{c-b}\cdot\frac{c-s}{a-s}=\frac{a'-b'}{c'-b'}, \quad\text{and}\quad \frac{a-b}{c-b}=\frac{a'-b'}{c'-b'}\cdot\frac{c'-z'}{a'-z'}.$$

To construct the solitary points from these equations,

first construct W from $\dfrac{a'-b'}{c'-b'}=\dfrac{w-b}{c-b}$, or $A'B'C' \triangle WBC$;

and then S from $\dfrac{c-s}{a-s}=\dfrac{w-b}{a-b}$, or $CSA \triangle WBA$;

and Z' from $\dfrac{c-s}{a-s}=\dfrac{a'-z'}{c'-z'}$, or $CSA \triangle A'Z'C'$.

(iii.) When S and Z' have been found from three stigmals, all other stigmals can be found from a subsidiary inval, thus: Suppose that the part of the plane containing the stigmata is slid over that containing the indices, by sliding $Z'S$ over $Z'S$ till Z' falls on S, and A' on A_1, B' on B_1, &c. Then $z'-s = a'-a_1 = b'-b_1 = \ldots\ldots = y-y_1$, and hence $s-a_1 = z'-a'$, ... $s-y_1 = z'-y$; and hence

$$(s-x)(z'-y) = (s-x)(s-y_1) = (s-a)(s-a_1) = (s-m)^2,$$

when M is properly determined. Hence the subsidiary inval $(s-x)(s-y_1) = (s-m)^2$ determines Y_1 from X, and then $Y_1Y = SZ'$ gives Y from Y_1. Hence also a hommal is merely an inval with its stigmod (or its indit) translated in the same plane without rotation, that is, a transordinated inval.

(iv.) There are now two easy constructions to find the double points E and F. First select O so as to bisect SZ', whence $s+z'=o$, and find O' the stigma of O considered as an index, whence

$$(s-e)(z'-e)=(s-o)(z'-o'), \quad \text{or} \quad e^2=i'.so'=z'o',$$

as shewn in the figure. Again,

$$(s-m)^2=(s-e)(s-e_1)=(s-e)(z'-e)=(e-s)(e+s)=e^2-s^2,$$

or $$e^2=s^2+(s-m)^2,$$

which is constructed as in art. 33. v.; by drawing USV perpendicular to SM, and making $US=SV$, both of the length of SM, so that

$$s-u=j(s-m), \qquad s-v=u-s,$$

which gives $$e^2=s^2-(s-u)^2=uv.$$

This shews that (uv), $(z'o')$ lie on inv (ee, ff).

(v.) It is convenient to call O (or common middle point of EF and SZ') the *centre*, EF the *double axis*, SZ' the *solitary axis*, and MN (where $m+n=2s$) the *subsidiary axis* of the hommal. For the hommal determined by three stigmals we may write hom (aa', bb', cc'), which for the solitary index and stigma may be written hom $(aa', S.., ..Z')$.

(vi.) The relative forms of the indit and stigmod are the same as for the inval (art. 44. v.), but the angular properties of the double points are peculiar to the hommal. See fig. 30.

First $$(eabc)=(ea'b'c'),$$

hence if A, B, C are collinear with each other and hence with S, in which case also $A'B'C'$ are collinear with each other and hence with Z'; then $\tan AEC=\tan A'EC'$, and $\tan AEA'=\tan CEC'$. Hence if two straight lines intersect at E, and are indefinitely produced each way, and then being clamped, are made to revolve, and to cut two given straight lines PQS and $P'Q'Z'$, they will intersect, the first in the indices and the other in the stigmata of a hommal, of which the solitary index S is in PQ, and solitary stigma Z' in $P'Q'$, and E is one of the double points. In fig. 30, the lines PQS, $P'Q'Z'$ are so chosen as to make (pp'), (qq') parts of the same hommal as before. In any such case Z', S are easily found, by making one arm of the biradial parallel to PQ and $P'Q'$ respectively, in which case the second arm cuts $P'Q'$ and PQ in Z' and S respectively. F is then the fourth point of the parallelogram $SEZ'F$. Also $\tan PFP'=\tan QFQ'$, but they are not generally $=\tan PEP'$. The same will be true if PQ, $P'Q'$ coalesce in SZ', and then E, F are the "imaginary" double points of the "real homography" on the line SZ'. This is a new demonstration of Chasles, *Géom. Sup.* art. 171, which it completes, shewing the nature of the points. But this property will be greatly generalised in art. 46. iii. By taking E as the centre of a circle, there will now be no difficulty in explaining and completing the result in *Géom. Sup.* art. 664.

(vii.) Observe that in applying the general property art. 42. iv. as Chasles has done to the construction of a homographic theory, we have from any stigmal (he), see fig. 31, a movable rayal cutting two primals which have the stinnal (kf). In this case the stigmins of the movable rayal on the first of the primals issuing from (kf), taken as indices have their stigmata formed by the stigmins of the same rayals with the second primal, and the stigmals thus formed make a hommal, of which the stigmata E, F of the two stinnals (he), (kf) are the double points. When the primals represent Cartesian straight lines (as in fig. 31), confining ourselves to the stigmods, we may say, if rays from E cut two rays issuing from F, the points of intersection form a hommal, of which E and F are the double points, and of which the solitary index and stigma are found by drawing rays from E parallel first to one and then to the other of the rays issuing from F. This view will be found to shed a new light upon many of Chasles's investigations (especially *Géom. Sup.* chap. vi., &c.), but was of course impossible so long as the points in an homography were considered to lie necessarily on the same straight line.

(viii.) Secondly, $(efab) = (efa'b')$; thirdly, $(efsa) = (ef..a')$; fourthly, $(esfa) = (e..fa')$; fifthly, $(eabs) = (ea'b'..)$; sixthly, $(esa..) = (e..a'z')$; from all of which angular properties may be readily deduced.

46. *Ray-hommals and Ray-invals, or the Chaslesian Homographic Relations of Rays, generalised.*—(i.) If the indices of a hommal are made direction points of the rayals emanating from a fixed stinnal (hk), and the stigmata of the same hommal are taken as the direction points of the rayals from another stinnal (mn), thus generating a *direction-hommal*, (art. 39. ix.), the rayals in these two pencils form a *double ray-hommal.* If the two stinnals (hk), (mn) are coincident, the result is a *single ray-hommal.* These rayals cut any primal in stigmals forming a homma-primal. The stigmo-(or indi-)rayals corresponding to those direction stigmata (or indices), which have solitary indices (or stigmata) respectively, will be parordinal.

(ii.) If (a_1a_2), (b_1b_2), (c_1c_2), (x_1y_2) be the stigmals on the direction hommal, and S_1, Z_2 the solitary points, then

$$(s_1-x_1)(z_2-y_2) = (s_1-a_1)(z_2-a_2), \text{ and } (a_1b_1c_1x_1) = (a_2b_2c_2y_2),$$

whence all properties may be deduced, (compare art. 39, ix. x.,) and the angular properties of the double points of hommals duly generalised.

(iii.) The following is the only case that can be noticed in this Tract. If from any stinnal there issue two rayals having their variable direction points X_1, Y_2 so related that tal X_1, Y_2 is constant, so that, for example, $\dfrac{x_1-y_2}{i-x_1y_2} = Rm$, or $x_1y_2+m(x_1-y_2)-i = o$, these pair of primals will be the analogues of the various positions assumed by the revolving lines in art. 45. vi. Now in this case the direction points of the double rayals determined by putting $x_1 = y_2 = e_1 = f_1$, give $e_1^2 = i = f_1^2$, so that they are I, I', and the rayals are parassals (art. 38. iii.), that is, parallel to the asymptals of a cyclal, or, as used to be said, "they pass through the circular points at infinity" (!); and this will also be true when some pairs of rayals are Cartesian; and will also be true although these parassals among other rayals will of course be incarprimals.

(iv.) Conversely, form a homma-primal from the indices and stigmata of a hommal $(ee, ff, S.., ..Z')$, by assigning Σ, Z″ as the indices of S, Z', where (σs), $(\zeta' z')$ are carstigmals in fig. 33. Let E', Φ be the indices of E, F, in which case (ϵe), (ϕf) are necessarily incarstigmals in the figure. Then it is always possible to give new indices P, Q to

E and F, so that rayals from (pe), (qf) to (ϵe), (ϕf) will be parassal, and in that case the tannal between any indi-rayal and stigmo-rayal will be constant. This condition gives

$$\frac{e-e}{p-\epsilon}=i-i=o,\quad \frac{e-f}{p-\phi}=i-i'=2i,\quad \frac{f-f}{q-f}=i-i=o,\quad \frac{f-e}{q-\epsilon}=i-i'=2i,$$

and hence $EF=2P\Phi$, $FE=2QE'$, and as Φ, E' are known, P and Q are determined. Let T be the direction point, and C the double point of the pri $(\sigma s, \zeta' z')$, and let $2n=e+f$, $2\nu=\epsilon+\phi$, and $e-n=n-f=k(n-c)$. Then

$$n-c=(\nu-c)(i-t),\quad f-tc=\phi.(i-t),\quad e-tc=\epsilon.(i-t),$$

whence $e-p=n-\phi=(k-t)(\nu-c)$, $c-p=(kt-i)(\nu-c)$,

$$f-q=n-\epsilon=(k+t)(c-\nu),\quad c-q=(kt+i)(c-\nu).$$

In the Cartesian case t, k are vectors. Hence C, N', P, Q are collinear, and EP, FQ perpendicular to CN', that is, (pe), (qf) are carstigmals.

The extremely perplexing investigation of this whole question in Chasles, *Géom. Sup.* arts. 171, 172, 181 (especially see table of errata for p. 126 in this art.), 651, and *Sect. Con.* art. 293, will serve to shew the great simplification introduced by stigmatic geometry. But in the present Tract a mere indication must suffice. The whole subject has been carefully examined in detail.

