Witold Więsław

Exempla mensurae capacitatis cuparumPříklady výpočtů objemů sudů

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EXEMPLA MENSURAE CAPACITATIS CUPARUM

Witold Więsław

1 Introduction

Some examples of calculating volume of casks are presented below. The examples are taken from selected ancient manuscripts and books printed many hundred years ago. We present here an extended version of our paper [21].

2 Antiquity

Ancient Egyptians and Babylonians used approximate algorithm for calculating the volume of a truncated cone (frustum cone) with a height h and bases with diameters d and D, averaging the truncated cone by a cylinder

$$V = \frac{1}{4}\pi h \left(\frac{d+D}{2}\right)^2. \tag{1}$$

R. C. Archibald claims that an algorithm equivalent with the formula (1) can be found in *The Oxyrhynchus Papyri* (see [17], p. 153 and 167). There are no exercises for calculating the volume of truncated cone in the most known Egyptian mathematical texts: neither in *The Rhind Mathematical Papyrus* nor in *The Moscow Papyrus*.

Babilonians used the algorithm following from (1) with $\pi = 3$ (sometimes $\pi = \frac{25}{8}$) ([12], Bd. 3, p. 61) which can be reduced to the formula

$$V = \frac{1}{4\pi} \cdot \left(\frac{p+P}{2}\right)^2 \cdot h, \tag{2}$$

equivalent to (1), where p and P denote perimeters of lower and upper discs of the truncated cone with the height h. They used sometimes another algorithm (see [12], Bd. 1, p. 176) which can be written as

$$V = \frac{1}{2} \cdot \left(\frac{p^2}{4\pi} + \frac{p^2}{4\pi}\right) \cdot h = \frac{s+S}{2} \cdot h, \tag{3}$$

where s and S denote areas of upper and lower disks, with $\pi = 3$. It means that the volume of truncated cone is approximated by a suitable cylinder, but differently than in Egypt.

There are no algorithms connected with calculations of a frustum cone in Euclid's *Elements*. However already Archimedes knew such algorithms. It is easy to obtain suitable algorithms for a truncated cone starting with Babylonian and Egyptian algorithms for the truncated pyramid and applying *the method of exhaustion*. It seems however that nobody in antiquity had written it *explicite*. For much more details see [18] and [19].

We omit here an easy and elementary exercise to see that the volume of a truncated cone is given by the formula

$$V = \frac{1}{3} \cdot h \cdot \pi \cdot \left(r^2 + rR + R^2\right) \tag{4}$$

where r and R are radii of its bases and h is the height. If we put $D = 2R$ and $d = 2r$, then the formula (4) takes either the form,

$$V = \frac{1}{12} \cdot h \cdot \pi \cdot \left(D^2 + Dd + d^2\right), \tag{5}$$

or the form

$$V = h \cdot \left(\frac{\pi}{4} \cdot \left(\frac{D+d}{2}\right)^2 + \frac{1}{3} \cdot \frac{\pi}{4} \cdot \left(\frac{D-d}{2}\right)^2\right). \tag{6}$$

A detailed explanation of formulae (5) and (6) in Babylonians tablets was given in ([18], p. 111-114).

Let us remark, that the formula (1) is exact, if and only if, the formulae (1) and (5) give the same, i.e. if and only if $4\left(D^2 + Dd + d^2\right) = 3\left(D^2 + 2Dd + d^2\right)$. It holds exacly in the case when $D = d$, i. e. in the case when truncated cone is a cylinder. Generally the formula (5) gives more than (1). Indeed, the difference of the right sides of (5) and (1) is equal to $\frac{1}{48} h\pi \left(D - d\right)^2$.

Now we can start with examples.

