Jarník's note of the lecture course Punktmengen und reelle Funktionen by P. S. Aleksandrov (Göttingen 1928)

Summary of results and bibliographical and historical comments

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SUMMARY OF RESULTS AND BIBLIOGRAPHICAL AND HISTORICAL COMMENTS

Since the turn of the century, ...
mathematics is more like the Nile Delta,
*its waters fanning out in all directions.**

Aleksandrov's course on point sets and real functions has several remarkable features which we want to mention at the beginning. The course

– reflects the state-of-the-art in important parts of these fields of mathematics in 1928

– is very *modern* in the sense that practically all results presented were less than thirty years old, the majority were less than ten years old, and some were not yet published

– centers around several excellent results established by P. S. Aleksandrov himself as well as by his collaborators and colleagues such as P. S. Urysohn and M. Ya. Suslin; in particular, we have in mind the proof of the Continuum Hypothesis for Borel sets, the A-operation, properties of analytic sets, topologically complete spaces and the role of zero-dimensional spaces

– includes principal results and concepts that have become standard facts presented today in basic university courses for mathematicians

– has a strong topological flavor even though it provides results in the context of the real line, Euclidean spaces, and metric spaces, and the notion of a topological space is not even mentioned; this reflects the fact that Aleksandrov was one of the leading architects in the construction of topology as a mathematical subject

– uses mathematical language which is fully set-theoretical and a style of exposition of results and their proofs that is similar to the contemporary way of presenting mathematics; perhaps the main difference is that the use of transfinite numbers and transfinite induction is much more frequent in comparison with the present state and the same applies to the use of continued fractions

– shows only a few differences in notation from today: $x \subset A$, $A + B$, AB, $A - B$, ΣA_n, ΠA_n were later replaced by $x \in A$, $A \cup B$, $A \cap B$, $A \setminus B$, $\bigcup A_n$, $\bigcap A_n$, respectively, the empty set is, unlike our $\emptyset$, written

* H. Weyl: *A half-century of mathematics*, Amer. Math. Monthly **58** (1951), 523–553; p. 523.

as 0, $\lim_{n=\infty}$ unlike our $\lim_{n\to\infty}$; also the distinction between f (= a function) and $f(x)$ (= the value of f at the point x) is not usually respected.

In summary, the course was delivered by a distinguished expert whose impact on contemporary mathematics is felt till today.

The aim of this chapter is to give an overview of the notions and results presented in the course. We give the definitions, theorems as well as remarks, and at several places add comments on the methods of proof. We tried to trace the origin of the results in order to place the material in historical context. Several quotations and bibliographical comments should illustrate the fact that the first quarter of the 20th century was a fascinating period for the development of descriptive set theory, real analysis and point-set topology.

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1. Point sets[**]

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[**] The numbering as well as pagination refer to the text of the course (pp. 51–90).

1. Point sets

1.1 *Metric spaces*[1] are defined in the usual way. Two definitions of a *complete metric space*[2] are given:

(i) the usual definition using Cauchy sequences (*Fundamentalfolgen*),

(ii) an alternative definition: a metric space R is said to be *complete*, if it is impossible to add a point ξ to R and introduce a new metric on $R \cup \{\xi\}$ that coincides with the old metric on R and for which ξ is not isolated.

It is proved that both definitions are equivalent. Using the notion of equivalent Cauchy sequences (*konfinale Fundamentalfolgen*), it is proved that an arbitrary metric space R admits a *completion*,[3] that is, there exists a complete space R' such that R is a dense subspace of R'. Uniqueness is only mentioned (*Volständige Hülle – Hausdorff*[4]).

It is remarked that a metric space can be equipped with different metrics such that the corresponding classes of convergent sequences coincide, that is, topological properties are preserved (*Raum eine "erlaubte Abänderung der Metrik erleidet"*).[***] However, such a change of metric need not preserve completeness.[5] A metric space (R, ρ) is *topologically complete*, if a topologically equivalent metric ρ' on R can be introduced in such a way that (R, ρ') is complete. Example: the interval $(0, 1)$ equipped with usual metric is not complete, but it is homeomorphic to the complete space $(-\infty, \infty)$.

The following statement is mentioned without proof (the names of Tychonov and Niemycki are mentioned[6]): For a metric space R, the following conditions are equivalent:

(i) *R is complete with respect to any topologically equivalent metric*;

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[1] The notion of a metric space appears in [Fr 1906] where the axioms of a metric (*écart*) are introduced. In [H 1914], Hausdorff replaced the Fréchet terminology *une class* (E), later *une class* (D) (*distance*), by the term *metric space*; see [H 1914], p. 211.

[2] The idea of completeness and of completion goes back to the so-called Bolzano-Cauchy convergence condition and to constructions of real numbers by means of equivalent Cauchy sequences of rational numbers; see [Me 1869], [Me 1872], [D 1978], pp. 81–83, [Kl 1972], pp. 982–987, 990–992, [St 2008], Chapter 16. M. Fréchet speaks of metric spaces which *admet une généralisation du théorème de Cauchy*, see [Fr 1906], p. 23. The term *complete* (*vollständig*) appears in [H 1914], p. 315. For more information on the development of the notion of completeness, see [D 1984].

[3] The result that every metric space admits a *completion* (*Vervollständigung*), that is, every metric space can be imbedded into a complete metric space as a dense subspace, is presented in [H 1914], pp. 315–316.

[4] In [H 1927], p. 107, it is shown that a completion is uniquely determined up to a metric isomorphism.

[***] We use the terminology: *topologically equivalent metrics*.

[5] For a discussion on completeness we refer to H. Herrlich, M. Hušek, G. Preuß: *Vervollständigung und totale Beschränktheit*, [H 2002], pp. 767–772.

[6] See [NT 1928]; also [H 1930] and a commentary on this paper by H. Herrlich, M. Hušek, G. Preuß in [H 2008], pp. 466–467.