(v.) *Ray-invals* result from similar considerations. Thus, $i^2=x_1y_2$ is a ray-inval, of which all the rays are orthal (art. 39. vi.), the double rayals being parassals, and the rayals corresponding to the solitary index and solitary stigma, or for $x_1=o$, $y_2=$ none, $y_2=o$, $x_1=$ none, being paraxals (art. 39. x.). As two invals have always a common stinnal (art. 44. iv.), any direction-inval, $t^2=x_1y_2$, will intersect $i^2=x_1y_2$, and hence the corresponding ray-inval will always contain two orthal rayals.

(vi.) A sheaf of parallel primals may be used in place of a pencil of rayals, provided their different original points be substituted for their common direction point.

47. *Transordination, or the Cartesian Transformation of Coordinates and of Curves, generalised.*—(i.) The general nature and object of this operation is explained in art. 36. ii. The change is not perfect unless every single indi-stigmal (that is, every single stigmal in the first stigmatic) corresponds to one and only one stigmo-stigmal (that is, to one and only one stigmal in the second stigmatic).

(ii.) This cannot be effected except by assuming relations of the first order, such as $x=b+(x'-a)$, or $x=\lambda x'+\mu y+\nu$, which, changing the index without changing the stigma, produce *indicial* transordination, and are the foundation of the ordinary Cartesian change of coordination. The values of the constants are assumed so as to facilitate subsequent calculation. Similar changes have already been made. Thus the hommal $(s-x)(z'-y)=(s-m)^2$, on putting $z'-y=s-y'$, becomes transordinated into the inval $(s-x)(s-y')=(s-m)^2$. Again, from this last equation, on taking $s-x=s-x'+(y'-x')$, we find $(s-x')^2-(y'-x')^2=(s-m)^2$, where $2x'=x+y'$ and is hence readily found. This however is a *cyclal* (art. 48. v.).

(iii.) More generally, assume such a relation as

$$ax+\beta y+\gamma = \alpha' x' + \beta' y' + \gamma', \qquad \lambda x + \mu x + \nu = \lambda' x' + \mu' y' + \nu',$$

which on elimination give results of the form

$$\pi . (y-x') = y + (t_1 - i)\, x - b_1 ; \quad \kappa . (y-y') = y + (t_2 - i)\, x - b_2,$$

and, on putting $y-x'=p$, $y-y'=q$, these lead at once to the distal transformation and biprimal coordination.

(iv.) Still more generally, putting for brevity $A = ax+a'y+a''$, $B = \beta x + \beta' y + \beta''$, $C = \gamma x + \gamma' y + \gamma''$, and $D=o$ for the result of eliminating x, y from the equations $A=o$, $B=o$, $C=o$, (that is, for the condition that the three corresponding primals have a common stinnal,) we may assume $Cx' = A$, $Cy' = B$. On determining the values of x, y in terms of x', y', they will be found to have a common denominator which will be a factor of the numerator when $D=o$, that is, when these primals have a common stinnal. Rejecting this case, the three primals form a trilateral such as $(u'u, v'v, w'w)$ with the conditions (art. 41, v.). Then, taking P', Q', R' to be co-stigmata for index X in these straight lines, and putting $A = y-p' = p$, $B = y - q' = q$, $C = y - r' = r$, we obtain a homogeneous distal equation between p, q, r, or πp, κq, ρr, which is the foundation of tri-primal coordination.

(v.) The primal (oo, xy), or $y + (t-i)\,x = o$ cuts the stigmatic $f(x, y) = o$ in (xy). Eliminating x, we obtain $\phi(y, t) = o$, which is the foundation of polar coordination.

(vi.) Taking a less perfect form of transordination, that is, one in which the condition (i.) is not perfectly satisfied, we may connect X with X', and Y with Y' by hommals, as

$$xx' + \lambda x + \mu x' + \nu = o, \quad yy' + \lambda' y + \mu' y' + \nu' = o.$$

In this case we shall occasionally have complete stigmals in one answering to defective stigmals (that is, solitary indices, or solitary stigmata) in the other. It was probably the desire to avoid these relations of continuities to discontinuities, that the extraordinary assumptions mentioned in art. 6. i., and Appendix I., were introduced, by which the real nature of the solitary points was illogically distorted. Thus it was not seen, or, if seen, repudiated, that it was possible to have analogies which held for all but a definite number of cases. The attempt to conceal this important logical fact by a mere juggle of language, shews the danger of studying logic from simple arithmetic and geometry, of which numerous instances could be cited besides those in Appendix I. The attempted passage from discontinuous arithmetic to continuous geometry (excepting only by Euclid's really "royal road"), like the attempted passage from discontinuous Cartesianism to some imagined continuity, has led to so much "stretching" of language, that the logical feeling of mathematicians, though dealing with "exact science," is in great danger of being entirely perverted. Thus Dean Peacock put forth his "permanence of equivalent forms," a logical fallacy long since exploded, but defended by him with great warmth and pertinacity. And "perspective projections," admirable as a piece of geometry, have landed us in the contradictions detailed in art. 6. i. and Appendix I. I have even heard these results defended by an excellent mathematician as "illogical, but convenient," as if want of logic, *i. e.* incorrect reasoning, were not the height of mathematical inconvenience.

(vii.) These hommal relations may be obtained from equations like

$$\frac{ax+by+c}{a''x+b''y+c''}=\frac{\alpha x'+\beta y'+\gamma}{\alpha''x'+\beta''y'+\gamma''}, \quad \frac{a'x+b'y+c'}{a''x+b''y+c''}=\frac{\alpha'x'+\beta'y'+\gamma'}{\alpha''x'+\beta''y'+\gamma''},$$

whence, on elimination, x, y, x', y' are obtained in similar forms, but then, on multiplying up, we find (xx'), (xy'), (yx'), (yy') given as stigmals on different hommals. In this case, by equating to o the denominators in the values of x, y, x', y' thus found, we obtain equations to primals in which (xy) and $(x'y')$ are stigmals, such that not one of the stigmals in either primal for the one stigmatic will have a corresponding stigmal in the other. Hence, relatively to each other, these stigmatics will have solitary indi-primals and solitary stigmo-primals. In this way homma-stigmatics are formed, which include the Cartesian case of homographic figures. And by proper changes of the constants these homma-stigmatics are brought into another relation which may be called hom'olo-stigmatics, and include the Cartesian case of homologic figures. In consequence of the old "imaginary" points, none of these relations are completely exhibited except in stigmatic geometry.

48. *Du'oqua'drals or Co'nals, or Conic Sections, generalised.*—(i.) Duoquadrals are derived from such forms of the general quadral equation (art. 43. i.) as always give *two* stigmata Y, Y' for each index X. When they have any Cartesian portion, these stigmatics give as the carstigmods (paths described by the stigmata of the Cartesian portion), the well known conic sections, and are hence also called *co'nals* (*con*-ics + *al*), a name which may then be applied generally to all duoquadrals.

(ii.) The extreme variety and the length of conal investigations preclude me from giving them in this Tract any even approximatively systematic form. I have myself carefully applied the present conception of stigmatic geometry, and the clinant calculus, to the treatment of conals, by generalising the usual Cartesian methods, and also those in Plücker's *System* and *Entwickelungen*, as well as those in Chasles's *Sections Coniques*, in great detail, and have always found satisfactory results, easier calculation, and complete geometrical realisation. The previous explanations of primals and uniquadrals render any other result impossible, and I shall therefore content myself with giving a few notes as to some methods, and a few results, together with the nomenclature which I have found it convenient to adopt, and inviting mathematicians to test the stigmatic theory by minuter applications. Several of these are contained in my second memoir on Plane Stigmatics, but with my old notation and nomenclature. If I may judge of the effect on others by that on myself, the continual explanation of formerly insuperable difficulties, the strictly geometrical meaning of calculations which seemed hopelessly analytical, and the absence of any difficulties in the assignment of positive and negative, will render such a process a source of intense delight to the geometer.

(iii.) When in the general quadral equation (art. 43. i.), $\beta^2-\alpha\gamma=o$, but $\alpha\epsilon-\beta\delta$ is *not* $=o$, the stigmatic is a *non-central*, and by indicial transordination (retaining the stigmata, but altering the origin and indices) may be reduced to the form $(y-x)^2+4sx=o$, which is here called a *parab'bal* (*parab*-ola + *al*). When s is scalar and x is also scalar, sx being tensor, $y-x$ is vector; or when S and X are both on the I side of O on OI, then XY, $X'Y'$ are parallel to OJ; or there is a Cartesian portion, and the carstigmod is a parabola. Y, Y' are constructed by art. 33. iii. Here O is the vertex, (oo) the *vertical*; S the *focus*, (ss) the *focal*. When O, S are known, we may write par (O, S), or par $(xy, S.)$ This case will not be further considered till art. 52., after the treatment of centrals.

(iv.) When *neither* $\beta^2-\alpha\gamma$, *nor* $(\gamma\delta-\beta\epsilon)^2-(\beta^2-\alpha\gamma)(\epsilon^2-\gamma\zeta)$ are $= o$, the conal is *central*, and by indicial transordination can be reduced to the form $g^2x^2+e^2(y-x)^2 = e^2y^2$, which embraces many cases according to the positions of E and G, as follows:—Generally let $e+f = g+h = s+z = o$, and $s^2 = e^2+g^2$, found as in art. 33. v. This may be called the central (ee, oo, og), or (E, O, G). There are no *solitary points*. E, F, in fig. 32, are the *double* or *major points*; G, H the *original* or *minor points*, and S, Z the *foci* of the central.