3 Mathematics in Nine Chapters [11]

In Chapter V we find Exercise 11:

Example I ([11], p. 474) *We have "a circle body"* [truncated cone]. *Its lower perimeter is equal 3 chun, the upper is equal 2 chun, and the height is 1 chun. The question is: what is its volume?*

The answer: $527\frac{7}{9}$ [cubic] *che.*
The rule: multiply the lower and upper perimeters, multiply each by itself, add together, multiply by the height, divide by 36.
Since 1 *chun = 10 che*, thus the volume is equal to $\frac{10}{36}(20^2 + 20 \cdot 30 + 30^2) = 527\frac{7}{9}$ [cubic] *che.* It is easy to see that Chinamen applied the following exact formula for the volume of truncated cone with lower perimeter P, upper perimeter p, and an altitude h:

$$V = \frac{h}{12\pi} \cdot \left(P^2 + pP + p^2\right).$$

In Chinese calculations $\pi = 3$. It is exactly the formula (5).

4 Codex Constantinopolitanus [5]

We start with examples attributed to Heron.

Example II ([5], page 44r5)
Let be given a container and let it have the diameter at the top 6 feet and the diameter below 8 feet and the height 10 feet.

To find how many keramia it shall contain.

I operate thus: I put together the diameter of the top and that below [result 14, of which one half becomes 7 feet. These] squared, result 49 feet. These eleven times, result 539. The 14-th of those, result $38\frac{1}{2}$ *feet. These I multiply into the height, in the 10 feet, result 385 feet. So many keramia shall it contain: 385.*

Example III ([5], page 43v:4; see also [19], p. 239)
Let a container be given and let it have the diameter below 5 feet and that above 3 feet and the altitude 8 feet. It was fulfilled by a wine up to 6 feet.

How many keramia shall it now contain?

I operate thus: I take the 3 away from the 5, remaining 2; these into 6, result 12; the eighth of those, result $1\frac{1}{2}$. *And I take the* $1\frac{1}{2}$ *from 5, result* $3\frac{1}{2}$. *That shall now be the width: till where the wine came up,* $3\frac{1}{2}$ *feet. And I put together the* $3\frac{1}{2}$ *and 5,* $8\frac{1}{2}$ *feet result, of which one half becomes* $4\frac{1}{4}$. *And these squared, result* $18\frac{1}{16}$. *These eleven times, result* $198\frac{1}{2}\frac{1}{8}\frac{1}{16}$. *I split off the 14-th of those, result* $14\frac{1}{7}\frac{1}{28}\frac{1}{112}\frac{1}{224}$. *These I multiply into the altitude, into the 6 feet, result* $85\frac{1}{7}\frac{1}{112}$ *feet. So many keramia shall it contain:* $85\frac{1}{7}\frac{1}{112}$ *feet.*

How can we explain the above examples? The cask is treated as a truncated cone with a height h and bases with diameters d and D. Heron used only an approximate algorithm, averaging the truncated cone by a cylinder, having as diameter the arithmetic mean of diameters d and D. Its volume V is equal to

$$V = \frac{1}{4}\pi h \cdot \left(\frac{dD}{2}\right)^2,$$

where he takes for π the Archimedean value $\frac{22}{7}$. It is exactly the formula (1) used by Egyptians. We see from Example III that Heron writes numbers as Egyptian fractions according to Egyptian tradition.

The third example from [5], althought with simpler calculations, is much more interesting, since it finally contains a cask.

Example IV ([5], page 44v:6)
Let be given a barrel and let it have the diameter above 6 feet, and the diameter in the middle 8 feet and the hight 10 feet.

To find how many keramia it contains.

I operate thus: I put together the diameter and that in the middle above. Together result 14 feet, of which one half is 7 feet.

These squared, result 49; these eleven times, result 539. Next I split off a 14-th of those, result $38\frac{1}{2}$*. These I multiply into the height, into the 10 feet, result 385 feet. So many keramia contains the barrel.*

Also here calculations are correct; π is approximated by the Archimedean value $\frac{22}{7}$.

5 Johann-Hartmann Beyer [3]

Example V ([3], pages 236-237)
We have a double cask (geminatae cupae) such that

Diametri maioris 42.8'. *area,*	1438.7'.8".	
minoris 36.2'. *area,*	1029.2'.5".	
79.0'.	2468.0'.3'.'0"'.	*area basios compositae.*
	1234.0'.1".5"'.	*area basios aequatae.*
aequatae 39.5'. *area,*	1225.4'.6".7"'.	*area diam. aequatae.*
	8.5'.4".8"'.	*differentia.* [difference]
	2.8'.4".9"'.	*triens.* [one third]
	1228.3'.1".6"'.	*area conocylindrica.*
	6 2. 4'.	*altitudo cupae.* [height of cask]
	4 9 1 3 2 6 4	
	2 4 5 6 6 3 2	
	7 3 6 9 8 9 6	
	766 4 6.9'.1".8"'.4"'.	*foliditas cupae gemin.*

The first line implies that $\pi = 3{,}141\,715$; the second implies $\pi = 3{,}141\,692$.