(ii) *R is compact, that is, every sequence of points in R possesses an accumulation point.*

1.2 Zero-dimensional space[7]

A metric space M is said to be *zero-dimensional*, if, for every $\varepsilon > 0$, there exist sets $M_1, M_2, \dots$ with union M such that

(a) $D(M_k) < \varepsilon$ (here $D(A)$ denotes the diameter of A),

(b) each M_k is relatively open in M,

(c) $M_i \cap M_j = \emptyset$ whenever $i \neq j$.[8]

A *neighborhood* of a point $p \in M$ (with respect to M) is an open set in M containing the point p.

It is noted that the sets M_k from the above definition are closed in M and that a subset of a zero-dimensional set is obviously zero-dimensional.

A *local base* (*Umgebungsbasis*) in a space M is a system of open sets in M such that, for every $p \in M$ and every $\varepsilon > 0$, there exists a set A in the system with $D(A) < \varepsilon$ and $p \in A$.

The following theorem is proved: *In every zero-dimensional set, there exists at most countable local base $U_1, U_2, \dots$ with the following properties:*

(a) *For $i \neq j$, $\quad U_i \subset U_j \quad$ or $\quad U_i \supset U_j \quad$ or $\quad U_i \cap U_j = \emptyset$,*

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[7] The exposition follows closely [AU 1928]. The text from a footnote on p. 89:

Die Resultate der vorliegenden Arbeit stammen im wesentlichen vom Frühjahr 1924. Der vorliegende Text ist aber erst im April 1926 vom Unterzeichneten entgültig redigiert worden. Paul Alexandroff.

The following interesting remark from [Pu 1998] is quoted in *Komentar zu* [H 1937] by H. Herrlich, M. Hušek, G. Preuß; see [H 2008], p. 550:

P. Alexandroff and P. Urysohn [AU 1928] *used continued fractions in 1928 to characterize the irrational numbers as a topologically complete, separable,* 0*-dimensional, metric space that contains no nonempty compact open set. In 1937 this result was rediscovered by Hausdorff* [H 1937] *using Baire's result of the homeomorphism between the irrationals and* $\mathbb{N}^{\mathbb{N}}$ *(which Hausdorff referred to as "der Bairesche Nullraum"). It is interesting to note that it was not unusual for Alexandroff and Hausdorff to independently prove the same result. For example they both verified the truth of the Continuum Hypothesis for the class of Borel sets on the real line* ([A 1916] *and* [H 1916]), *and they both proved that every non empty compact metrizable space is the continuous image of the Cantor set* ([A 1926] *and Theorem V in section 35, p. 197 in* [H 1927] *as well as the announcement in* [A 1925]).

For further comments on zero-dimensional spaces we refer to *Komentar zu* [H 1937] quoted above as well as to [Ku 1933], §20, §21.

The importance of the Baire space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ is illustrated by the following Aleksandrov-Urysohn's theorem (see [Ke 1995], p. 37): *The Baire space $\mathcal{N}$ is the unigue, up to homeomorphism, nonempty Polish zero-dimensional space for which all compact sets have empty interior.* (Recall that a separable completely metrisable space is called *Polish.*)

[8] See [AU 1928], p. 90.

(b) *every increasing sequence*

$$U_{i_1} \subset U_{i_2} \subset \dots$$

reduces to a finite sequence.[9]

Further, the following assertions are proved:

(a) *The set of irrational numbers is zero-dimensional and topologically complete.*

(b) *Every zero-dimensional space is homeomorphic to a subset of the irrational numbers.*[10]

(c) *A zero-dimensional space M is not compact if and only if there is a partition of M into infinitely many nonempty disjoint open subsets.*[11]

(d) *A subset of irrational numbers is compact if and only if it is bounded, closed, and nowhere dense.*

To prove (a), a metric ρ on the set of irrational numbers is introduced as follows. Irrational numbers are written as continued fractions. If

$$\xi := m_1 + \frac{1|}{|m_2} + \frac{1|}{|m_3} + \dots, \qquad \eta := n_1 + \frac{1|}{|n_2} + \frac{1|}{|n_3} + \dots,$$

then $\rho(\xi,\xi) := 0$ and, for $\xi \neq \eta$, $\rho(\xi,\eta) := \frac{1}{2^k}$, where k is the smallest index i with $m_i \neq n_i$.

The following notions are introduced: A space M is said to be *locally compact at a point* (*kompakt in einem Punkte*),[12] if there exists a neighborhood of this point having a compact closure. A space M is called *locally compact* (*in kleinem kompakt*), if it is locally compact at every point.

A characterization of locally compact sets of irrational numbers is proved: *A locally compact subset of irrational numbers is a difference of two nowhere dense closed subsets of the real line.*

In view of the fact that local compactness is invariant with respect to homeomorphisms, the following statement holds: *Every zero-dimensional locally compact space is homeomorphic to a nowhere dense closed set of irrational numbers.*

The next result proved reads as follows: *Every complete homogeneous zero-dimensional space which is not locally compact is homeomorphic with the set*

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9 See [AU 1928], Satz I, p. 90.

10 See [A 1928], Satz II, p. 36. The fact that the Baire space is homeomorphic to the space of irrational numbers was proved in [Ba 1909], p. 103.

11 See [AU 1928], Satz II, p. 92.

12 See [A 1924a], p. 294.

of irrational numbers.[13] Here a space is called *homogeneous*, if, for every pair of points, there are homeomorphic neighborhoods.

1.3 Classes of sets

Firstly, the notion of relatively open sets and relatively closed sets is recalled. It is also mentioned that, for a continuous function f, the set $\{a \leq f \leq b\}$ is closed and the set $\{a < f < b\}$ is open.

Given a system of sets $\mathfrak{M}$, the systems $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\delta$ are defined in a usual way:[14] $M \in \mathfrak{M}_\sigma$, if there are $M_1, M_2, \ldots$ from $\mathfrak{M}$ such that $M = \bigcup_{n=1}^{\infty} M_n$; $M \in \mathfrak{M}_\delta$, if there are $M_1, M_2, \ldots$ from $\mathfrak{M}$ such that $M = \bigcap_{n=1}^{\infty} M_n$.