(v.) *Cy'clal* (κύκλ-ος + *al*), E on OI, G on OJ, $Te = Tg$, $e^2+g^2 = o$, equation $x^2-(y-x^2) = e^2$. This may be called cyc (O, E). The equation gives $(y-x)^2 = x^2-e^2 = (x-e)(x+e) = (x-e)(x-f)$; which gives the contraction of Y, Y' from X immediately, and shews that Y, Y' lie harmonically with respect to E, F. When X is on I between E and F, then XY, XY' are parallel to OJ, and the carstigmod or locus of Y, Y' is a *circle* of which O is the centre and EF the diameter (fig. 34). When the indit is MN, or X lies on the line MN, as at X_1, M, X_2, on MN, the stigmod consists of two branches proceeding from Y_1 and Y_2 so that the *circle* is but an extremely small part of the cyclal. If OE had been taken on OJ at OG, so that $g = je$, we should have $x^2-(y-x)^2+g^2 = o$, whence $(y-x)^2 = x^2+g^2$; hence when X is on OI, Y is always on OI; when X is on OJ, and $Tx < Ty$, XY being parallel to OI, Y will describe the same characteristic circle as before, but every stigmal (xy) is non-cartesian. This is Chasles's "imaginary" circle, more particularly referred to in art. 40. v. (2). Also since $e^2 = y(2x-y)$, the primals, that is, the *assals* $y = o$ and $2x-y = o$ are the *asymp'tals* (*asympt*-otes + *al*) of the cyclal; see art. 38. iii. These have *no* carstigmod. The nature of their asymptoticity is easily seen, for as X retreats in any direction, the angle EF diminishes, EX, FX become more nearly of the same length, Y approximates to O, and Y' to a point Y_o, where $XY_o = OX$, while O, Y_o are the stigmata of X in the assals. The asymptals of all concentric cyclals are parassal, and hence paraprimal.

Since in the cycal $2x = y+y'$, we can eliminate x from the equation $e^2 = x^2-(y-x)^2 = 2xy-y^2 = (y+y')y-y^2 = yy'$. Hence the pairs of co-stigmata form an inval of which O is the solitary point; E, F are the double points. The stigmods of Y and Y' for a given indit are therefore related as the indit and stigmod of an inval. There are really always two branches, which are disguised in the Cartesian case, because they are then two semicircles united at their extremities by the double points E and F. This gives an easy way of finding X from Y, and shews that though each index has two stigmata, each stigma has but one index, which is also apparent from the original equation being only of one dimension in y. We have already found that $(x-y)^2 = (x-e)(x-f)$, which also shews that if we form an inval of which X is the solitary point, and (ef) a variable stigmal, each stigmal determines a new circle having the common stigmals (xy), (xy') with each of the others. Compare art. 49. v. (1).

(vi.) *E'quiper'bal* (*equi*-lateral or *equi*-angular + hy-*perb*-ola + *al*), E and G are coincident and both lie on OI, (no figure), $e^2+g^2=2e^2=s^2$, equation $x^2+(y-x)^2=e^2$, whence $(y-x)^2=e^2-x^2=i'.(x-e)(x-f)$, so that Y, Y' in the equiperbal are found by turning YXY' in the cyclal through a right angle. This is the foundation of Poncelet's supplemental circle. When X is on I, beyond E and F, then XY, XY' will be parallel to OJ, and the carstigmod, or the locus of Y, Y', is an equilateral hyperbola, where the two branches are visibly separated. Also, since $e^2=[x+j(y-x)].[x-j(y-x)]$, the asymptals are $x+j(y-x)=o$, and $x-j(y-x)=o$, which have a Cartesian part, and their carstigmods will be the loci of P and Q, the extremities of PXQ, the YXY' of the asymptals to the cyclal, turned through a right angle about X. See the more general case of the hyperbal, in (viii.)

(vii.) *Ellip'sal* (*ellips*-e + *al*), E on OI, G on OJ, $Tg<Te$. In this case (no figure) let $kj=g$, so that $g^2+k^2=o$, $k^2.Re^2$ is a tensor, and K lies upon OI. The equation becomes $e^2(y-x)^2-k^2x^2+e^2k^2=o$, whence $e^2(y-x)^2=k^2.(x^2-e^2)=k^2.(x-e)(x-f)$; hence XY, XY' are immediately found, by forming XU, the mean bisector of XE, XF, as in the cyclal, and altering its length so that len XU : len XY :: len OE : len OG. When X is on OI between E and F, then XY, XY' are parallel to OJ, and the carstigmod or the locus of Y, Y' is an *ellipse*, of which EF is the *major* axis, and GH the *minor axis*, and S, Z the *foci*. Also, since $e^2k^2=[kx-e(y-x)].[kx+e(y-x)]$, the primals $kx-e(y-x)=o$, $kx+e(y-x)=o$, will be the asymptals of the ellipsal, and will have no carstigmod. The ellipsal includes the cyclal as a particular case. If in fig. 32, OE', OG' (not OE, OG) are taken as the semi-major and semi-minor axes; S, Z will be foci, and (mn) a carstigmal in the characteristic ellipse.

(viii.) *Hyper'bal* (*hyperb*-ola + *al*), E and G both on OI, so that $e^2.R^2g$ is a tensor; no particular relation is needed between len OE and len OG, $s^2=e^2+g^2$. The equation remains $g^2x^2+e^2(y-x)^2=e\,g^2$, whence $e^2(y-x^2)=g^2.(e^2-x^2)=i'.g^2(x-e)(x-f)$, and hence YXY' is found by turning the corresponding line of the ellipsal, for which $g^2=k^2$, through a right angle. Hence Poncelet's supplemental ellipses and hyperbolas. When X is on I, beyond EF, then XY, XY' are parallel to OJ, and the carstigmod or locus of Y, Y' is an hyperbola, of which EF is the *major*, and GH the *minor*, or "imaginary," axis. It has been usual to represent the minor axis by a line perpendicular to EF, and call it imaginary. In fact (og), (oh), which are the stinnals of the ordinal with the hyperbal, are incarstigmals, and both points G, H lie on the line EF. If, in figure 32, OE'' is taken as the semi-*real* axis, and S the focus of the flat hyperbola there (very indifferently indicated rather than) drawn, OG'' will be the minor semi-axis, (og'') being the stinnal of the ordinal with the hyperbal, determined by making $g''^2=s^2-e''^2$. The primal (oo, og'') through (oo) will be the ordinal, and have OG_2 for its carstigmod, and OG_2 is parallel to the carstigmod of the tangental at E''. If len OG_2 = len OG'', OG_2 is the line usually drawn as the "imaginary" semi-minor axis. Similarly,

OE being any semi-diameter, OK is usually drawn as the "imaginary" conjugate semi-diameter, being parallel to the tangent at E, whereas it is only the carstigmod of the symmetral (art. 50.) to the diametral, of which OE is the carstigmod, and the proper stinnal $(g_1 g)$ of that primal with the curve is found by turning OK through a right angle to OG, and drawing GG_1 perpendicular to OG'. We shall find in art. 50. iii. (3) that $s^2 = e''^2 + g''^2 = e''^2 - g_2^2 = e^2 + g^2 = e^2 - k^2$.

Since $e^2 g^2 = [gx - je(y+x)] . [gx - je(y-x)]$, the asymptals of the hyperbal are $gx + je(y-x) = o$, and $gx - je(y-x) = o$, and have a carstigmod, which will be found by turning the YXY' of the asymptals of the ellipsal through a right angle. Thus, in fig. 32, OL is an asymptote to the flat hyperbola on the right, where $F''L = OG_2$.

(ix.) *Hy'perel* (*hyper*-bola + *el*-lipse, the final-*al* omitted for euphony), E and G lie anywhere on the plane. This is the general case, to which all properties of centrals belong. The equations have the same forms as in (viii.) Given X (fig. 32), join XE, XF, make $XF_1 = FX$, draw XU the mean bisector of XE, XF_1, and revolve XU through $\angle UXY = \angle EOG$, altering its length so that len XU : len XY :: len OE : len OG. When X lies on EF between E and F, as at X_1, this construction gives Y as at Y_1, Y'_1 on an ellipse of which OE, OG are conjugate semi-diameters. But if X lie beyond E, F, as at X_2, the same construction gives Y as at Y_2, Y'_2 on a confocal hyperbola passing through E (the same as that described in viii.). From this circumstance is derived the name *hyperel*, which thus becomes synonymous with the general central quadral. If the ordinate X_2Y_2 be revolved through a right angle to X_2Y_3, its termination will lie on one of Poncelet's supplementary hyperbolas, which is however quite useless in this case, as the stigmod is sufficiently clear in itself.

The equations to the asymptals are the same as before; but if we put them into the proper distal form (art. 40.), using (xp'), (xq') for the costigmals in the asymptals, with (xy) in the central, they become

$$y - p' = y - x + j' . Re . gx, \quad y - q' = y - x - j . Re . gx,$$

whence $(y-p')(y-q') = g^2$, or the mean bisectors of $P'Y$, $Q'Y = OG$ and GO, as in fig. 32, where pri (oo, xp') and pri (oo, xq') are the asymptals. Now $2(y-x) = y - y'$, hence

$$y' - q' = (y - q') - 2(y - x) = i' . (y - p'), \text{ or } Q'Y' = YP',$$

a well known property in the hyperbola, but seldom directionally stated. (In the ordinary hyperbola, the parallelogram $P'YQ'Y'$ becomes a straight line.) Also if $y - p_1 = \pi(y - p')$, $y - q_1 = \pi . (y - q')$, we have $(y - p_1)(y - q_1) = \pi^2 . g^2$. Hence the above property holds for the stigmins of *any* transversal drawn through (xy) and cutting both the central and the asymptals. Also if $y - p_1 = p$, $y - q_1 = q$, $pq = \pi^2 . g^2$, or (pq) is the stigmal of an inval depending on the direction of the transversal. And so on for the generalisation of all other properties deduced in Plücker's *System*, p. 91.

(x.) The unreduced duoquadral equations to the cyclal takes one of the forms $2xy - y^2 + 2\delta' x + 2\epsilon' y + \zeta' = o$,

or $x^2 - (y-x)^2 + 2\rho' x + 2\sigma' . (y - x) + \zeta' = o$.

