Chapter 4

Operator-valued functions

In the previous chapters we have encountered various definitions of product integral of a matrix function $A : [a,b] \to \mathbf{R}^{n\times n}$. It is known that every $n \times n$ matrix represents a linear transformation on the space $\mathbf{R}^n$, and that the composition of two linear transformations corresponds to multiplication of their matrices.

It is thus natural to ask whether it is possible to define the product integral of a function A defined on $[a,b]$ whose values are operators on a certain (possibly infinite-dimensional) vector space X. This pioneering idea (although in a less general scope) can be already found in the second part of the book [VH] written by Bohuslav Hostinský. He studies certain special linear operators on the space of continuous functions (he calls them "linear functional transformations") and calculates their product integrals as well as their left and right derivatives. Hostinský imagines a function as a vector with infinitely many coordinates; the linear operator on $\mathbf{R}^n$ given by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

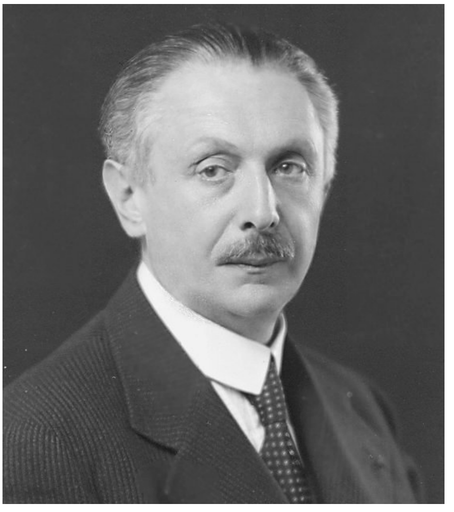
then goes over to the operator on continuous functions given by

$$y(t) = \int_p^q A(t,u)x(u)\,\mathrm{d}u.$$

This idea was not a new one – already Volterra noted[1] that integral equations can be treated as limiting cases of systems of linear algebraic equations. His observation was also later used by Ivar Fredholm, who obtained the solution of an integral equation as a limit of the solutions of linear algebraic equations.

Bohuslav Hostinský was born on the 5th December 1884 in Prague. He studied mathematics and physics at the philosophical faculty of Charles University in Prague and obtained his doctoral degree in 1907 (his dissertation thesis was devoted to geometry). He spent a short time as a high school teacher and visited Paris in 1908–09 (he cooperated especially with Gaston Darboux). Since 1912 he gave lectures as privatdozent at the philosophical faculty in Prague and was promoted to professor of theoretical physics at the faculty of natural sciences in Brno in 1920; he held this position until his death on the 12th April 1951.

1 [Kl], Chapter 45



Bohuslav Hostinský[1]

Hostinský initially devoted himself to pure mathematics, especially to differential geometry. However, his interest gradually moved to theoretical physics, in particular to the kinetic theory of gases (being influenced mainly by Borel, Ehrenfest and Poincaré). Of course, this discipline requires a good knowledge of probability theory; Hostinský is considered one of the pioneers of the theory of Markov processes and their application in physics. Apart from that he also worked on differential and integral equations and is known for his critical attitude towards the theory of relativity. A good overview of the life and work of Bohuslav Hostinský is given in [Ber].

4.1 Integral operators

In the following discussion we focus our attention to the space $\mathcal{C}([p,q])$ of continuous functions defined on $[p,q]$. Every mapping $T : \mathcal{C}([p,q]) \to \mathcal{C}([p,q])$ is called an operator on $\mathcal{C}([p,q])$; we will often write Tf instead of $T(f)$. The operator is called linear if

$$T(af_1 + bf_2) = aT(f_1) + bT(f_2)$$

for each pair of functions $f_1, f_2 \in \mathcal{C}([p,q])$ and each pair of numbers $a, b \in \mathbf{R}$. The inverse operator of T, which is denoted by T^{-1}, satisfies

$$T^{-1}(T(f)) = T(T^{-1}(f)) = f \tag{4.1.1}$$

for every function $f \in \mathcal{C}([p,q])$; note that the inverse operator need not always exist. The last equation can be shortened to

$$T^{-1} \cdot T = T \cdot T^{-1} = I,$$

[1] Photo provided by Tomáš Hostinský

where $\cdot$ denotes the composition of operators and I is the identity operator, which satisfies $If = f$ for every function f.