Starting with $\mathfrak{F}$ (= the system of closed sets) and $\mathfrak{G}$ (= the system of open sets), one defines, declaring $\mathfrak{F}$ and $\mathfrak{G}$ as 0-th clases, two classifications:

$$\mathfrak{F}, \quad \mathfrak{F}_\sigma, \quad \mathfrak{F}_{\sigma\delta}, \quad \mathfrak{F}_{\sigma\delta\sigma}, \quad \mathfrak{F}_{\sigma\delta\sigma\delta}, \quad \ldots,$$

$$\mathfrak{G}, \quad \mathfrak{G}_\delta, \quad \mathfrak{G}_{\delta\sigma}, \quad \mathfrak{G}_{\delta\sigma\delta}, \quad \mathfrak{G}_{\delta\sigma\delta\sigma}, \quad \ldots;$$

this procedure is extended to the transfinite: given an infinite countable ordinal (*eine Zahl 2. Zahlenklasse*) α of the form $\alpha = \beta + n$ (where β is a limit ordinal and n is finite), α is called *even* or *odd*, if n is *even* or *odd*, respectively. For α even (resp. odd), $\mathfrak{F}_\alpha$ stands for the class of sets obtained by countable intersection (resp. union) from sets of classes $\mathfrak{F}_\beta$ with $\beta < \alpha$. Similarly for $\mathfrak{G}_\alpha$ with intersection and union interchanged. It is proved that $\mathfrak{F}_\alpha \subset \mathfrak{G}_{\alpha+1}$, $\mathfrak{G}_\alpha \subset \mathfrak{F}_{\alpha+1}$ and that *every closed set is a G_δ-set* (that is, it belongs to $\mathfrak{G}_\delta$) and *every open set is an F_σ-set* (that is, belongs to $\mathfrak{F}_\sigma$).[15]

It is emphasized that a particularly important role is played by F_σ-sets and G_δ-sets. As a first illustration it is shown that, for an arbitrary function, *the set of all continuity points is a G_δ-set.*[16] The significance of F_σ-sets and G_δ-sets in measure theory is explained (every measurable set can be squeezed between an F_σ-set and a G_δ-set of the same measure). It is noted that this lies behind continuity properties of measurable functions.[17]

1.4 Extension properties and absolute G_δ sets

The following results are proved: *If f is continuous on a set M in a Euclidean space, then f can be continuously extended to a G_δ-*set *containing M. If, moreover, f is a homeomorphism, then f can be extended to a homeomorphism between two G_δ-sets containing the domain of f and the range of f, respectively.*[18]

[13] See [AU 1928], Satz IV, p. 95.

[14] See [H 1914], p. 304.

[15] See [H 1914], p. 306. The notation G, F, G_δ, F_σ was introduced by Hausdorff and recalls the terms *Gebiet* (an open set) and *fermé* (closed). The indices δ and σ are used on p. 23 and recall *Durchschnitt* (an intersection) and *Summe* (union), respectively.

[16] The result on continuity points goes back to W. H. Young [Y 1903b]; see [H 1927], p. 251.

[17] This refers to [Le 1902] and to Luzin's theorem; cf. [Le 1903] and [Lu 1912]. Interesting comments and further references are given on pp. 426–427 in [Bg 2007a].

[18] This result was proved by M. Lavrentieff in [La 1924]; see also [H 1927], p. 216.

Then the following question is advanced: Why are G_δ-sets so important? The genuine reason lies in the following result (**A**) by Aleksandrov: *G_δ-sets are nothing else than topologically complete spaces.*[19]

One can ask: Which topological properties may be used for a characterization of the topological completeness?[20]

A *base* of a metric space is a system of open sets such that all open sets can be expressed as a union of sets from the system. A base is said to be *complete* if, for every decreasing sequence of sets of the base, there exists a point belonging to the closure of each set in the sequence.

As an example, a complete base for the space of irrational numbers is constructed (using a continued fractions expansion).

It is mentioned that a space is topologically complete if and only if every base contains a complete base,[21] which occurs exactly for G_δ-sets. However, the proof of the statement (**A**) is proved directly without recourse to properties of a base.[22]

Theorem. *Let M be a subset of metric space R. Suppose that M can be endowed by a complete topologically equivalent metric. Then M is a G_δ-set in R.*[23]

The converse, namely that a G_δ-set can be transformed into a complete space by means of a topologically equivalent metric, is obviously false. In fact, every metric space is a G_δ-set in itself, but it need not be topologically complete.

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[19] Let us quote from p. 448 in *Commentary on* [H 1924] by H. Herrlich, M. Hušek, G. Preuß; see [H 2008]: *Topologists realized the importance of competely metrizable spaces (mainly around 1920) and tried to find purely topological characterizations (first in realm of metrizable spaces). In* [Fr 1921], *M. Fréchet asked whether there exists a metric space not admitting a complete topologically equivalent metric. He did not realize that the answer follows from his own result published in* [Fr 1910], *p. 8, namely that each non-empty complete dense-in-itself metric space must be uncountable, so that the metric space of rational numbers does not admit a complete metric. In fact, for rational numbers an even older result from* [YY 1906] *could be used.*

[20] A topological description of complete spaces is given in [A 1924b]. See also [W 1930].

[21] The paper [A 1924b] is considered as the beginning of the study of completeness in topological context. Aleksandrov's proof of the characterization of complete metrizability in terms of a complete base uses separability. This assumption was removed by Hausdorff (1926, unpublished) and by Vedenissov [W 1930]; see Commentary on [H 1924] by H. Herrlich, M. Hušek, G. Preuß in [H 2008], p. 450. Why Hausdorff has never published his result is explained there (a quotation from a letter by Hausdorff to Aleksandrov and Urysohn).