If T, T' be the direction points of two intersecting rayals $(\beta b, xy)$, $(\delta d, xy)$, proceeding from fixed stigmals (βb), (δd), then

$$(b-y)+(t-i)(\beta-x)=o, \text{ and } (d-y)+(t'-i)(\delta-x)=o.$$

Hence the condition tal $TT'=\mu$, giving $\mu=\dfrac{R(t-i)-R(t'-i)}{1+R(t-i)+R(t'-i)}$, is easily reduced to an equation in x and y, which on multiplying out will be found to be one of these two general forms of the cyclal. This generalises a portion of art. 34. v., and admits of the complete application of ray-hommals in the same way as Chasles uses the homographic properties of rays in a circle. This shews also that three stigmals, forming a trilateral $(\alpha a, \beta b, \gamma c)$ determine a cyclal. To construct it from them, it is necessary to find the *axis*, that is, the stigmals of the centre, and the major points. On drawing orthals through the middle stigmals of two of the laterals, their stinnal is the stigmal of which the centre is the stigma. Transordinate so as to make the central stigmal (oo), then $(x'y)$ being one of the transordinated stigmals, draw $X'Y'$ so that $2x'=y+y'$, and find E, F as double points of the inval (oo, yy'). On making this construction first in a Cartesian case, carefully marking the indices, its nature will be quite clear. A cyclal thus given may be noted as cyc $(\alpha a, \beta b, \gamma c)$.

(xi.) For conals generally, if from (μm), (νn) rayals be drawn intersecting in fixed stinnals (αa), (βb), (γc), and a variable stinnal (xy), and the direction points of the rayals from (μm) be A_1, B_1, C_1, X_1 and from (νn) be A_2, B_2, C_2, Y_2 respectively, then we may find a_1-i, a_2-i, b_1-i, b_2-i, &c., in the same way as in (x.), whence we can form $a_1-b_1=(a_1-i)-(b_1-i)$, and so on. Then if the movable rayals form a ray-hommal with the fixed rayals, we have $(a_1b_1c_1x_1)=(a_2b_2c_2y_2)$. Substituting the values of a_1-b_1, &c., thus found, we obtain as the locus of (xy) a general quadral, of which it is easy to investigate the particular cases. Also if there be four fixed stigmals (αa), (βb), (γc), (δd), whence rayals are drawn to a movable stinnal (xy), and A_1, B_1, C_1, D_1 be their variable direction points; the condition $(a_1b_1c_1d_1)=\lambda$, reduced as before, gives a general quadral. In the latter case, $(abcd)$ is also constant; hence $\lambda=\mu\ (abcd)$, where μ is a constant, or the anral of the rayals, now called *chordals* (*chord* + *al*) of the quadral, divided by the anral of the stigmata of the fixed stigmals is constant. These contain stigmatic generalisations of Chasles's fundamental propositions, *Sections Coniques*, arts. 8. and 4. respectively. They can also be deduced in other ways. The deduction in Chasles is made from perspective projections of a circle; but this is inapplicable stigmatically when the centre of projection is not in the same plane as the curve. Hence it is not possible to pass in that way from the properties of general stigmals of a circle (non-Cartesian as well as Cartesian, "imaginary" as well as "real" points) by such projections. For the same reason it will be necessary to establish a stigmatic theory of contact before the corresponding generalisation of the fundamental proposition of tangents can be undertaken. That proposition is proved in art. 51. iv. After these chief propositions have been proved, the whole of the demonstrations in Chasles's *Sections Coniques* can be adapted stigmatically by mere alteration of terminology.

49. *Intersections of Duoquadrals by Primals.*—(i.) The intersections of a hyperel $e^2(y-x)^2+g^2x^2 = e^2g^2$ by a primal $y-x+tx = b$ give at once $(g^2+e^2t^2)x^2-2bte^2x = (g^2-b^2) \cdot e^2$ ....(1),
whence $(g^2+e^2t^2)x = bte^2 \pm eg\sqrt{(g^2+e^2t^2-b^2)}$..................(2),
which is constructed by putting

$$et = m, \quad g^2+m^2 = n^2, \quad n^2-b^2 = r^2, \quad bm = nm', \quad gr = nr',$$

whence $nx_1 = em'+er', \quad nx_2 = em'-er'$.

In particular cases this construction may be greatly simplified.

(ii.) There is no intersection, if $g^2+e^2t^2 = o$ and $b = o$; for the equation (1) in (i.) then reduces to $o = e^2g^2$, an impossibility. In this case, the primal is an asymptal, as already found.

(iii.) If $g^2+e^2t^2 = o$, but b not $= o$, the equation (1) in (i.) reduces to $(g^2-b^2)+2btx = o$, giving only one value of x, or a *par*asymptal cuts the hyperel in one stigmal only.

(iv.) If b does not $= o$, but $g^2+e^2t^2 = b^2$, then there is also only one value of x, produced however not by the reduction of the equation (1) in (i.) to a simple form, but to a complete square. This makes the primal a tangental at (xy), and on determining t from this condition, and from the equations to the primal and the hyperel, we find $te^2(y-x) = g^2x$, so that (x_1y_1) being any other stigmal on the tangental, its equation is

$$e^2(y-x) \cdot (y_1-x_1)+g^2x \cdot x_1 = e^2g^2.$$

Hence tangentals to a central can be drawn through *any* stigmal, except the centre stigmal (oo). The whole theory of the tangental and polar can now be deduced; see arts. 50. and 51.

(v.) For the particular case of the cyclal proceed thus, fig. 34, where the lettering must first be understood in a general, not a Cartesian, sense.

Primal $y-x+tx = b = ct$; cyclal $x^2-(y-x)^2 = e^2$;

$$\text{whence} \quad x = \frac{bt \pm \sqrt{(b^2+e^2-e^2t^2)}}{t^2-i}, \quad y = \frac{b \mp \sqrt{(b^2+e^2-e^2t^2)}}{t+i}.$$

Put (x_1y_1), (x_2y_2) for the two values of the stinnals.

The orthal from (oo) on the primal is $t(y-x)+x = o$, and if its stinnal with the primal be (mn), and with the cyclal be (δd), $(\delta' d')$,

$$\text{we have} \quad n = \frac{b}{t+i}, \quad m = \frac{bt}{t^2-i}, \quad d^2 = \frac{t-i}{t+i} \cdot e^2,$$

whence $2n = y_1+y_2, \quad 2m = x_1+x_2, \quad y_1y_2 = d^2$,

so that (nm) is the middle stigmal of chordal (x_1y_1, x_2y_2), and Y_1, Y_2 lie harmonically with respect to D, D'.

$$\text{Also} \quad \frac{m-x}{m} = \frac{\pm\sqrt{(b^2+e^2-e^2t^2)}}{bt}, \quad \frac{n-y}{n} = \frac{\mp\sqrt{(b^2+e^2-e^2t^2)}}{b},$$

$$\text{and} \quad (n-y)^2 = \frac{b^2+e^2-e^2t^2}{(t+i)^2} = n^2+e^2 \cdot \frac{i-t}{i+t} = n^2-d^2 = (n-d)(n-d').$$

(1) First particular case. The primal and cyclal are Cartesian, $e = Se$, $b = Vb$, $t = Vt$, or $Ve = Sb = St = o$; $e^2 = T^2e$, $b^2 = i' \cdot T^2b$, $t^2 = i' \cdot T^2t$, $b^2+e^2-e^2t^2 = i' \cdot T^2b+T^2e+T^2e \cdot T^2t = (i+T^2t)\{T^2e-T^2n\}$, since $T^2b = (i+T^2t) \cdot T^2n$. If then $Te > Tn$, or the line CB cuts the circle (this case is not drawn in the figure),

$$U(b^2+e^2-e^2t^2) = i, \quad \text{and hence} \quad V\frac{m-x}{m} = o, \quad \text{and} \quad S\frac{n-y}{n} = o,$$

or XMO is a straight line, and ONY a right angle. This corresponds to the case of art. 34. x. But if $Te < Tn$, as in fig. 34,

$$U(b^2+e^2-e^2t^2) = i', \quad \text{and hence} \quad S\frac{m-x}{m} = o, \quad V\frac{n-y}{n} = o,$$

or OMX is a right angle, (and hence X_1MX_2 a straight line perpendicular to OM,) and ONY or Y_1NY_2O is a straight line. In this case $T^2(n-y) = T^2n - T^2e$. Hence set off NZ' or NS of the same length as OE, and with centre Z' and radius of the same length as ON describe a circle which will cut ON in Y_1, Y_2. It is easily seen that this construction is the same as that for finding the double points in the hommal resulting from the intersections with CB of rays from K, L, the extremities of the diameter parallel to C, B passing through any points in the circle. Thus the tangents KZ', LS determine the solitary stigma and index, and the rays KD, LD two other points, (drawn but not lettered in the figure,) whence Y_1, Y_2 are found. Chasles's definition of the imaginary points of intersection corresponds to their being the double points thus obtained. Then X_1, X_2 are found by making $CX_1Y_1 \triangle CX_2Y_2 \triangle COB$. It is well to verify by construction that (x_1y_1), (x_2y_2) are really stigmals belonging to the cyclal. If X_1Y_1 and X_2Y_2 are produced to the same length backwards, they will fall on other parts of the stigmod corresponding to the indit X_1X_2. This is seen to be a two-branched curve in the figure. The stigmods described by two different stigmata for any indit are necessarily so; but the two branches of the carstigmod in this case, as mentioned in art. 48. v., coalesce and form the circle, whereby, as so frequently happens in Cartesian geometry, the real relations are completely disguised.

Observe that since $(n-y)^2 = (n-d)(n-d')$, if we were to suppose e, and hence d, d', to vary, (dd') will become the stigma on an inval of which N is the solitary point and Y_1, Y_2 the double points. This would give a series of cyclals having the common chordal (x_1y_1, x_2y_2) on the primal, of which CB is the carstigmod, and hence being the only part hitherto recognisable, was used to *represent* that chordal and called the *radical axis*. Since (art. 50. ii.) the symmetrals of a cyclal are orthal, no generality is lost by considering this chordal to be the ordinal, and taking the origin O at N, and the equation to the cyclal as $(c-x)^2-(y-x)^2 = (c-h)^2 = (c-k)^2$, so that C is its centre, and HK its axis. Let (oe), (of) be the stinnals of the ordinal with this cyclal, then the inval becomes $e^2 = f^2 = hk$, and all the general cyclals which the ordinal intersects in (oe), (of) will be found from their axis HK, which forms an ordinate in this inval. This at once generalises and simplifies the investigation of the properties of this *common chordal*.