It is easy to see that Johann-Hartmann Beyer applied here an algorithm based on the formula (4) for calculating volumes of casks. He treated a cask as a body formed from two frustum cones put together with diameters d and D respectively. His examples imply that the number π was approximated in different ways. Once (Caput XII) he approximates π by $3{,}141\,694\,3$. In other places (Caput XIII) he takes π equal to $3{,}141\,720\,2$. Beyer understood, that his calculations give only an approximate value of the volume.

Example VI ([3], pages 26-27)
Let a simple barrel be given, with greater diameter equal 30.3' [i.e. 30,3], smaller 25.1', and the height 26.0'. Calculate its volume and density.

1. *Area* [of the circle] *with diameter 30.3' equals*
   $721.0'.9''.5'''.5''''.0^v.0^{v'}.5^{v''}$. [...]

STEREOMETRIÆ
INANIVM NOVA ET
FACILIS RATIO,
GEOMETRICIS DEMONSTRATIO-
nibus confirmata: & necessariis obscuriorum quorundam delineationibus illustrata:
QVA CORPORVM REGVLARIVM OMNI-
VM, TAM RECTILINEORVM QVAM CVRVILINEO-
rum capacitates promtissime explorantur.

Auctore
IOHANNE-HARTMANNO BEYERO, REI-
publicæ Francofurtensis Medico.

FRANCOFVRTI,
In Officina Paltheniana, Impensis Ionæ Rhodii.
ANNO M. DC. III.

The title page of [3]

Exemplum aliud. Conicum esto poculum: cuius basios diameter poll. $4.2'$: altitudo, poll. $11.7'$. Heic basios diameter poll. $4.2'$, in circulari tabula planitiē suppeditat $13.8'.5''.4'''$. — quæ cum tertia altitudinis parte, $3.9'$, facit soliditatem cuborum $54.0'$.— qui sunt Quadr. $1\frac{1}{4}$

poll. $13.\ 8'.\ 5''.\ 4'''$. planus basios.
$3.\ 9'$. triens altitudinis.

124686
41562

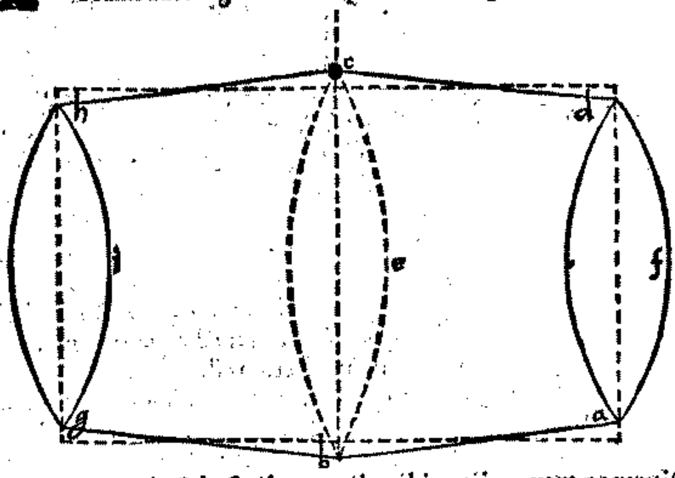
$54.0'.3''.0'''.6''''$. soliditas.

CAPVT XII.

De Coni curti, seu cupæ vinariæ stereometria.

CVpa vinaria alia Simplex est, vt a b c d, cui vna basis lignea: alia Geminata, vt a b g h c d, cui duæ bases ligneæ.

Praxis autem geodæsiæ cuparum fere septimembris existit.