[22] To prove the completeness, Aleksandrov indicates in [A 1924b] a construction of a complete base which makes it possible to find a complete metric. To that end, his argument includes the following important result: *Every open cover of a separable metric space admits an open refinement that is locally finite.* In other words, Aleksandrov showed, in modern terminology, *paracompactness* of separable metric spaces. This result was rediscovered by J. Dieudonné in 1944 and A. H. Stone removed the hypothesis of separability in 1948. It seems that Aleksandrov never published a detailed proof developing the idea from [A 1924b].

[23] This result is contained in [A 1924b]. The proof presented in the lecture was suggested by Urysohn and is contained in a letter (dated by May 21, 1924) of Aleksandrov and Urysohn to Hausdorff; for details, see *Commentary on* [H 1924] by H. Herrlich, M. Hušek, G. Preuß [H 2008], pp. 449–450.

The correct statement reads as follows: *Let M be a G_δ-set in* a complete *space. Then there exists a topologically equivalent metric making M a complete space.*

We give some comments on the method of the proof presented in the lecture course.[24] Let (R, ρ) be a complete metric space and M be a G_δ-set in R. For a closed subset F of R, denote $G := R \setminus F$ and, for x, $y \in G$, define

$$\rho_F(x, y) := \frac{\rho(x, y)}{\rho(x, y) + \rho(x, F) + \rho(y, F)}\,,$$

where $\rho(p, F)$ is the distance of the point p from the set F. Then ρ_F is a topologically equivalent metric on (G, ρ). Now, M is a G_δ-set, that is, there are open sets G_n in R such that $M = \bigcap_1^\infty G_n$. Denote $F_n := R \setminus G_n$ and, for x, $y \in M$, define

$$\rho^*(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_{F_n}(x, y).$$

It is proved that ρ^* is a topologically equivalent metric on (M, ρ). Defining, for x, $y \in M$,

$$\rho'(x, y) := \rho(x, y) + \rho^*(x, y),$$

ϱ' is obviously a topologically equivalent metric on (M, ρ). It is not difficult to see that ρ' is a complete metric on M.

Consequently, a G_δ-set M in a complete space is topologically complete, whence M is a G_δ-set in every (in particular in every complete) space in which M is imbeded. Thus sets which are G_δ in a complete space (and thus in every complete space) are called *absolute G_δ-sets.*

The notion of an absolute G_δ-set is topologically invariant. More precisely: *Let M be a G_δ-set in a complete space, let M^* be contained in a metric space R^* and let M^* be a homeomorphic image of M. Then M^* is a G_δ-set in R^*.*

2. Baire functions and Borel sets

Real functions on $[0, 1]$ are considered. *A* function system is called a *Baire system*, if

(a) together with f_1, f_2, also $f_1 \pm f_2$, $f_1 \cdot f_2$, f_1/f_2 belong to the system (provided the ratio is well defined);

(b) together with $f_1, f_2, \dots$ also $\lim_{n\to\infty} f_n$ belongs to the system (pointwise convergence).

An intersection of an arbitrary family of Baire systems is a Baire system and the family of all functions on $[0, 1]$ is a Baire system. Hence, for every family Σ

[24] The proof used by Aleksandrov in his lecture copies that of Hausdorff from [H 1924]. The Hausdorff proof does not require separability assumption.

of functions, there exists the smallest Baire system containing Σ (namely the intersection of all Baire systems containing Σ).

A system of subsets of the real line or of $[0,1]$ is said to be a *Borel system*, if it contains $\bigcap_{n=1}^{\infty} M_n$ and $\bigcup_{n=1}^{\infty} M_n$, whenever $M_1, M_2, \dots$ are sets of the system.

An intersection of an arbitrary family of Borel systems is a Borel system; all sets form a Borel system. Hence, for every family Σ of sets, there exists the smallest Borel system containing Σ (namely the intersection of all Borel systems containing Σ).

The smallest Baire system containing continuous functions is obtained as follows:[25] We call all continuous functions on $[0,1]$ *functions of the* 0-*th class.* If α is a countable ordinal and if all classes $< \alpha$ have already been defined, then the α-*th class* is the family of all functions which can be represented as pointwise limits of sequences of functions belonging to classes $< \alpha$. It is proved that the sum, difference, product, absolute value, maximum and minimum of functions of the α-th class is a function of the same class. It is mentioned that a corresponding result for the ratio causes a difficulty and will be postponed.****

It is explained that the family of all functions belonging to some α-th class for countable ordinals α is the smallest Baire system containing the continuous functions.

The classes defined above are briefly called *Baire classes* and the functions are called *Baire functions* or *analytically representable functions.* Especially, the functions of the α-th class are called the *Baire functions of the* α-*th class.*

Analogously, starting with open and closed sets as sets of the 0-*th class*, we define the set of α-*th class* for every countable ordinal (the so called *Borel sets of the* α-*th class*). We arrive at the system of *Borel sets* as the smallest Borel system containing open and closed sets.

There is a close relation between Baire functions and Borel sets. Firstly, the following result is proved: *If M is a Borel set, then there exists a Baire function f such that $M = \{f > 0\}$.* In the course of the proof it is established

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[25] The hierarchy of Baire functions was introduced in [Ba 1899] and extensively studied in [Le 1905]; see also [Po 1916], [D 1981]. In *Topologie* by Aleksandrov and Hopf [AH 1935], p. 20, the significance of Lebesgue's work is expressed as follows: *Die deskriptive Mengenlehre wurde (anschließend Arbeiten über unstetige Funktionen) von Lebesgue begründet. Ihre weitere Entwicklung beginnt elf Jahre später mit dem Mächtigheitsatz für die Borelschen Mengen …* [this refers to [A 1916]]. Borel sets appear implicitly (as measurable sets) in connection with the introduction of the Borel measure; see [Bo 1898], pp. 46–47 and [Bo 1905a], p. 17. In [Le 1905], Lebesgue speaks of B-measurable sets (*ensembles mesurables* B) as sets that can be obtained from intervals from $\mathbb{R}$ by means of repeated application of countable unions and differences. The terminology used nowadays goes back to [H 1914], p. 305.