(2) Next suppose the primal to be Cartesian, but the cyclal to be $x^2-(y-x)^2 = g^2$, where $g = je$, and is hence a vector. This may be distinguished as the *vec-cyclal*, and corresponds to Chasles's "imaginary circle," (see below, p. 78, col. 1, at bottom,) which here becomes a geometrical reality; see art. 48. v. In this case,

$$b^2+g^2-g^2t^2 = i'.\, T^2b - T^2g - T^2g\,.\,T^2t,$$

and hence $S\sqrt{(b^2+g^2-g^2t^2)} = o$ in all cases. Hence we have as before $S\frac{m-x}{m} = o$, $V\frac{n-y}{n} = o$; but $T^2(n-y) = T^2n + T^2g$.

Hence make len NY' = len $Y''N$ = len OZ', and the stigmins Y', Y' are determined. Then find X', X'' from $CX'Y' \triangle CX''Y'' \triangle COB$. The figure shews that $X''Y''$ is a mean bisector of $X''G$, $X''H$, and hence that $(x''y'')$ is a stigmal in the vec-cyclal as well as in the carprimal. This will suffice to initiate the very interesting relations of this case.

(vi.) *Carnot's Transversals* for conals may be considered thus:—(1) Let two primals through any stigmal (xy) cut the conal whose equation is $\phi(x, y) = o$, in (x_1y_1), (x_2y_2) and $(x'y')$, $(x''y'')$ respectively. Then if λ be the coefficient of y^2 in $\phi(x, y)$ and κ, κ_1 be coefficients depending on the direction points of the primals (put the equation in the distal form, and apply art. 40. iv.), each of the following expressions represents $\phi(x, y)$, and we have consequently

$$\kappa . (y-y_1) . (y-y_2) = \kappa_1 . (y-y') . (y-y'') \quad \ldots\ldots\ldots\ldots (1).$$

If the second primal is tangental, the second side becomes

$$\kappa_2 . (y-y')^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If the second primal is a parasymptal, it cuts the conal in one stigmal only, and the second side becomes $\kappa_3 . (y-y')$ .... (3).

If the second primal be an asymptal, it does not cut the conal at all, and

$$(y-y_1) . (y-y_2) = \kappa_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

(2) If two primals be drawn intersecting each other in (xy) and the conal in (x_1y_1), (x_2y_2) and $(x'y')$, $(x''y'')$ respectively. And two others parallel to the former respectively and intersecting each other in $(\xi\eta)$ and the conal in $(\xi_1\eta_1)$, $(\xi_2\eta_2)$, and $(\xi'\eta')$, $(\xi''\eta'')$ respectively, then

$$\kappa\,(y_1-y)\,(y_2-y) = \kappa'\,(y'-y)\,(y''-y),$$

and

$$\kappa\,(\eta_1-\eta)\,(\eta_2-\eta) = \kappa'\,(\eta'-\eta)\,(\eta''-\eta),$$

so that, on eliminating κ, κ',

$$\frac{(y_1-y)\,(y_2-y)}{(\eta_1-\eta)\,(\eta_2-\eta)} = \frac{(y'-y)\,(y''-y)}{(\eta'-\eta)\,(\eta''-\eta)}.$$

(3) Let the laterals $(\beta b, \gamma c)$, $(\gamma c, \alpha a)$, $(\alpha a, \beta b)$, of the tri $(\alpha a, \beta b, \gamma c)$ intersect the conal in (λl), $(\lambda' l')$, in (μm), $(\mu' m')$, and in (νn), $(\nu' n')$, respectively, and let κ_1, κ_2, κ_3 be the coefficients due to their direction points respectively, then

$$\kappa_1 . (c-l) . (c-l') = \kappa_2 . (c-m) . (c-m'),$$

$$\kappa_2 . (a-m) . (a-m') = \kappa_3 . (a-n) . (a-n'),$$

$$\kappa_3 . (b-n) . (b-n') = \kappa_1 . (b-l) . (b-l');$$

whence eliminating κ_1, κ_2, κ_3 we have

$$\frac{(c-l)\,(c-l')}{(c-m)\,(c-m')} \cdot \frac{(a-m)\,(a-m')}{(a-n)\,(a-n')} \cdot \frac{(b-n)\,(b-n')}{(b-l)\,(b-l')} = i,$$

and this expression holds for non-Cartesian as well as for Cartesian intersections. Thus, in fig. 34, the laterals (oo, cc), (cc, ob), (ob, oo) of the Cartesian trilateral (oo, cc, ob) intersect the Cartesian cyclal in (ee), (ff), in (x_1y_1), (x_2y_2), and in (og, oh), respectively, and hence

$$\frac{gh}{ef} \cdot \frac{(e-c)\,(f-c)}{(y_1-c)\,(y_2-c)} \cdot \frac{(y_1-b)\,(y_2-b)}{(g-b)\,(h-b)} = i.$$

The position of the triangle then shews that $U\dfrac{(y_1-b)(y_2-b)}{(y_1-c)(y_2-c)} = i$, and hence $\angle CY_1B = \angle BY_2C$, which on account of the perpendicularity of Y_1Y_2 on BC is easily verified, and shews a real geometrical relation of the "imaginary points" Y_1, Y_2.

In a similar way all the other transversal relations may be generalised.

50. *Symmetrals, or Conjugate Diameters generalised.*—(i.) A primal, drawn through the stigmal of which the centre of a central is the stigma, cutting the central in two known stinnals, is called a *diametral*, and those stinnals its *terminals*. The major and minor axals (ee, ff), (og, oh) of a central, which in this form are the abscissal and ordinal, are such diametrals, of which the stigmals just named are the terminals. The central expressed as $g^2x^2+e^2(y-x)^2 = e^2y^2$ has then this property, that for any value of x the two values of $y-x$ are equal and opposite. The equations to these principal diametrals are $x = o$ and $y-x = o$.

(ii.) Now, transordinate indicially (art. 47. ii.), putting

$$x = ax' + b(y-x') \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

whence $$y-x = (i-a)x' + (i-b)\,.\,(y-x') \quad\ldots\ldots\ldots\ldots(2).$$

Then, putting alternately $x'=o$, $y-x'=o$, for the equations to new diametrals, they give, in the old coordination, $x=by$, $x=ay$ respectively. If T_1, T_2 be the two direction points of these primals, then $b(i-t_1) = i$ and $a(i-t_2) = i$. Putting these values for a and b, and then the resulting values for x and $y-x$ in the equation to the central, and reducing, we find

$$\frac{i-t_1}{i-t_2}(g^2+e^2t_2^2)\,x'^2 + 2(g^2+e^2t_1t_2)\,.\,x'(y-x') + \frac{i-t_2}{i-t_1}(g^2+e^2t_1^2)(y-x')^2 = e^2g^2 \quad\ldots\ldots(3).$$

This therefore will have the same form as before, if $g^2+e^2t_1t_2 = o$. Hence the pairs of diametrals satisfying this condition form a rayinval, and the two rayals in each rayar pair (art. 39. ix.) may be called *symmetrals* (*con*-jugate, *con*- represented by *sym*-, and dia-*metral*). The double rayals are determined by $g^2+e^2t^2 = o$, but these are not diametrals, for, putting $t = t_1 = t_2$, this condition reduces equation (3) to $o = e^2g^2$, which is clearly impossible. But these double rays are the asymptals (see art. 48. viii.), and, calling their direction points A_1, A_2, we have $t_1t_2 = a_1^2 = a_2^2$, which gives an easy construction, when the asymptals and one symmetral is known, to find the other symmetral. In the cyclal, since $e^2+g^2 = o$, we have $t_1t_2 = i$, or the symmetrals of any pair in the cyclal are orthal.

(iii.) Let $(uc, u'c')$, $(vd, v'd')$ be two symmetrals expressed by their terminals, having the direction points T_1, T_2. Let a primal from (uc) orthal to $(vd, v'd')$ cut the latter in $(m'm)$ having the direction point P_1 so that $t_2p_1 = i$.

Then $$g^2u^2+e^2(c-u)^2 = e^2g^2, \qquad g^2v^2+e^2(d-v)^2 = e^2g^2,$$

$$ut_1 = u-c, \qquad vt_2 = v-d, \qquad i'\frac{g^2}{e^2} = t_1t_2 = \frac{(c-u)(d-v)}{uv}.$$

Substituting from the third and fourth in the first and second, and reducing by the fifth of these equations,

$$g^2t_1^2e^2 = (c-u)^2.(g^2+t_1^2e^2), \quad e^2t_1^2e^2 = v^2.(t_1^2e^2+g^2),$$

$$u.(c-u)+v.(d-v) = o \quad \text{.......} \quad (1),$$

$$u^2+v^2 = e^2, \quad (c-u)^2+(d-v)^2 = g^2 \quad \text{.......} \quad (2),$$

$$c^2+d^2 = e^2+g^2 = s^2 \quad \text{.......} \quad (3),$$

$$v.(c-u)-u(d-v) = \pm e.g \quad \text{.......} \quad (4),$$

$$\frac{c-m}{c} = \frac{p_1-i}{t_1-i} \cdot \frac{t_1-t_2}{p_1-t_2}, \text{ by art. 41. ii.,}$$

$$= \frac{Rt_2-i}{t_1-i} \cdot \frac{t_1-t_2}{Rt_2-t_2} = \frac{d}{c} \cdot \frac{v(c-u)-u(d-v)}{(d-v)^2-v^2},$$

whence $$(c-m).(2v-d) = \pm eg \quad \text{.......} \quad (5).$$

These are generalisations of mostly well known properties, but (3) was I believe never noticed till my second memoir on Plane Stigmatics (14 June, 1866), though it gives a very neat and useful construction by art. 33. v. for finding the focus from any pair of symmetrals of which the terminals are known, or the terminal of a second symmetral from the foci and one symmetral. Compare especially Chasles, *Sect. Con.*, art. 205, and observe that that article applies only to the ellipse and to the case of "real" or Cartesian symmetrals, whereas the present equation applies generally. The reduction of these to the usual tensor relations in the Cartesian case of either ellipse or hyperbola presents no difficulty.