1\. Diameter maioris basios b c, regula cubimetrica quam accuratissime exploratur: & planities eius e cyclico canone mutuata, in abacum refertur.

D

A page from [3]

2. *Area* [of the circle] *with diameter 25.1' equals* $494.8'.2''.8'''.8''''.0^{v}.3^{v'}.6^{v''}$. [...]

3. *Joint* [area] *is equal 1215.9', and half of it 608.0' is equal to the base* [of the cylinder]

4. *Diameter consisting from greater and smaller equals 55.4',* [and] *its half 27.7'.*

5. *Area* [of the circle with diameter 27.7'] *equals 602.7'.*

6. *Area 602.7' subtrahend from the area 608.0' gives a difference 5.3', one third of which is equal to 1.8'.* [It] *added to the area 602.7' gives 604.5'.*

7. *Multiply area of truncated cone 604.5' by its height 26.0'. It gives the volume of the barrel: 15717.0'.*

Finally Beyer states:

the circle with greater diameter	30.3'	[has]	*area*	721.1'—	
the circle with smaller diameter	25.1'	[has]	*area*	494.8' —	
	55.4'			1215.9'	*composed area of the base*
				608.0'	*area of the base equals*
the circle with [the middle] *diameter*	27.7'	[has]	*area*	602.7'	
				5.3'	*difference*
				1.8'	*one third*
				604.5'	*area of trucarted cone*
				26.0'	*height of the barrel*
				362700	
				12090	
				15717.0'.0".	*volume of the barrel*

How to interpret the above given calculations?

Let the greater circle have a diameter D, and the smaller one a diameter d. Let us put $\tilde{V} = \frac{1}{2} \cdot \left(\frac{\pi d^2}{4} + \frac{\pi D^2}{4} \right)$, and $\tilde{\tilde{V}} = \frac{\pi}{4} \cdot \left(\frac{D+d}{2} \right)^2$. If h is the height (the length) of the cask, then the algorithm described by Beyer in Examples V and VI can be reduced to the formula for the volume

$$V = h \cdot \left(\tilde{\tilde{V}} + \frac{1}{3} \cdot \left(\tilde{V} - \tilde{\tilde{V}} \right) \right), \tag{7}$$

which implies (5), after suitable calculations. Consequently Beyer [3] used a correct algorithm for calculating the volume of the cask, treating the cask as two truncated cones put together.

6 Johann Keppler [7]

He treated in [7] the problem of calculating the volume of a rotation body, and in particular the volume of a barrel. At the beginning Kepler recalls and discusses results known already to Archimedes, using in his discussions geometrical form of infinitesimals. A circle is for him the regular polygon with $n = \infty$ sides.

In another place of [7] Kepler states, that among all right cones with a fixed diagonal the cone in which the diameter to the height is equal to $\sqrt{2}$ has the greatest volume ([7], Pars II, Theorema I). Kepler also proves that among the right parallelepipeds with quadratic base inscribed in a given sphere the greatest volume has a cube ([7], Pars II, Theorema IV). Kepler formulates the following problem: calculate the volume of a body obtained by rotation of a conic around a straight line. In this way he obtains many rotation solids called by him the pearl, the lemon etc. He classifies the constructed solids and compares their volumes. However he gives no algorithms. The solids contain elliptic, hyperbolic and the circle barrels. He tries to answer the question when the volume of a circle barrel is maximal.

7 Adrianus Metius [1-2]

He presents heuristic algorithms for calculating the volume of a wine in a barrel in dependence upon the height of wine there. He gives practical tables for calculating the volume of a cask and gives numerical examples of applying the tables.

NOVA
STEREOMETRIA
DOLIORVM VINARIORVM, INPRIMIS Auſtriaci, figuræ omnium aptiſsimæ;

ET

USUS IN EO VIRGÆ CUBICÆ compendioſiſſimus & plane ſingularis.

Acceſſit

STEREOMETRIÆ ARCHIMEDEÆ Supplementum.

Authore

Ioanne Kepplero, Imp. Cæſ. Matthiæ I. ejuſq; fidd. Ordd. Auſtriæ ſupra Anaſum Mathematico.

Cum privilegio Cæſareo ad annos XV.