**** In fact, in the notes, this is proved later on for $\frac{1}{f}$, if f is strictly positive. However, for a non-vanishing function f, $f = \lim f_n$, one can write $\frac{1}{f} = \lim \frac{f_n}{f_n^2 + \frac{1}{n}}$ and the result follows immediately.

that, for a Borel set M, the function equal 1 on M and 0 elsewhere is a Baire function. Next, the following result is proved: *If f is a Baire function, a and b are real numbers, then the sets*

$$\{a \leq f \leq b\} \quad \text{and} \quad \{a < f < b\}$$

are Borel sets. Hence, every set defined by an analytic condition is a Borel set and the converse holds are well: *Let f be a function. If $\{a \leq f \leq b\}$ is a Borel set for every a and b, then f is analytically representable.* This illustrates how closely Borel sets and Baire functions are related.[26]

The next result shows that the classes of Baire functions and Borel sets are rich: *For every infinite countable ordinal α, there exists a Baire function which belongs to the α-th class but does not belong to any lower class, and a Borel set which belongs to the α-th class but not to any lower class.*[27]

Firstly, it is explained that it is sufficient to establish the result for Borel sets only. Next, the main ingredient is the following auxiliary result (attributed to Sierpiński[28]): *For every infinite countable ordinal α, there exists a set-valued mapping Φ_α, defined on sequences of sets, such that, whenever $M_1, M_2, \dots$ runs through all sequences of sets of a system $\mathfrak{M}$, then $\Phi_\alpha(M_1, M_2, \dots)$ runs exactly through all sets of most α-th class over $\mathfrak{M}$.*

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[26] A parallel study of Baire functions and Borel sets is due to Lebesgue [Le 1905]; see also [H 1914], p. 391, [H 1927], pp. 232–243. Those are functions and sets which can be obtained by an *analytic construction*, that is, can be represented analytically (*represéntable analytiquement*). Let us note that the beginning of the 20$^{\text{th}}$ century was marked by a crisis in the foundations of mathematics (the legitimity of Zermelo's well-ordering principle and the axiom of choice) and discussions on which definitions of mathematical objects are acceptable was quite controversial; see, for instance, [BBHL 1905], [Bo 1905b], [Md 1976], [Md 1991] (and also the commentary in *Deskriptive Mengenlehre in Hausdorffs Grundzügen der Mengenlehre* by V. Kanovei and P. Koepke, [H 2002], pp. 773–787). For instance, Lebesgue's point of view was that, say, sets and functions have to be properly defined (*définies*) or at least described (*décrites*). (The last word was the motivation for the terminology *descriptive set theory*.)

A certain tension among mathematicians concerning *well defined* objects and differences of notions as *défini*, *décrit* and *choisir*, *nommer* is explained in [Us 1985], pp. 85–91; cf. also the Preface to [Lu 1930] written by Lebesgue.

In fact, already the title of [So 1917] illustrates the fact that the authors of that period were attentive to the use of set-theoretic tools; cf. [Us 1985], pp. 85–91. We quote from Suslin's paper, p. 90: *… nous avons trouvé, sans utiliser l'axiome de M. Zermelo et les nombres transfinis, un ensemble* (A) *tel que son complémentaire rélativement à l'intervalle* $(0, 1)$ *n'est pas un ensemble* (A).

Also Luzin writes in [Lu 1917], p. 93: *Si l'on introduit les nombres transfinis en infinité* (énumérable), *on peut supprimer* [footnote: *Cette remarque est due à M. Souslin.*] *de l'énoncé du théorème I (et I')* [see our footnote [45]] *les mots* à un ensemble énumérable de points près. *On démontre le corollaire sans employer l'axiome de M. Zermelo, mais il faut employer, pour former effectivement la série de polynômes du corollaire, les nombres transfinis en infinité énumerable (c'est-à-dire ceux qui sont inférieurs à l'un d'eux).*

[27] See [Le 1905], pp. 208–211.

[28] See [Si 1920] and a detailed discussion on pp. 570–574 in [H 2008].

3. Analytic sets[29]

Suslin sets or $\mathcal{A}$-*sets* over a set system $\mathfrak{M}$ are defined as follows: One takes from $\mathfrak{M}$ a countable system of set and label the sets in the following way:

$$(*) \qquad M_1, M_2, \ldots; M_{1,1}, M_{1,2}, M_{2,1}, \ldots; M_{1,1,1}, M_{1,1,2}, \ldots, M_{i_1,i_2,\ldots,i_k}$$

($k, i_1, i_2, \ldots, i_k$ mutually independent natural numbers). From this countable system one selects the so-called *chain*

$$M_{i_1}; M_{i_1,i_2}; M_{i_1,i_2,i_3}; \ldots$$

(i_1 fixed through the whole sequence; the same for i_2 beginning the second member, etc.).

The intersection of all sets of this chain is called the *kernel* of the chain. Also, the unions of the kernels of such chains which can be formed from the system $(*)$ are called *Suslin sets* over $\mathfrak{M}$ and any set of this form is a result of

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[29] We choose the following interesting quotation on the birth of analytic sets from [Ku 1980], pp. 68–69:

Here I would like to emphasize the collaboration of Sierpiński, Lusin and Suslin. Suslin was at that time a most gifted student of Lusin at Moscow University. On Lusin's recommendation Suslin was studying Lebesgue's celebrated article, "Sur les fonctions représentables analytiquement", published in the Journal de Mathématique, 1905.

That article contained a false statement namely that the projection (and therefore any continuous image) of a Borel set is a Borel set. As Professor Sierpiński told me, he was a witness of a conversation between Suslin and Lusin, in which Suslin communicated his discovery of that error.