(iv.) Putting for t_1, t_2 the values in (iii.), we have for the transordination in (ii.),

$$x = \frac{x'}{i-t_1} + \frac{y-x'}{i-t_2} = \frac{u}{c} \cdot x' + \frac{v}{d} \cdot (y-x'),$$

$$y-x = \frac{t_1}{t_1-i} \cdot x' + \frac{t_2}{t_2-i} \cdot (y-x') = \frac{c-u}{c} \cdot x' + \frac{d-v}{d}(y-x'),$$

and then substituting in the equation to the central and reducing by (iii.), we find $$d^2x'^2+c^2(y-x')^2 = c^2d^2,$$
so that the central referred to symmetrals has always the same form.

51. *Tangentals, Polals, Polarals, Focals, Confocal Centrals, and Curvacyclals, or the Relations of Tangents, Poles, Polars, Foci, Confocal Conics, and Circle of Curvatnre, generalised.*—(i.) Notation as in art. 50. If T_o be the direction point of the tangential to a central at (uc), and T_1, T_2 those of the diametral (oo, uc) and its symmetral, it appears by the equation to the tangental in art. 49. iv. that

$$t_o = \frac{g^2}{e^2} \cdot \frac{u}{c-u} = i' \frac{g^2}{e^2 t_1}, \text{ by art. 50. iii.,} = t_2, \text{ by art. 50. ii.}$$

The tangental is consequently parallel to the symmetral.

(ii.) If the double point of the tangental at (uc) be W, it appears by the equation in art. 49. iv. that $uw = e^2$, or U, W are harmonically situate with respect to E, F. As the stigma C does not appear, the co-stigmal (uc') will have a tangental with the same double point. On account of art. 49. iv. the same is true for the co-stigmals of any index, when the central is referred to symmetrals. Hence, to draw two tangentals to a central through a given stigmal $(w'w)$, first draw a diametral (oo, uc) through that stigmal, then its symmetral (oo, vd), and then determine the terminals (uc), (vd) of both. Find X' so that $x'.w = c^2$, and taking X' as the index of a stigmal referred to the symmetrals as axals, find its stigmata Y_1, Y_2, and then find the indices X_1, X_2 of these stigmata referred to the old axals. The two tangents referred to the symmetrals are $(ww, x'y_1)$ and $(ww, x'y_2)$, and referred to the old axals are $(w'w, x_1y_1)$, $(w'w, x_2y_2)$.

(iii) The primal (x_1y_1, x_2y_2) or *contact-chordal* is the *polaral* (*polar* + *al*) of the stigmal $(w'w)$ in reference to the central, and this stigma is the *polal* (*pol*-e + *al*) of that chordal. The properties of these stigmals and primals depend upon the inval equation $x'w = c^2$ by which they were determined in (ii.).

(iv.) "If through four fixed stigmals in a central there be drawn any four tangentals, intersecting any fifth tangental, and also four chordals meeting in any fifth stigmal of the central, the anral of the four stigmins of the four first with the fifth tangental will be equal to the anral of the direction points of the four chordals." This is the stigmatic expression of Chasles's fundamental property (*Sections Coniques*, art. 2.) referred to in art. 48. xii. The following is the demonstration I gave in 1866, in my second memoir on Plane Stigmatics, art. 110, reduced to the present terminology.

The anral of the four chordals remains unaltered, whatever be the fifth stigmal to which they are drawn (art. 48. xi.); hence it is sufficient to prove the proposition for any particular position of the fifth stigmal. Assume it to be the contact stigmal of the fifth tangental with the central, and through the stinnals of the four tangentals with the fifth, draw four rayals to the stigmal of which the centre of the central is the stigma. These will be symmetrals to the diametrals which are parallel to the four chordals (as they are all contact chordals), and their direction points will have the same anral as the direction points of these diametrals (on account of the inval, art. 50. ii.), and hence as the anral of the direction points of the four chordals. But the direction points of the four rayals have also the same anral as the four stigmins of the four tangentals with the fifth, through which the rayals were drawn (art. 42. iii.). Hence the proposition is established in all its generality for all central quadrals, Cartesian or non-Cartesian, and consequently all deductions made from it, by adapting the reasoning in Chasles's *Sections Coniques* to the stigmatic generalisations, must also be necessarily correct. For non-central quadrals, see art. 52. xii.

(v.) If B be the original point of the tangental, and T its direction point, then, by art. 49. iv., $g^2 + e^2t^2 = b^2$. Hence, if tangentals be parallel to the asymptals of a cyclal, that is, be parassal, so that $t^2 = i$, we have $b^2 = g^2 + e^2 = s^2 = z^2$. Hence *all* such tangentals contain the stigmals (os) or (oz). In this case then the equation to the tangental at (xy) reduces to $y = s$ or z, and $2x - y = s$ *or* z.

Now the double points in both cases are (ss) or (zz). Consequently there are four primals (ss, os), (zz, os), (zz, oz), (zz, os), having either

S or Z as the double point, and also either S or Z as the original point, which possess the property of being at once parassal and tangental to the central. These *two points*, S, Z, are known as the *foci*, and the four stigmals (ss), (os), (zz), (oz), may be termed the *focals*. By confusing foci with focals (*i.e.*, stigmata with stigmals, as usual in Cartesian geometry), Plücker (*System*, p. 106, l. 6) recognises four *Brennpuncte* or *foci* in a central; two real, lying on the major axis,—these are the focals (ss) and (zz); and two imaginary, lying on the minor axis,—these are the focals (os), (oz). This results from his definition of *focus*, which is really only that of focal. Salmon (*Conics*, 3rd ed. p. 233, 4th ed. p. 242) also says that the two imaginary points, meaning the two stigmals (os), (oz), "may be considered as imaginary foci of the curve." He also speaks of a quadrilateral, corresponding to that stigmatic quadrilateral of which the four are the four tangentals just named. Chasles (*Sections Coniques*, art. 294) speaks of this quadrilateral, but recognises as foci two only of its apicals (ss), (zz), as will be found only translating his language stigmatically. His words are: "Les foyers d'une conique dont *les deux sommets réels* du quadrilatère imaginaire circonscrit à la courbe, et dont les points du concours des côtés opposés sont les deux points imaginaires situés à l'infini sur un cercle." *Points*, which are either indices or stigmata, should be kept distinct from *stigmals*, which consist of stigmata referred to indices. If we use *foci* for the *points*, there are but two in a central, determined by $s^2 = z^2 = e^2 + g^2 = c^2 + d^2$, but there are four *focals*, which, referred to the principal axals, are (ss), (zz), (os), (oz), the first two on the abscissal and the second two on the ordinal. In fig. 32, S' is so taken that $ss' = e^2$, hence the ordinal through $(s's')$ is the contact-chordal for tangentals from (ss). Consequently $(s's)$, which is a stigmal in the parunal through (ss), must be the stigmal of contact. It is readily seen by actual construction that $(s's)$ is a stigmal in the central. If for any indit through S' we find the corresponding stigmod for the central, and also for the parunal, the latter would remain the point S, and hence the fact of contact would not appear to the eye. But on turning all the ordinates through a right angle, we obtain supplementary figures in which the contact is visible. For illustration this is shewn in fig. 32 for the car-ellipsal $e'^2(y-x)^2 + g'^2x^2 = e'^2g'^2$, in the tangental from (zz), of which the contact-chordal is the parordinal $(z'z', z'z_1)$, where $zz' = e'^2$. The ordinates turned through a right angle generate one of Poncelet's supplementary hyperbolas, and the tangent to this from z represents the stigmod of the actual tangental, and is seen also to be a tangental from (zz). It must be remembered that this arrangement in the figure does not represent the actual state of things, but merely serves to make it clearer to the eye by separating points which would have otherwise coalesced, or have lain on the same straight line.

(vi.) "If pairs of rayals be drawn from any focal of a central to the corresponding stinnals of a movable tangental and two fixed tangentals, the tannal of the direction points of the rayals will be constant." This is a generalisation of Chasles (*Sections Coniques*, art. 293), and applies to all four focals; the demonstration follows from art. 45. iii.

(vii.) "The sum of the tannals of the direction points between the rayals drawn from any stigmal in a central to the two focals (ss), (zz), or of those drawn to the two focals (os), (oz), and the normal (or orthal to the tangental at the point) is null." This is a generalisation of the property whence the foci received their name. The existence of this property for *both* pairs of stigmals (ss), (zz) and (os), (oz), justifies therefore the application of the term *focal* to all four.

Let N_1 be the direction point of the normal (that is, the orthal to the tangental) at (xy), and S_1, Z_1; S_2, Z_2, the direction points of the rayals from (xy) to (ss), (zz), (os), (oz) respectively. Then, art. 49. iv.,

$$n_1 = \frac{(y-x)\,.\,e^2}{x\,.\,g^2}, \quad \text{while} \quad s_1 = \frac{y-x}{s-x}, \quad z_1 = \frac{y-x}{z-x} = \frac{x-y}{s+x},$$

$$s_2 = \frac{x-(y-s)}{x} = \frac{s-(y-x)}{x}, \quad z_2 = \frac{x-(y-z)}{x} = \frac{s+(y-x)}{i'\,.\,x}.$$

Hence $\quad \text{tal}\ S_1N_1 = \dfrac{s_1-n_1}{i-s_1n_1} = \dfrac{s\,.\,(y-x)}{g^2} = \dfrac{n_1-z_1}{i-z_1n_1} = \text{tal}\ N_1Z_1,$

and $\quad \text{tal}\ S_2N_1 = \dfrac{s_2-n_1}{i-s_1n_1} = \dfrac{s\,.\,x}{e^2} = \dfrac{n_1-z_2}{i-n_1z_2} = \text{tal}\ N_1Z_2.$

(viii.) The equations $s^2 = e^2+g^2 = e'^2+g'^2 = e''^2+g''^2$, fig. 32, point to a series of conals with a common centre O and common foci S, Z. These are called confocal centrals. If we put $e^2 = x^2$, $g^2 = (y-x)^2$, these equations reduce to $s^2 = x^2+(y-x)^2$, which gives an equiperbal (art. 48. vi.) whence, given S, Z, the whole system can be found. If we assume any pair of values of e, g, to give a standard hyperel, then by art. 50. iii. (3), another pair, as c, d, will give terminals of symmetrals, which must be referred to indices by being taken as clinants of stigmata in the hyperel determined by the other.