Bohuslav Hostinský was concerned especially with integral operators of the first kind

$$Tf(x) = \int_p^q K(x,y)f(y)\,\mathrm{d}y$$

and with the operators of the second kind

$$Tf(x) = f(x) + \int_p^q K(x,y)f(y)\,\mathrm{d}y,$$

where the function K (the so-called kernel) is continuous on $[p,q] \times [p,q]$. Thus if $f \in \mathcal{C}([p,q])$, then also $Tf \in \mathcal{C}([p,q])$. These operators play an important role in the theory of integral equations and have been studied by Vito Volterra, Ivar Fredholm, David Hilbert and others since the end of the 19th century (see [Kl], Chapter 45).

Hostinský starts[1] with a recapitulation of the basic properties of the integral operators. If $K(x,y) = 0$ for $y > x$, we obtain either the Volterra operator of the first kind

$$Tf(x) = \int_p^x K(x,y)f(y)\,\mathrm{d}y,$$

or the Volterra operator of the second kind

$$Tf(x) = f(x) + \int_p^x K(x,y)f(y)\,\mathrm{d}y.$$

Composition of two integral operators of the second kind

$$T_1 f(x) = f(x) + \int_p^q K_1(x,y)f(y)\,\mathrm{d}y,$$

$$T_2 f(x) = f(x) + \int_p^q K_2(x,y)f(y)\,\mathrm{d}y,$$

produces another operator of the second kind

$$(T_2 \cdot T_1)f(x) = f(x) + \int_p^q J(x,y)f(y)\,\mathrm{d}y,$$

whose kernel is

$$J(x,y) = K_1(x,y) + K_2(x,y) + \int_p^q K_2(x,z)K_1(z,y)\,\mathrm{d}z. \qquad (4.1.2)$$

[1] [VH], p. 182–186

An important question is the existence of inverse operators (which corresponds to solvability of the corresponding integral equations). The Volterra operator of the second kind always has an inverse operator

$$T^{-1}f(x) = f(x) + \int_p^x N(x,y)f(y)\,\mathrm{d}y,$$

which is again a Volterra operator of the second kind whose kernel $N(x,y)$ (usually called the resolvent kernel) can be found using the method of successive approximations. Several sufficient conditions are known for the existence of the inverse operator to the Fredholm operator of the second kind; we do not dwell into the details and only mention that the inverse operator is again a Fredholm operator of the second kind

$$T^{-1}f(x) = f(x) + \int_p^q N(x,y)f(y)\,\mathrm{d}y,$$

with a kernel $N(x,y)$. Equations (4.1.1) and (4.1.2) imply that the kernel satisfies

$$K(x,y) + N(x,y) = -\int_p^q K(x,t)N(t,y)\,\mathrm{d}t = -\int_p^q N(x,t)K(t,y)\,\mathrm{d}t. \tag{4.1.3}$$

Generally, the operators of the first kind need not have an inverse operator.

4.2 Product integral of an operator-valued function

We now assume that the integral operator kernel depends on a parameter $u \in [a,b]$:

$$T(u)f(x) = f(x) + \int_p^q K(x,y,u)f(y)\,\mathrm{d}y$$

Thus T is a function defined on $[a,b]$ whose values are operators on $(\mathcal{C}([p,q]))$.