Lusin thought it unbelievable that the great Lebesgue could commit a mistake. This was all the more surprising because Lebesgue deduced his result from a trivially false statement, viz. that the projection of an intersection of two (plane) sets is the intersection of projections of those sets (Lebesgue was very possibly misled by the fact that in the case of union such a commutativity does indeed hold). A counterexample to this hypothesis would be a system of two parallel segments situated one above the other.

The discovery of Borel sets whose continuous image is not a Borel set was a matter of fundamental importance for the development of descriptive set theory. As it turned out shortly after Suslin's discovery, the family of continuous images of Borel sets, also called the family of analytic sets ("Suslin sets" is the term now in use) [in the footnote: *As Suslin has proved, analytic sets can also be defined as sets which can be obtained by starting with closed sets by means of the so-called operation* (A) *(thus named in honour of his colleague P. S. Alexandrov).*] *has many important properties, such as measurability in Lebesgue's sense of the term, Baire's property, and inclusion of a Cantor set (in the case of not being countable).* A personal testimony of Sierpiński on a conversation between Suslin and Luzin is contained in Sec. 28 of [Si 1964].

The Preface to Lusin's *Leçons sur les ensembles analytiques et leurs applications* ([Lu 1930]) was written by Lebesgue, from which we quote: *… une Préface m'a semblé être le seul endroit où je pourrais avouer très haut ce que M. Lusin a soigneusement caché: l'origine de tous les problèmes dont il va s'agir ici est une grossière erreur de mon Mémoire sur les fonctions représentables analytiquement. Fructueuse erreur, que je fus bien inspiré de la commettre!*

the so-called *A-operation.*[30] A description of all $\mathcal{A}$-sets is given in a similar way as above for the system of Borel sets of α-th class. It is derived that there are

[30] The idea of the A-operation goes back to the works of Aleksandrov [A 1916] and Hausdorff [H 1916] in 1916. They proved the Continuum Hypothesis for uncountable Borel sets using a special representation of Borel sets. The A-operation was explicitly introduced in [So 1917] by Suslin under the supervision of Luzin. (As far as the relation of Luzin and Suslin is concerned, see interesting comments in [ZD 2007], pp. 13–17.) The main Suslin contributions are the following: every Borel set on the real line is an A-set (the terminology used later on is *analytic set* or *Suslin set*); Borel sets are precisely A-sets whose complement is an A-set as well; there are non-Borel A-sets; projections of A-sets are A-sets. There is an extensive literature on the history of the discovery of A-sets, see [T 1993], [Lo 2001], [BK 2005], [ZD 2007], pp. 13–17, 37, 119–120. The book [ZD 2007] reveals also a complicated personal relationship of Luzin and Aleksandrov. It may be suitable to add two more quotations concerning the origin of the A-operation in connection with the discovery of analytic sets. In [A 1978], p. 39, in a commentary to the Russian translation of [A 1916], Aleksandrov explains that his scheme (e) used in the paper is nothing else than the A-system. (To what extent such a statement is justified is a subtle question discussed in depth in [BK 2005], [Lo 2001] and [T 1993] where related references are to be found.) He also provides the following information: *The original text of this work was written by me in Russian during the summer 1915. The translation of the work to French was done by N. N. Lusin who introduced some changes in terminology … and edited the whole work again making no changes in its mathematical contents.* Aleksandrov gave a talk on his result at a student seminar on October 13, 1915; see [I 1996], p. 6.

On p. 173 of [A 1978], one also finds the following commentary by Kolmogorov: *Somewhat aside there is the student work No 1 of the present publication* [it is meant [A 1916]] *written and published in 1916 dealing with a solution of the question of cardinality of B-sets posed to the author by his teacher N. N. Lusin. But the most unexpected turned out to be the existence of a very simple construction providing an arbitrary B-set of as high transfinite class as wanted – the famous A-operation. Soon afterwards M. Suslin discovered that this operation applied to open or closed sets (in the case of sets on the real line – to intervals) leads to a wider class of sets, namely to the class of A-sets – one of the central objects of study in descriptive set theory in the forthcoming years.*

The following text is taken from [Bg 2007a], p. 420: *W. Sierpiński who was not only an eye-witness of the first steps of this theory but also one of its active creators, wrote: "Some authors call analytic sets Souslin; it would be more correct to call them Souslin-Lusin sets".*

The definiton usually adopted in contemporary mathematics reads as follows (see [Ke 1995], p. 85): Let X be a Polish space. A set $A \subset X$ is called *analytic* if there is a Polish space Y and continuous function $f : Y \to X$ with $f(Y) = A$. (The class of analytic sets is denoted by $\sum_1^1(X)$. Let us remark that, for $A \neq \emptyset$, one can take $Y = \mathcal{N}\,(:= \mathbb{N}^{\mathbb{N}})$.) It should be mentioned that analytic sets are considered also in a more general topological context, see, for instance, [AS 1980], Part I: K-analytic sets by C. A. Rogers and J. E. Jayne or [Ch 1969].

In order to show (in a modernized notation) a connection with the approach chosen in Aleksandrov's course, recall the following definitions: $\mathbb{N}^{<\mathbb{N}}$ is the set of all finite sequences of natural numbers. If $x \in \mathcal{N}$, $x = (x_0, x_1, \dots)$ and $n \in \mathbb{N}$, let $x|n := (x_0, \dots, x_{n-1})$. *Definition* ([Ke 1995], p. 198): Let $(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a *Souslin scheme* on a set X, i.e., a family of subsets of X indexed by $\mathbb{N}^{<\mathbb{N}}$. The *Souslin operation* $\mathcal{A}$ applied to such a scheme produces the set

$$\mathcal{A}_s P_s := \bigcup_{x\in\mathcal{N}} \bigcap_n P_{x|n}.$$

The basic representation of analytic sets in a Polish space X reads as follows ([Ke 1995], p. 199): *For any $A \subset X$, A is analytic if and only if $A = \mathcal{A}_s F_s$* with F_s closed.