To find the stinnals of two confocal hyperels (ee, oo, og) and $(e'e', oo, og')$,

$$g^2x^2+e^2(y-x)^2 = e^2g^2, \qquad g'^2x^2+e'^2(y-x)^2 = e'^2g'^2,$$

where $\qquad s^2 = e^2+g^2 = e'^2+g'^2$, fig. 32.

These equations give $\quad s^2x^2 = e^2e'^2, \quad s^2(y-x)^2 = g^2g'^2.$

If then T, T' be the direction points of the tangentals to these hyperels at (xy), we have $\quad t = \dfrac{x}{y-x}\,.\,\dfrac{g^2}{e^2}, \qquad t' = \dfrac{x}{y-x}\,.\,\dfrac{g'^2}{e'^2},$

so that $tt' = i$, or the tangentals are orthal. This stinnal is very nearly the (x_2y_2) of fig. 32. If in the same figure we take the Cartesian ellipsal $(e'e', oo, og')$, and the confocal Cartesian hyperbal $(e''e'', oo, og'')$, their stigmin is E, and the perpendicularity of the carstigmods of the two Cartesian tangentals at E is evident.

(ix.) The theory of transversals in art. 49. vi. is sufficient to determine the *curva-cyclal* (*curva*-ture + *cyclal*) to any conal whatever.

Let (αa), $(\alpha' a')$, $(\beta' b')$ be three stigmals in a central. (The reader should draw a Cartesian case; there was no room for the figures.) Draw the chordal $(\alpha a, \alpha' a')$, and through $(\beta' b')$ draw an orthal to this chordal, cutting it in (λl), and also cutting the central again in (βb), and the cyclal drawn through the three first stigmals, in (δd). Take $2\mu = \beta+\beta'$, $2m = b+b'$, and through (μm) draw a primal parallel to the chordal $(\alpha a, \alpha' a')$, and cutting the central in (γc), $(\gamma' c')$. Let $(\omega' \omega)$ be the stigmal of which the centre of the cyclal is the stigma, and draw the symmetrals $(\omega' \omega, p'p)$, $(\omega' \omega, q'q)$, parallel to the chordals $(\alpha a, \alpha' a')$ and $(\beta b, \beta' b')$, so that $(\omega-p)^2+(\omega-q)^2=o$, because, being orthal, they are symmetrals in a cyclal, art. 50. iii. Then by transversals,

in the cyclal $$\frac{(b'-l)(d-l)}{(a-l)(a'-l)}=\frac{(\omega-q)^2}{(\omega-p)^2}=i' \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

in the central $$\frac{(b-l)(b'-l)}{(a-l)(a'-l)}=\frac{(b-m)(b'-m)}{(c-m)(c'-m)}=\frac{i'(b-m)^2}{(c-m)(c'-m)}\ldots (2),$$

and by division $$\frac{b-l}{d-l}=\frac{(b-m)^2}{(c-m)(c'-m)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This holds for all circles. Now take the circle which is the limit as A, A', B' approach L. The tangental at (λl) will be the limit of the chordal $(aa, a'a')$, and since the normal to it in the cyclal will be a diametral, $(\omega'\omega)$ will lie on $(\lambda l, \delta d)$, and $d-l=2(\omega-l)$. Also $b-l=2(b-m)=i'.2(l-m)$. Hence the last equation becomes

$$l-\omega=\frac{(c-m)(c'-m)}{l-m} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

a new expression, giving an easy construction for the axis of the curva-cyclal at (λl) in the general case by making $UMC \;\Delta\; C'ML$ and $L\Omega=UM$.

For the general form of the usual expression for centrals, from (oo) draw an orthal to the tangental cutting it in (ρr), and, parallel to the tangental, a diametral to the conal cutting the latter in (νn), then $\omega-l=n^2.Rr$. Make $VON \;\Delta\; NOR$, and $L\Omega=OV$.

52. *Parab'bals.*—(i.) There is no figure. If the reader will draw an ordinary Cartesian parabola with vertex O, focus S, parameter $OE=4OS$, directing point D, when $DO=OS$, axis OE, ordinate XY, he will probably experience no difficulty.

(ii.) Putting $4s=e$, the general equation to the parabbal (art. 48. iii.) is $(y-x)^2+ex=o$. To construct Y, join XO, draw $OF=EO$, and make XY equal to the mean bisector of OF, OX. If X is on OE, the stigmod is the usual parabola. As long as X is on any straight line through O, as OX_1, the ordinates remain parallel to each other and len XY = len X_1Y_1, where X_1Y_1 is the Cartesian ordinate at X_1 and len OX_1 = len OX. Hence the locus of Y is again an ordinary parabola, with "diameter" OX, and tangent at O parallel to XY. If the index X move on OF, away from S, then XY, XY' lie on OF, and one of the stigmata will encroach on OS, but never farther than S. If these ordinates be turned through a right angle, the result is an ordinary parabola with focus D and axis OD. If X fall on S, $(y-s)^2+4s^2=o$, and len YY' = len OE. If X fall on D, $(y-d)^2=4s^2$, and if $2d=s+s'$, then (ds), (ds') are the two stigmins of the directrix $d-x=o$ with the parabbal. In all cases

$$(d-x)^2=(s+x)^2=(s-x)^2+4sx=(s-x)^2-(y-x)^2,$$

which is the generalisation of the property whence the directrix was named, giving in the Cartesian case, len SY = len DX.

(iii.) To determine the stinnals of the primal $y-x+tx=b$ with the parabbal $(y-x)^2+4sx=o$, we find

$$t^2x^2-2tbx+b^2=4dx \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

whence $$t^2x = bt + 2d \pm 2\sqrt{(btd+d^2)} \quad\text{.....................}\ (2).$$
In the general case this is best constructed as in (viii.).

(iv.) If $t = o$, or the primal is parabscissal, (i.) becomes $b^2 = 4dx$, and hence there is only one stinnal. Such primals are termed *paraxials* (*par*-a + *axi*-s + *al*) in preference to *diametrals*, a term applicable to central quadrals only. There is no asymptal.

(v.) If $bt = s$, there will be only one stinnal by the reduction of (1) to a complete square. In this case $t(y-x) = 2s$ gives T, the direction point of the tangental at (xy), of which, if (x_1y_1) be any stigmal upon it, the equation is $(y_1-x_1)(y-x) + 2s(x+x_1) = o$.
If N, P be the double and original points of tangental at (xy),

$$n + x = o, \quad p = \tfrac{1}{2}(y-x) = s\,.\,Rt, \quad t = s\,.\,Rp,$$
$$n = p\,.\,Rt = s\,.\,R^2t = p^2\,.\,Rs, \quad p^2 = sn = i'\,.\,sx.$$

If T' be the direction point of pri (ss, op), then $\tau = p\,.\,Rs = Rt$, or $\tau t = i$, so that this primal is orthal to the tangental. Also, since $s(s-y) = s[s-x-(y-x)] = s(s+p^2\,.\,Rs-2p) = (s-p)^2$, SP is the mean bisector of SY, SO. These generalise known properties.
The value of N being independent of Y, two tangentals can be drawn from (nn), and the ordinal (xy, xy') will be the contact chordal.

(vi.) Transordinate indicially; assuming $x = u + a(x'-v) + b(y-x')$. The equations to the new axals found by putting $y = x'$, and $x' = v$ alternately, are $x = u + a(y-v)$, $x = u + b(y-v)$, which intersect in (uv). Substituting in $(y-x)^2 + 4sx = o$, and assuming $(v-u)^2 + 4su = o$, $a = i$, $(v-u-2s)\,b = v-u$, in which case (uv) is a stigmal on the parabbal, and the new axals are a paraxial and a tangental at (uv), we find $(y-x')^2 + 4(s-v)\,.\,(x'-v) = o$, an equation of precisely the same form as before. To find Y from X', draw $VZ = 4SV$, and take $X'Y$ equal to mean bisector of VX', VZ.

(vii.) Let $(x''y'')$ be a stigmal referred to the axals in (vi.), and let $2v = x'' + x'$, then

$$(y''-x'')^2 = i'\,.\,4(s-v)\,.\,(x''-v) = i'\,.\,4(s-v)(v-x') = i'\,.\,(y-x')^2,$$

and hence these ordinates are of equal length and at right angles, so that $(x''y'')$ can be constructed from $(x'y)$.

(viii.) To determine intersections of pri (aa, ob) with the parabbal, see (iii.). Draw tangental (nn, op) parallel to (aa, ob), touching parabbal at (uv). It is determined by $bp = as$, $bn = ap$, $u + n = o$, $v - u = 2p$; see (v.). Through (uv) draw a paraxial, cutting pri (aa, ob) in (wx'') and find $(x''y_1)$ and $(x''y_2)$ as $(x''y'')$ was found in (vii.). In the Cartesian case Y_1Y_2 is perpendicular to AB. Then $(x''y_1)$, $(x''y_2)$ are the stinnals referred to the paraxial and tangental as axes, and Y_1Y_2 are the required stigmins. To these the indices X_1, X_2 referred to the old axes may now be found from the primal. But since $w-u = x''-v$, $(v-u-2s)(x_1-w) = (v-u)(y_1-x'')$, we find on substituting in $(y_1-x'') + 4(s-v)(x''-v)$, that $(x_1-w)^2 + 4u(u-w) = o$, so that (wx_1), (wx_2) are stigmals on a parabbal of which (uu) is the vertical, and (oo) the focal. In the Cartesian case the same equations shew that if Y_1W_1 be drawn perpendicular to the carordinate WX'', then $w_1 - x'' = x_1 - w = w - x_2$, which give X_1 and X_2 immediately.