Hostinský now proceeds[1] to calculate its left derivative. He doesn't state any definition, but his calculation follows Volterra's definition of the left derivative of a matrix function. Assume that K is continuous on $[p,q]\times[p,q]\times[a,b]$ and that the derivative

$$\frac{\partial K}{\partial u}(x,y,u)$$

exists and is continuous for every $u \in [a,b]$ and $x,y \in [p,q]$. We choose a particular $u \in [a,b]$ and assume that the inverse operator $T^{-1}(u)$ exists. According to Equation (4.1.2), the kernel of the operator $T(u+\Delta u)\cdot T^{-1}(u)$ is

$$J(x,y,u,\Delta u) = K(x,y,u+\Delta u) + N(x,y,u) + \int_p^q K(x,t,u+\Delta u)N(t,y,u)\,\mathrm{d}t =$$

$$= K(x,y,u+\Delta u) - K(x,y,u) + K(x,y,u) + N(x,y,u) +$$

[1] [VH], p. 186–188

$$+\int_p^q K(x,t,u+\Delta u)N(t,y,u)\,\mathrm{d}t =$$

$$= K(x,y,u+\Delta u) - K(x,y,u) + \int_p^q \left(K(x,t,u+\Delta u) - K(x,t,u)\right) N(t,y,u)\,\mathrm{d}t$$

(we have used Equation (4.1.3)). Denote

$$M(x,y,u) = \lim_{\Delta u\to 0} \frac{J(x,y,u,\Delta u)}{\Delta u} =$$

$$= \frac{\partial K}{\partial u}(x,y,u) + \int_p^q \frac{\partial K}{\partial u}(x,t,u)N(t,y,u)\,\mathrm{d}t. \tag{4.2.1}$$

According to Bohuslav Hostinský, the left derivative of T at u is an operator of the second kind whose kernel is $M(x,y,u)$. By a similar method he deduces that the right derivative of T at u is an operator of the second kind with the kernel

$$M^*(x,y,u) = \frac{\partial K}{\partial u}(x,y,u) + \int_p^q N(x,t,u)\frac{\partial K}{\partial u}(t,y,u)\,\mathrm{d}t.$$

These statements are somewhat confusing, because the left derivative should be the operator

$$\lim_{\Delta u\to 0} \frac{T(u+\Delta u)\cdot T^{-1}(u) - I}{\Delta u},$$

which is an operator of the *first* kind with kernel $M(x,y,u)$. Similarly, the right derivative should be rather

$$\lim_{\Delta u\to 0} \frac{T^{-1}(u)\cdot T(u+\Delta u) - I}{\Delta u},$$

i.e. an operator of the *first* kind with kernel $M^*(x,y,u)$.

The next problem tackled by Hostinský is the calculation of the left integral of the operator-valued function

$$T(u)f(x) = f(x) + \int_p^q K(x,y,u)f(y)\,\mathrm{d}y.$$

We again assume that the function K is continuous on $[p,q]\times[p,q]\times[a,b]$. As in the case of derivatives, Hostinský provides no definition and starts[1] directly with the calculation, which is again somewhat strange: He chooses a tagged partition $D: a = t_0 < t_1 < \cdots < t_m = b$, $\xi_i \in [t_{i-1}, t_i]$, and forms the operator

$$T(\xi_m)\Delta t_m \cdots T(\xi_1)\Delta t_1,$$

[1] [VH], p. 188–191

where $\Delta t_i = t_i - t_{i-1}$. He now claims it is an operator of the second kind, which is not true. The correct procedure that he probably has in mind is to take the operator-valued function

$$S(u)f(x) = \int_p^q K(x,y,u)f(y)\,\mathrm{d}y$$

and to form the operator

$$P(S,D) = (I + S(\xi_m)\Delta t_m)\cdots(I + S(\xi_1)\Delta t_1),$$

which is an operator of the second kind with the kernel

$$\sum_{s=1}^{m} \sum_{1\le i_1<\cdots<i_s\le m} \int_p^q \cdots \int_p^q K(x,z_1,\xi_{i_s})\cdots K(z_{s-1},y,\xi_{i_1})\Delta t_{i_s}\cdots\Delta t_{i_1}\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}$$