ANNO *M. DC. XV.*

LINCII

Excudebat JOANNES PLANCVS, ſumptibus Authoris.

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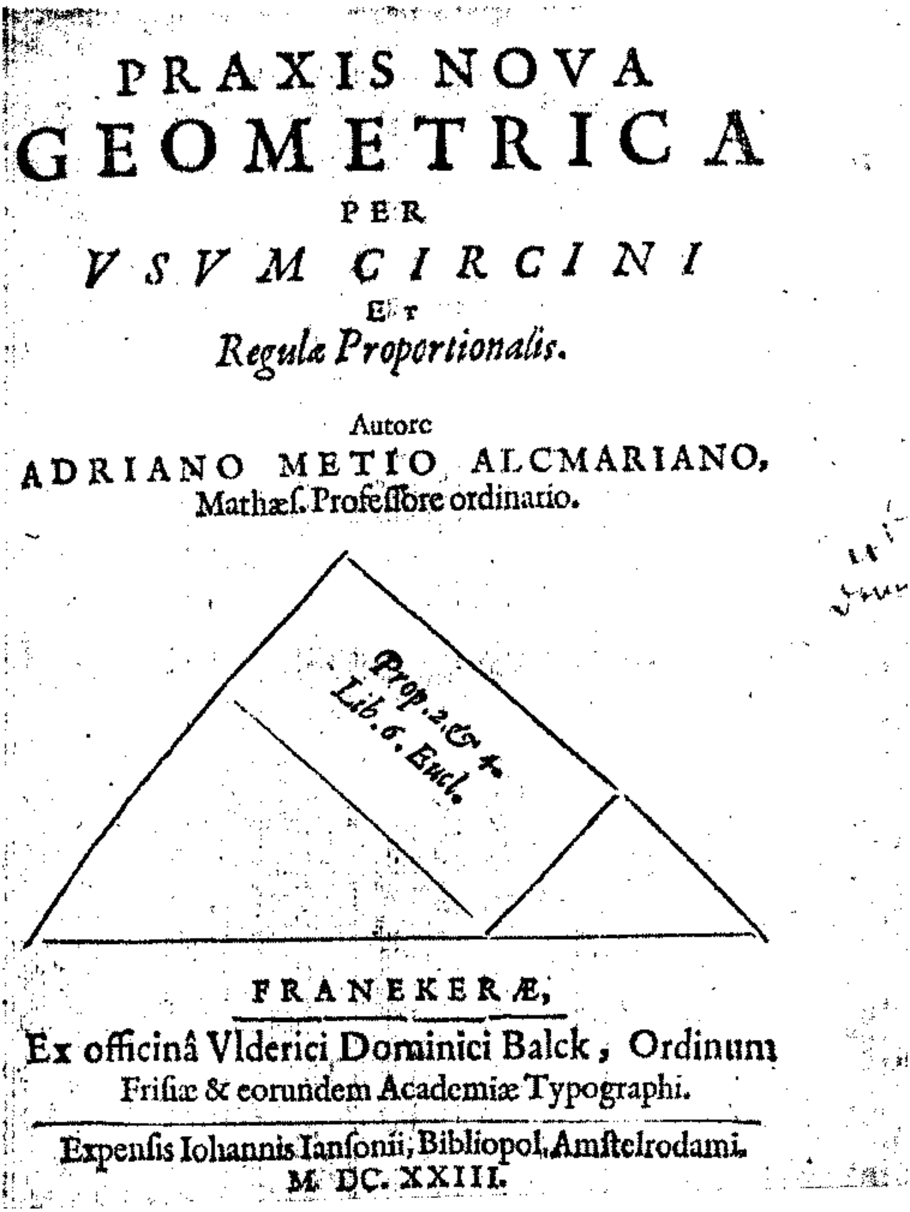


PRAXIS NOVA GEOMETRICA

PER

VSVM CIRCINI

ET

Regulæ Proportionalis.

Autore

ADRIANO METIO ALCMARIANO,

Mathæs. Professore ordinario.

Prop. 2. & 4. Lib. 6. Eucl.