(ix.) For tangentals from (hk). Through (hk) draw a paraxial cutting parabbal in (uv), take $2v = x''+k$, and find $(x''y_1)$, $(x''y_2)$, as in (viii.), then $(kk, x''y_1)$,$(kk, x''y_2)$ are the tangentals referred to the paraxial and tangental at its extremity, and (hk, x_1y_1), (hk, x_2y_2) the same referred to old axes, and (x_1y_1, x_2y_2), that is (aa, ob) in the chordal of contact, or polaral of the polar (hk). The paraxial through the stinnal of the tangentals cuts the chordal of contact at its middle stigmal.

(x.) For focal. If in iv. (2) we put $bt = s$, for tangental, and make $t = i$ or i', we obtain as the equations to the parassal tangentals (see art. 51. v.) $y = s$, and $2x-y = s$, or the primals (ss, os), and (ss, od). There is therefore only one focal (ss) where these two tangentals intersect. The stigmals of contact are respectively (ds), (ds') where $s+s' = 2d$, and hence (compare ii.) the contact chordal is the directrix.

(xi.) If N_1 be the direction point of the normal or orthal on tangental at (xy), and S_1 of the pri (ss, xy) from the focal, then $2sn_1 = y-x$, $(s-x)\, s_1 = y-x$, whence tal $S_1N_1 = n_1 = \tan N_1O$, which is the generalisation of the property that gave its name to the focus ; see art. 51. vii.

(xii.) To demonstrate (art. 51. iv.) for parabbals, proceed thus. From any stigmal on a parabbal draw chordals to four other stigmals on it, and draw tangentals at all the five stigmals, and through the stinnals of the last four tangentals with the fifth draw paraxials (having therefore the same anral as the stigmins of these tangentals), these will pass through the middle stigmals in the four chordals of contact, and hence have the same anral as the original points of four paraxials drawn from the first four stigmals of contact (art. 46. vi.). But this last anral is equal to the anral of the four chordals, which is again equal to the anral of four chordals drawn from the same four stigmals to any other stigmal.

(xiii.) The anral of the stigmins of four tangentals with a fifth is equal to the anral of the direction points of these four tangentals ; see Chasles *Sec. Con.* art. 58, where, as the tangentals have no common stinnal, he has been obliged to invent a new name, not here required.

Let the four stigmals of contact be (aa), (βb), (γc), (δd), and the four stinnals $(a'a')$, $(\beta' b')$, $(\gamma' c')$, $(\delta' d')$, and the four direction points of the tangentals at the four first stigmals be A_1, B_1, C_1, D_1 ; and the original points of the paraxials be A'', B'', C'', D''. Then, by (v.),

$$2a'' = a-a = 2s\,.\,Ra_1, \quad 2b'' = b-\beta = 2s\,.\,Rb_1, \ \ \&\text{c.},$$

$$\text{hence} \quad (a'b'c'd') = (a''b''c''d'') = \frac{(Ra_1-Rb_1)\,(Rc_1-Rd_1)}{(Ra_1-Rd_1)\,(Rc_1-Rb_1)} = (a_1b_1c_1d_1).$$

53. *Multindicials, or the meaning in Plane Geometry of Algebraical Equations with several Independent Variables.*—(i.) In stating the general conception in art. 36. i., only one index, X, was mentioned, for clearness. But it is evident that in the equation $f(x_1, x_2, \ldots x_n, y) = o$, the points X_1, X_2, ... X_n may be assumed as indices respectively, and the resulting values of y determined, giving stigmata of which each one corresponds to many indices. Such stigmatics are distinguished as *mult-indicials.* Hence there is no need to proceed beyond plane geometry for the perfect treatment of the relations of all such equations as are now referred to real geometry of three dimensions or ima-

ginary geometries of n dimensions. As long as commutative algebra only is used, the stigmatic conception, with the algebra of clinants, allows of every result being clearly and distinctly considered as the algebraical expression of a geometrical relation of points on a plane.

(ii.) But multindicials as well as *sol-indicials* (having one index) may be treated in the manner which originally suggested itself to me (Appendix III.) by assuming $O\Xi_1$, $O\Xi_2$, $O\Xi_3 \ldots O\Xi_n$, OH', as unit radii, and determining a point R, by the condition $r = x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n + y\eta$. This is what is in fact done in Cartesian geometry, in the form $r = x\xi + y\eta$, only scalar values of x and y being then admissible, whereas clinant values give the complete generalisation. We have thus *derived* stigmatics, of which the most general form would be

$$r = F\,[f_1(x_1, x_2 \ldots x_n, y)\,.\,\xi_1, \quad f_2(x_1, x_2 \ldots x_n, y)\,.\,\xi_2, \ldots\ldots].$$

Some of these I investigated in my original papers of 1855 and 1860, (see Appendix III.,) and the results are sometimes very curious.

54. *Solid Stigmatics.*—(i.) The Cartesian solid geometry results from a species of the *derived* stigmatics just mentioned, OI, OJ, OK being three unit radii (here supposed to be rectangular) of a unit sphere, and R the point that we wish to investigate; on assuming $OR = x\,.\,OI + y\,.\,OJ + z\,.\,OK$, any equation $f(x, y, z) = o$, will, for any given values for x, y, determine values of z. If the given values of x, y, and the determined values of z, be *all* scalar, the point R can be drawn. But if they be *not* scalar the conception is insufficient to determine R, until it is supplemented in various ways, and hence the custom of supposing R to become an "imaginary point," the fact being that no provision had been made for this case.

(ii.) Among such provisions as might be suggested, the following would always give a position for R, which would agree with that now assigned so far as the Cartesian case is concerned. Suppose OIJ to be the clinant plane, but suppose it also to be movable, and that it can be placed so as to make OI, OJ coincide with OI, OK, or with OK, OI respectively. This amounts to saying, allow OJ, OK on the plane JOK, and OK, OI on the plane KOI to *function as* OI, OJ on the plane OI. In this case, $x\,.\,OI$ gives a line OX_1 on the plane IOJ; $y\,.\,OJ$ gives a line OY_1 on the plane JOK; and $z\,.\,OK$ gives a line OZ_1 on the plane KOI, with perfect certainty and distinctness; and then, as before, $OR = OX_1 + OY_1 + OZ_1$, by the usual operations of directional addition of directed lines in space, R being the summit opposite to O of the parallelopipedon of which OX_1, OY_1, OZ_1 are adjacent sides. This is only one out of numerous possibilities. It is clearly *not* a general conception. It is merely one of those geometric contrivances *ad hoc*, useful enough as illustrations, but not suitable for universal adoption, like Poncelet's supplementary ellipses and hyperbolas, all very well in their way, but needing no farther notice in a Tract on principles.

(iii.) Clinant or purely commutative algebra is not adapted for the purposes of solid geometry, which involves non-commutative operations, when the plane on which the similar triangles are to be constructed, is constantly movable. The required instrument is furnished by *quaternions*, but the resultant stigmatic geometry differs from the former, owing to the variability of plane. In clinants, two points, O and I, could be considered fixed, and one only, X, being variable, could pass into any point of the plane, and hence determine any triangle on that plane. Now it might also pass into any point in space, but in doing so it would determine triangles only on such planes as intersect in OI. To complete the geometry of space, the standard line must be itself movable, but its *origin* may be fixed, and the *length* of its initial limit may be unchanged. Let then OM be a unit radius in the same unit circle as before, so that $OM = m \, . \, OI$, and $Tm = i$, where m is a clinant. OM may be called the (*unit*) *base*, M the *base point*. Let X be any point in space, which may be called the *vertex*. Then MOX will be any triangle on, or parallel to, any plane in space; and if OA be any line parallel to the plane of MOX, it is possible to construct $AOB \, \Delta \, MOX$, and thus determine B. The operation thus performed is called a *quaternion*, and may be represented by x_m, the subscript letter referring to the clinant m, so that $OB = x_m \, . \, OA$. This is the operation, differently conceived, of which Sir W. R. Hamilton has investigated the laws, and we see that *clinants are quaternions with a constant base point and constant plane of rotation*, or for which x_m always $= x_i = x^{\cdot}$ on the plane IOJ. Now assume the laws of quaternions as established by Sir W. R. Hamilton, and let y_n be some other quaternion, and let $\phi \, (x_m, y_n) = o$. Then, so far as this equation can be solved, (which is not very far, for Sir W. R. Hamilton only solved the equation of the first degree completely,) the assumption of any two points M, X, forming a *quin* (*qu*-aternion *in*-dex) will determine two other points N, Y, forming a *quas* (*qua*-ternion *s*-tigma). The relation then is not one between *two points*, index and stigma, forming a stigmal, but between two *pairs* of points, quin and quas, forming a *qual* (*qu*-aternion stigm-*al*), and hence partakes of the character of the relation between an indi-stigmal and a stigmo-stigmal in the case of a transordinated stigmatic, (art. 47. i.) This bare statement of the conception must here suffice. Solid stigmatics, and the correspondence of points lying in different planes, lie beyond the scope of this Tract, although the geometry here developed allows of such correspondence being expressed in various particular cases, by the aid of conventions similar to those in (ii.) and those indicated in the first case of art. 44. iv.

CONCLUSION.

55. Such is my Stigmatic Geometry. The sketch is rough, and bare of detail, but the outline is, I trust, sufficiently firm and true for Mathematicians to recognise the main features of my Theory, and to justify my own confidence that Clinants and Stigmatics are a New Power in Mathematical Analysis, a New Instrument for Geometrical Investigation, and a New Form of Life for Algebra.