(we have used Equation (4.1.2)). Passing to the limit for $\nu(D)\to 0$ we calculate that the left product integral of the function S is an operator of the second kind with the kernel

$$L(x,y,a,b) = \tag{4.2.2}$$

$$= \sum_{s=1}^{\infty} \int_p^q \cdots \int_p^q \int_a^b \int_a^{t_s} \cdots \int_a^{t_2} K(x,z_1,t_s)\cdots K(z_{s-1},y,t_1)\,\mathrm{d}t_1\cdots\mathrm{d}t_s\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}.$$

In a similar way can calculate the right product integral of S, which is obtained as the limit of operators

$$P^*(S,D) = (I + S(\xi_1)\Delta t_1)\cdots(I + S(\xi_m)\Delta t_m)$$

for $\nu(D)\to 0$; the result is an operator of the second kind with the kernel

$$R(x,y,a,b) = \tag{4.2.3}$$

$$= \sum_{s=1}^{\infty} \int_p^q \cdots \int_p^q \int_a^b \int_a^{t_s} \cdots \int_a^{t_2} K(x,z_1,t_1)\cdots K(z_{s-1},y,t_s)\,\mathrm{d}t_1\cdots\mathrm{d}t_s\,\mathrm{d}z_1\cdots\mathrm{d}z_{s-1}.$$

In analogy with product integrals of matrix functions we use the symbols

$$\prod_a^b (I + S(t)\,\mathrm{d}t) \quad \text{and} \quad (I + S(t)\,\mathrm{d}t)\prod_a^b$$

to denote the left and right product integrals, respectively. Let us briefly summarize their properties:

If the operator-valued function S is defined on $[a,c]$ and $b\in[a,c]$, then[1]

$$\prod_a^c (I + S(t)\,\mathrm{d}t) = \prod_b^c (I + S(t)\,\mathrm{d}t)\cdot\prod_a^b (I + S(t)\,\mathrm{d}t).$$

[1] [VH], p. 193

The left side of the last equality represents an operator of the second kind whose kernel is $L(x, y, a, c)$, while the right side is a composition of two second-kind operators with kernels $L(x, y, a, b)$ and $L(x, y, b, c)$. Thus according to Equation (4.1.2) we have

$$L(x, y, a, c) = L(x, y, a, b) + L(x, y, b, c) + \int_p^q L(x, z, b, c)L(z, y, a, b)\,\mathrm{d}z. \quad (4.2.4)$$

Similarly, the right integral satisfies

$$(I + S(t)\,\mathrm{d}t)\prod_a^c = (I + S(t)\,\mathrm{d}t)\prod_a^b \cdot (I + S(t)\,\mathrm{d}t)\prod_b^c,$$

and we consequently

$$R(x, y, a, c) = R(x, y, a, b) + R(x, y, b, c) + \int_p^q R(x, z, a, b)R(z, y, b, c)\,\mathrm{d}z. \quad (4.2.5)$$

Hostinský next demonstrates[1] that the derivative and the integral are reverse operations; this means that if we consider the left (right) integral as a function of its upper bound, then its left (right) derivative is the original function.
If f is an arbitrary continuous function, then

$$\lim_{c\to b} \int_b^c f = f(b).$$

Using this result and the definition of L we conclude

$$\lim_{c\to b} \frac{L(x, y, b, c)}{c - b} = K(x, y, b)$$

(the series in the definition of L is uniformly convergent and we can interchange the order of limit and summation). Consequently, Equation (4.2.4) gives

$$\frac{\partial L(x, y, a, b)}{\partial b} = \lim_{c\to b} \frac{L(x, y, a, c) - L(x, y, a, b)}{c - b} =$$

$$= \lim_{c\to b} \frac{1}{c - b}\left(L(x, y, b, c) + \int_p^q L(x, z, b, c)L(z, y, a, b)\,\mathrm{d}z\right) =$$

$$= K(x, y, b) + \int_p^q K(x, z, b)L(z, y, a, b)\,\mathrm{d}z.$$

If $N_L(x, y, a, b)$ is the resolvent kernel corresponding to $L(x, y, a, b)$, then, according to Equation (4.2.1), the left derivative of the left integral of function S is an operator of the first kind with kernel

$$\frac{\partial L}{\partial b}(x, y, a, b) + \int_p^q \frac{\partial L}{\partial b}(x, t, a, b)N_L(t, y, a, b)\,\mathrm{d}t =$$