FRANEKERÆ,

Ex officinâ Vlderici Dominici Balck, Ordinum Frisiæ & eorundem Academiæ Typographi.

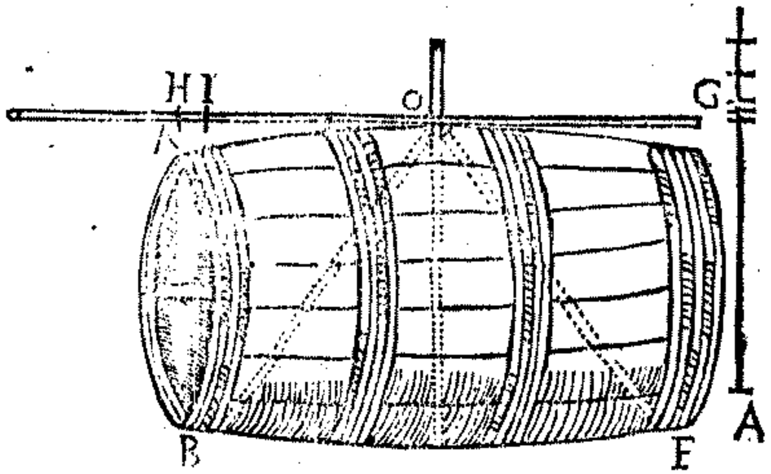
Expensis Iohannis Iansonii, Bibliopol. Amstelrodami.

M. DC. XXIII.

The title page of [1]

PROPORTIONALIS. 41

cant, capiatque illud exempli gratia 120. cantaros ſive 6. **Anckers**, ſi jam deſideras menſuram 20. cant. ſive **ein**



Ancker / tunc per prob. 14. quærendum latus cubi, cujus cubus ſe habet ad cubum lateris O B, (utpote longitudinis vaſis tranſverſi) quemadmodum ſeſe habent 20. ad 120. vel 1. ad 6. hoc eſt cubus O B lateris erit in ſextupla proportione minuendus.

Ad eundem quoque modum & ſphæræ menſura inveſtigabitur, quando nempe ſphæræ axin una cum capacitate ejus exploratam habemus. Vel per numeros, qualium menſura lineæ primæ 100. habet, earum axis ſphæræ ſimilis menſuræ 124. continebit.

PROBL. 25.

Lineam 10. pondo metallorum conſtruere.

COntinet hæc linea axes ſive Diametros globorum 10. ℔ in ſeptem metallorum generibus; quarum inventio

E talis

A page from [1]

8 John Newton [16]

In the introduction to the book [16] John Newton[1] wrote:

TO THE READER. [...] *And besides the Gauging of Casks, I have here showed the Gauging of Brewers Tuns, either in the whole, or Inch by Inch, whether such Tuns be taken for the* ***Frunctum*** *of Pyramides, Cones, or Cylindroides, and illustrated the Precepts given by so many Examples, that I hope no difficulty will be found in it, by any that are but meanly acquainted with Arithmeticks, and for the more ready Gauging of a Cask filled but in part, I have added a Table showing the Area of the Segments of a Circle, whose area is Unity;* [...]

John Newton presents different algorithms for calculating the volume of casks. He states on the page 9:

Take the difference between the contents of the two circles, answering to the two Diameters of the Head and Bung: one third part of that difference, being Subtracted from the great Cylinder, or two thirds being added to the lesser, the sum or remainder being Multiplied by the length will give the content.

Example VIII. ([16], p.9)

The content of a circle whose Diameter is 21.5 is	1.5716
And the content of a circle whose Diameter is 18.3 *is*	1.138 6
Their difference is	4 330
1/3 of the difference is	1 443
Two thirds is	2 887
The greater circle 1.571 6 *The less*	1.138 6
1/3 Subract, 1443 2/3 Adde,	0.288 7
Difference, 1.427 3 *Summe,*	1.427 3

Or thus, according to Mr. Oughtred, take two thirds	
of the great circle,	1.047 8
One third of the lesser,	0.379 5
Their sum,	1.427 3
is the content as before, which being Multiplied by the length	
will give the content.	