Borel sets not belonging to the α-th class and also sets which are not $\mathcal{A}$-sets.[31] It is noted that the complement of Borel set is a Borel set. A further result: *Every Borel set is an $\mathcal{A}$-set.*[32] To prove this, two statements are shown:

(a) *The union operation and the intersection operation for countably many sets are special cases of the A-operation.*[33]

(b) *The A-operation applied to $\mathcal{A}$-sets gives again an $\mathcal{A}$-set.*[34]

Theorem.[35] *Every uncountable $\mathcal{A}$-set contains a perfect subset* (hence has the power of continuum).

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[31] The result on Baire classes goes back to [Le 1905]. The existence of analytic non-Borel sets is due to Suslin [So 1917], Théorème II. The Proceedings [Pi 1994] contains *Guidelines 1900–1950* where, year by year, the most influential publications for the development of mathematics are included. For the year 1917, among 10 articles, Suslin's note [So 1917] is listed.

[32] See [So 1917], Théorème I. Mikhail Yakovlevich Suslin (1894–1919) (sometimes his name is transliterated Souslin) was born in Krasavka (now the region of Saratov), he studied mathematics at Moscow University. For a detailed biography of Suslin see [I 1996]. It can be mentioned that in 1916, as a student of Luzin, he found an error in Lebesgue's article [Le 1905] in which Lebesgue "proved" that, for any Borel set in $\mathbb{R}^2$, the projection on the real axis was also a Borel set. He reported on his discovery at a student seminar on November 28, 1916; see [I 1996], p. 7. In 1917, Suslin graduated and immediately started lecturing. He died of spotted fever in 1919 during the Russian Civil War. He made a major contribution to descriptive set theory. Because of his early death he was able to publish only one paper [So 1917] during his life, but his other results appeared in [H 1927] and [Lu 1930]. His name is associated with the so-called *Suslin hypothesis* concerning the problem of the existence of a *Suslin line*, that is of a totally ordered set enjoying several natural conditions satisfied for the real line but *not isomorphic* to $\mathbb{R}$. It has been shown to be independent of ZFC; see [ST 1971] and [UK 1988]. He formulated his problem shortly before his death (see M. Souslin: *Problème 3*, Fundamenta Mathematicae 1(1920), p. 223). An account on Suslin's mathematical achievements can be found in [UK 1988], where a detailed analysis of set theoretic aspects of the Suslin problem is also discussed.

[33] See [So 1917], Lemme 2, Lemme 3.

[34] See [LS 1918], the footnote on p. 48.

[35] Luzin, in [Lu 1917], p. 94, attributes the result to Suslin. For which (uncountable) sets of real numbers can one prove the Continuum Hypothesis, that is, which sets have cardinality $2^{\aleph_0}$? The first result, for closed sets, goes back to G. Cantor [C 1884]. A detailed discussion can be found in [Sc 1913], pp. 269–299. For G_δ-sets, the result is due to W. H. Young [Y 1903a]; see also [YY 1906], p. 64. In [H 1914], pp. 465–466, Hausdorff established this fact for $G_{\delta\sigma\delta}$-sets. In 1916, independently, Aleksandrov in [A 1916] and Hausdorff in [H 1916] showed that every uncountable Borel set has cardinality $2^{\aleph_0}$. Aleksandrov's proof based on a special process (a germ of the A-operation) of constructing arbitrary Borel sets, which was an essential ingredient for the result of Suslin that the same cardinality result holds for analytic sets. The role of the A-operation in creation of the theory of analytic sets is discussed in detail in [T 1993], [Lo 2001], [BK 2005], [ZD 2007], pp. 13–17, 37, 119–120. A deep analysis of the common part as well as of the differences in arguments by Aleksandrov and Hausdorff is given in *Commentary on* [H 1916] by V. Kanovei and P. Koepke in [H 2008], pp. 439–442.

Concerning the development of descriptive set theory, we quote from the review MR561709 by P. G. Hinman of the book [Mo 1980]: *Descriptive set theory was founded around the turn of the century and enjoyed a very active life into the 1930's as a mixture of point-set topology, real analysis, and sets theory. Then for over thirty years there were only sporadic results because, as we know now, the most interesting remaining problems have turned out to be unsolvable on the basis of commonly accepted mathematical principles as formulated in any standard axiomatic set theory.*

It is proved that *every $\mathcal{A}$-set is a continuous image of the set of irrational numbers.* The converse holds as well but this will come later on.[36]

It is explained that *every Borel set is an $\mathcal{A}$-set*[37] *such that the complement is also an $\mathcal{A}$-set.* Next, a converse is established: *If both M and the complement of M are $\mathcal{A}$-sets, then M is a Borel set.*[38] To show this, the following notion is introduced: It is said that two sets A, A' can be *separated by Borel sets*, if there are two Borel sets B, B', such that

$$B \supset A, \quad B' \supset A', \quad B \cap B' = \emptyset.$$

Then it is shown: *If A, A' are two $\mathcal{A}$-sets in a complete space such that $A \cap A' = 0$, then A, A' can be separated by Borel sets.*[39] This already implies the above characterization of Borel sets within $\mathcal{A}$-sets.

It is recalled that a function f is analytically representable, if and only if, for every a, $\{f > a\}$, $\{f \leq a\}$, $\{f < a\}$, $\{f \geq a\}$ are Borel sets. Hence, in view of the result on the complement of $\mathcal{A}$-sets, we have: *f is analytically representable if and only if, for every a, the four sets*

$$\{f > a\}, \quad \{f \leq a\}, \quad \{f < a\}, \quad \{f \geq a\}$$

are $\mathcal{A}$-sets.

The next result reads as follows: *A continuous image of an $\mathcal{A}$-set is an $\mathcal{A}$-set.*[40] So the previous result can be modified by saying "the four sets ... are continuous images of the set of irrational numbers".