[1] [VH], p. 199–200

$$= K(x,y,b) + \int_p^q K(x,z,b)L(z,y,a,b)\,\mathrm{d}z +$$

$$+ \int_p^q \left(K(x,t,b) + \int_p^q K(x,z,b)L(z,t,a,b)\,\mathrm{d}z \right) N_L(t,y,a,b)\,\mathrm{d}t = K(x,y,b)$$

(we have applied Equation (4.1.3) to functions L and N_L), which completes the proof.

Finally let us note that in case when S is a constant function, i.e. if the kernel K does not depend on u, we obtain

$$L(x,y,a,b) = R(x,y,a,b) = \sum_{s=1}^{\infty} \frac{(b-a)^s}{s!} K^s(x,y), \tag{4.2.6}$$

where

$$K^s(x,y) = \int_p^q \cdots \int_p^q K(x,z_1)K(z_1,z_2)\cdots K(z_{s-1},y)\,\mathrm{d}z_1 \cdots \mathrm{d}z_{s-1}.$$

Remark 4.2.1. There exists a close relationship between the formulas derived by Hostinský and those obtained by Volterra. Recall that if $A : [a,b] \to \mathbf{R}^{n\times n}$ is a continuous matrix function, then

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = I + L(a,b),$$

where $L(a,b)$ is a matrix with components

$$L_{ij}(a,b) = \sum_{s=1}^{\infty} \left(\sum_{z_1,\dots,z_{s-1}=1}^{n} \int_a^b \int_a^{t_s} \cdots \int_a^{t_2} a_{i,z_1}(t_s)\cdots a_{z_{s-1},j}(t_1)\,\mathrm{d}t_1 \cdots \mathrm{d}t_s \right).$$

This resembles Equation (4.2.2) – the only difference is that in (4.2.2) we integrate over interval $[p,q]$ instead of taking a sum over $\{1,\dots,n\}$. Similarly, the right integral of a matrix function satisfies

$$(I + A(t)\,\mathrm{d}t) \prod_a^b = I + R(a,b),$$

where

$$R_{ij}(a,b) = \sum_{s=1}^{\infty} \left(\sum_{z_1,\dots,z_{s-1}=1}^{n} \int_a^b \int_a^{t_s} \cdots \int_a^{t_2} a_{i,z_1}(t_1)\cdots a_{z_{s-1},j}(t_s)\,\mathrm{d}t_1 \cdots \mathrm{d}t_s \right),$$

which is an analogy of Equation (4.2.3).

Also the formula

$$\prod_a^c (I + A(t)\,\mathrm{d}t) = \prod_b^c (I + A(t)\,\mathrm{d}t) \prod_a^b (I + A(t)\,\mathrm{d}t)$$

implies

$$I + L(a, c) = (I + L(b, c))(I + L(a, b)),$$

i.e. the components satisfy

$$L_{ij}(a, c) = L_{ij}(a, b) + L_{ij}(b, c) + \sum_{z=1}^{n} L_{i,z}(b, c) L_{z,j}(a, b).$$

In a similar way we obtain that the components of the right integral satisfy

$$R_{ij}(a, c) = R_{ij}(a, b) + R_{ij}(b, c) + \sum_{z=1}^{n} R_{i,z}(a, b) R_{z,j}(b, c).$$

These relations resemble Equations (4.2.4) and (4.2.5).

Remark 4.2.2. Product integration of operator-valued functions can be used to solve certain integro-differential equations. Hostinský considers[1] the equation

$$\frac{\partial f}{\partial t}(x, t) = \int_p^q K(x, y) f(y, t)\,\mathrm{d}t,$$

whose kernel is independent of t; as he remarks, its solution was found by Volterra (without using product integration) in 1914. More generally, we can consider the integro-differential equation

$$\frac{\partial f}{\partial t}(x, t) = \int_p^q K(x, y, t) f(y, t)\,\mathrm{d}t$$

with the initial condition

$$f(x, t_0) = f_0(x), \quad x \in [p, q].$$

This equation can be rewritten using the operator notation to

$$\frac{\partial f}{\partial t}(t) = S(t) f(t),$$

where $S(t)$ is an integral operator of the first kind for every t. Thus, for small Δt we have

$$f(t + \Delta t) = f(t) + \Delta t S(t) f(t) = (I + S(t)\Delta t) f(t).$$

[1] [VH, p. 192]

Arguing similarly as in the case of matrix functions, we conclude that

$$f(t) = \prod_{t_0}^{t} (I + S(u)\,\mathrm{d}u) f_0,$$

i.e.

$$f(x,t) = f_0(x) + \int_p^q L(x,y,t_0,t) f_0(y)\,\mathrm{d}t.$$

However, the effort of Bohuslav Hostinský is not primarily directed towards integro-differential equations. He shows that the right product integral kernel $R(x,y,a,b)$ can be used to obtain the solution of a certain differential equation known from the work of Jacques Hadamard.
The last chapter of [VH] is devoted to integral operators of the first kind. By an appropriate choice of the kernel, Hostinský is able to produce infinitesimal operators of the first kind, i.e. operators that differ infinitesimally from the identity operator (thus the kernel is something like the Dirac δ-function). The composition of such infinitesimal operators leads to an operator whose kernel K provides a solution of the equation

$$K(x,y,u,v) = \int_p^q K(x,z,u,w) K(z,y,w,v)\,\mathrm{d}z.$$

This equation is known as the Chapman or Chapman-Kolmogorov equation and is often encountered in the theory of stochastic processes (see also Section 1.4 and Equation (1.4.7), which represents a discrete version of the Chapman equation). The topic is rather special and we don't discuss it here.

4.3 General definition of product integral

Although Hostinský is interested only in integral operators on the space $\mathcal{C}([p,q])$, it is possible to work more generally with linear operators on an arbitrary Banach space X. Before proceeding to the corresponding definitions we recall that the norm of a linear operator T on the space X is defined as

$$\|T\| = \sup\{\|T(x)\|;\ \|x\| = 1\}.$$

Let $\mathcal{L}(X)$ denote the space of all bounded linear operators on X, i.e. operators whose norm is finite; $\mathcal{L}(X)$ is a normed vector space.

Definition 4.3.1. Let X be a Banach space, $A : [a,b] \to \mathcal{L}(X)$, $t \in [a,b]$. Assume that $A(t)^{-1}$ exists. We define the left and right derivatives of A at t as

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \lim_{h \to 0} \frac{A(t+h)A(t)^{-1} - I}{h},$$

$$A(t)\frac{\mathrm{d}}{\mathrm{d}t} = \lim_{h \to 0} \frac{A(t)^{-1}A(t+h) - I}{h},$$

provided the limits exist. Of course, at the endpoints of $[a,b]$ we require only the existence of the corresponding one-sided limits.