It is summarized that $\mathcal{A}$-sets are invariant with respect to countable union and intersection operations, the A-operation and to continuous images. In particular, the $\mathcal{A}$-sets are topologically invariant.

As we know, the topological invariance holds for G_δ-sets and is trivial for F_σ-sets. More generally: *A homeomorphic image of a Borel set*[41] *is again a Borel set of the same class.*[42]

For further development of the analytic sets theory we refer to [Lu 1930], [Lu 1958], [Ch 1969], [AS 1980], [Ka 1985], [Us 1985], [Ke 1995], [LL 1996], [Fm 2003], Chapter 42, [Bg 2007b], [Mo 2009].

[36] See [H 1927], p. 211; for a discussion, see also the comment [105] on p. 386 in [H 2008]. Cf. [Ke 1995], pp. 85, 109, and footnote [40].

[37] See the footnote [32].

[38] See [So 1917], Théorème III.

[39] See [Lu 1927], Sec. 42, [Lu 1930], p. 156. In [AS 1995], on p. 6, it is mentioned: ... *the first separation theorem (essentially due to Souslin (1917))* ...

[40] See [H 1927], p. 209 and references to §37 on p. 281; for a discussion, see also the comment [103] on p. 385 in [H 2008].

[41] The reference is as in [40]; in the comment [103] on p. 385 in [H 2008], a relation to the results of Suslin, Luzin and Sierpiński is given; cf. also the comment [106] on p. 386 in [H 2008].

[42] See [Si 1920] and [H 1927], p. 214; see also the comment [109] on p. 387 in [H 2008].

The concluding part of the text is sketchy and the following information is provided without proofs.

(a) (*Uncountable*) *Borel sets are exactly one-to-one continuous images of the set of irrational numbers.*[43]

(b) *The range of any analytically representable function is an $\mathcal{A}$-set.*[44]

(c) *Every $\mathcal{A}$-set is the range of an everywhere defined Baire* 1 *function which is discontinuous at rational points only.*[45]

Urysohn constructed a non-Borel $\mathcal{A}$-set as

– a set of boundary values of a power series[46]

– a set of points in which a closed set is accessible from outside[47]

– the range of the limit of a convergent series of polynomials.[48]

Very little is known about complements of $\mathcal{A}$-sets. Every $\mathcal{A}$-set as well as every complement of an $\mathcal{A}$-set is a union of at most $\aleph_1$ disjoint Borel sets (hence of cardinality $\aleph_1$ or $\aleph$ or at most countable).[49]

One can define (a reference to Aleksandrov's Fundamente V[50]) the complements of $\mathcal{A}$-sets as follows: Let $F_{i_1 i_2 \ldots i_k}$ be closed sets; the $\mathcal{A}$-sets are defined using unions of chains where the Baire neighborhoods corresponding to indices $i_1, i_1 i_2, i_1 i_2 i_3, \ldots$ contain a unique point. Similarly one can express complements of $\mathcal{A}$-sets considering those chains of neighborhoods $[i_1^1]$, $[i_1^2, i_2^2]$, $[i_1^3, i_2^3, i_3^3]$ for which their union covers the whole Baire space.

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[43] Cf. [H 1927], p. 211; for a discussion, see also comment [105] on p. 386 [H 2008]. Concerning the statement, the situation is as follows: *A set in* $\mathbb{R}$ (*or in a Polish space*) *is Borel precisely when it is the image of a* closed *subset* $F \subset \mathcal{N}$ *under a continuous injective mapping*; see [Bg 2007b], p. 31 and [Ke 1995], p. 83. When one can take $F = \mathcal{N}$ is cleared up by the following result from [Si 1927] (see also [Si 1976], vol. II, pp. 715–718): *Pour qu'un ensemble linéaire soit une image continue et biunivoque de l'ensemble de* tous les nombres irrationnels, *il faut et il suffit qu'il soit un ensemble mesurable B condensé* (c'est-à-dire dont chaque point est un point de condensation).

[44] See [Lu 1917], Théorème III.

[45] See [Lu 1917], Théorème I. For a more precise result, see [Lu 1927], pp. 12–15, and [Si 1927b], [Si 1927c]; see also [Si 1976], pp. 648–650, 643–647.

[46] See [U 1926], [U 1951], vol. II., pp. 819–822. The result was established in 1924. Boundary values are understood as the cluster set of the corresponding holomorphic function.

[47] See [U 1928], [U 1951], vol. II, pp. 807–818. The result was established for $\mathbb{R}^n$ with $n \geq 3$ in 1923 and presented at the Moscow Mathematical Society at the session of October 21, 1923. For the plane case, the problem is stated to be open for closed sets. However, a construction of a G_δ-set is given (see p. 817). For a construction of a *closed* plane set, see [Ni 1928]. A related result to that of [U 1926] can be found in [Lu 1928]; see also [Lu 1958], pp. 462–463.

[48] See [Lu 1917], Corollaire on p. 93; we were not able to trace the result back to Urysohn.

[49] See [LS 1918], Sec. 3 and [LS 1923], Sec. 4. The basic facts on *co-analytic* sets can be found in [Ke 1995], pp. 242–312.

[50] See [A 1924c].

Sierpiński defined (a reference to Poln. Akad. Okt. 1927[51]) a prototype of a complement of an $\mathcal{A}$-set as follows: With every natural number

$$p = 2^{m-1}\bigl(2^{n_1} + 2^{n_1+n_2} + \cdots + 2^{n_1+\cdots+n_k} - 1\bigr)$$

with $n_i > 0$ one associates the Baire neighborhood $[n_1, n_2, \ldots, n_k]$. One considers all irrational numbers $x := 1 + \frac{1|}{|p_1} + \frac{1|}{|p_2}$ and x is a point of the set provided those Baire neighborhoods corresponding to $p_1, p_2, \ldots$ cover the whole Baire-space.

⋆ ⋆ ⋆ ⋆ ⋆

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