To every function $A:[a,b]\to\mathcal{L}(X)$ and every tagged partition

$$D: a=t_0<t_1<\cdots<t_m=b,\ \xi_i\in[t_{i-1},t_i],$$

we assign the operators

$$P(A,D)=\prod_{i=m}^{1}(I+A(\xi_i)\Delta t_i)=(I+A(\xi_m)\Delta t_m)\cdots(I+A(\xi_1)\Delta t_1),$$

$$P^*(A,D)=\prod_{i=1}^{m}(I+A(\xi_i)\Delta t_i)=(I+A(\xi_1)\Delta t_1)\cdots(I+A(\xi_m)\Delta t_m).$$

Definition 4.3.2. Consider function $A:[a,b]\to\mathcal{L}(X)$. The left and right product integrals of A are the operators

$$\prod_a^b(I+A(t)\,\mathrm{d}t)=\lim_{\nu(D)\to 0}P(A,D),$$

$$(I+A(t)\,\mathrm{d}t)\prod_a^b=\lim_{\nu(D)\to 0}P^*(A,D),$$

provided the limits exist.

Example 4.3.3. Recall that for every $A\in\mathcal{L}(X)$ we define the exponential $e^A\in\mathcal{L}(X)$ by

$$e^A=\sum_{n=0}^{\infty}\frac{A^n}{n!},$$

where $A^0=I$ is the identity operator, $A^1=A$ and $A^{n+1}=A^n\cdot A$ for every $n\in\mathbf{N}$. It can be proved (see Theorem 5.5.11) that

$$\prod_a^b(I+A\,\mathrm{d}t)=(I+A\,\mathrm{d}t)\prod_a^b=e^{A(b-a)}$$

(the special case for integral operators with a constant kernel follows also from Equation (4.2.6)).

We state the next theorem without proof; a more general version will be proved in Chapter 5 (see Theorem 5.6.2).

Theorem 4.3.4. Let $A:[a,b]\to\mathcal{L}(X)$ be a continuous function. Then the functions

$$Y(t)=\prod_a^t(I+A(u)\,\mathrm{d}u),$$

$$Z(t) = (I + A(u)\,\mathrm{d}u)\prod_a^t$$

satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = A(t), \quad Z(t)\frac{\mathrm{d}}{\mathrm{d}t} = A(t)$$

for every $t \in [a,b]$.

Definition 4.3.2 represents a straightforward generalization of the Riemann product integral as defined by Volterra. In the following chapter we provide an even more general definition applicable to functions $A : [a,b] \to X$, where X is a Banach algebra. It is also possible to proceed in a different way and try to generalize the Lebesgue product integral from Chapter 3; the following paragraphs outline this possibility.

Definition 4.3.5. A function $A : [a,b] \to \mathcal{L}(X)$ is called a step function if there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ and operators $A_1, \ldots, A_m \in \mathcal{L}(X)$ such that $A(t) = A_i$ for $t \in (t_{i-1}, t_i)$, $i = 1, \ldots, m$. For such a step function we define

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \prod_{i=m}^{1} e^{A_i(t_i - t_{i-1})},$$

$$(I + A(t)\,\mathrm{d}t)\prod_a^b = \prod_{i=1}^{m} e^{A_i(t_i - t_{i-1})}.$$

Definition 4.3.6. A function $A : [a,b] \to \mathcal{L}(X)$ is called product integrable if there exists a sequence of step functions $A_n : [a,b] \to \mathcal{L}(X)$, $n \in \mathbf{N}$ such that

$$\lim_{n\to\infty} (L)\int_a^b \|A_n(t) - A(t)\|\,\mathrm{d}t = 0$$

(where (L) denotes the Lebesgue integral of a real function). We then define

$$\prod_a^b (I + A(t)\,\mathrm{d}t) = \lim_{n\to\infty} \prod_a^b (I + A_n(t)\,\mathrm{d}t),$$

$$(I + A(t)\,\mathrm{d}t)\prod_a^b = \lim_{n\to\infty} (I + A_n(t)\,\mathrm{d}t)\prod_a^b.$$

The method from Chapter 3 can be again used to show that the value of product integral does not depend on the choice of a particular sequence of step functions $\{A_n\}_{n=1}^{\infty}$. The above defined integral is called the Lebesgue product integral or the Bochner product integral; the class of integrable functions is larger as compared to Definition 4.3.2. More information about this type of product integral can be found in [DF, Sch1].