[1] John Newton – an astronomer from Cambridge. He is the author of many books from mathematics and astronomy, published in the years 1654-1692. *Short title Catalogue* of Donald Wing (vol. II (1641-1700), Columbia Univ. Press 1948) quotes 22 books of him.

What does it mean? How can we interpret the above stated calculations? John Newton states on the page 4, that 1 *Wine-gallon content is 231 cubic inches.* It is easy to see that he applies the formula

$$V = \frac{h}{231} \cdot \left(\frac{1}{3}p + \frac{2}{3}P\right) = \frac{h}{C} \cdot \left(5d^2 + 10D^2\right), \tag{8}$$

where p and P denote the areas of the head and the bung of the cask respectively, d and D their diameters, h the height of the cask and C is a constant. The calculations imply that he takes $\pi = 3{,}141\,5$. He claims without proof (page 12) that

To find the content of the whole Cask, being as before supposed to be the *Fructum* of a Spheroid, the rule is this: *Ten times the square of the greater Diameter BD more 5 times the square of PQ, the lesser Diameter, Multiplied by ML the length of the Vessel, and the Product divided by 15 times the quotient of the Diameter of a circle divided by the Area, will give the content of the whole Vessel.*

In the text above *the quotient of the Diameter* is a constant. Thus we have the formula (8). John Newton writes in another place (page 16):

But if it is taken for the middle ***Fructum*** *of a* ***Parabolical Spindle*** *intercepted between two Paralell plains Equidistant from the Center, cutting the Axis at right Angles, the work will be somewhat differing from the former, not more difficult, but at the experience of some practical Gaugers hath clearly proved it more exact.*

The way is thus.

Let D represent the great Diameter, d the lesser, L the length, hence to find the content of the whole Cask, I say

$$\frac{(8DD + 7dd) \times L}{Dr.} = Q.$$

That is in words. Eight times the square of the greater Diameter, more seven times the square of the lesser, being multiplied by the length, and divided by the common Divisor, the quoties shall be the content.

John Newton considers two cases

Case I. He assumes at first, that the cask is constructed by turning an arc of a circle with its centre in the symmetry point of the cask. In this case he applies the formula (8), in which we can take either the constant $C = \dfrac{60}{\pi}$, or another one, if we want to have the result in

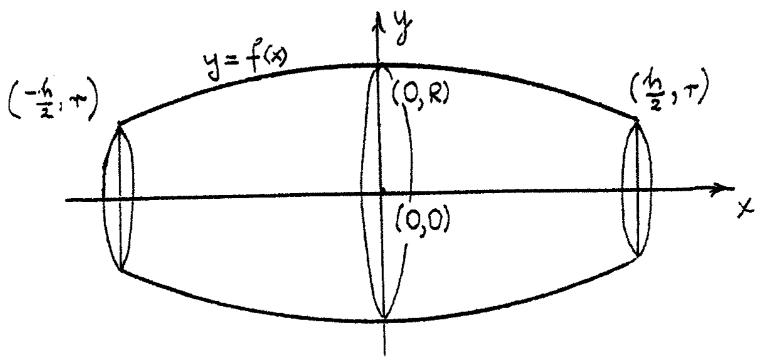


Fig. 1

gallons, not in cubic inches. He mentions John Wallis, however without a concrete source. Let us analyse the text. Intersecting the barrel we have the circle $x^2+y^2 = R^2$, with points $\left(\frac{h}{2}, r\right)$ and $\left(-\frac{h}{2}, r\right)$, i. e. $\left(\frac{h}{2}\right)^2 + r^2 = R^2$. Thus the volume V of the barrel can be calculated by the well-known formula:

$$
\begin{aligned}
V = \pi \cdot \int_{-h/2}^{h/2} y^2 \mathrm{d}x = 2\pi \cdot \int_0^{h/2} y^2 \mathrm{d}x = 2\pi \cdot \int_0^{h/2} \left(R^2 - x^2\right) \mathrm{d}x = \\
= 2\pi \left(R^2 \frac{h}{2} - \frac{1}{3} \cdot \left(\frac{h}{2}\right)^3 \right) = \pi \cdot \frac{h}{3} \cdot \left(3R^2 - \left(\frac{h}{2}\right)^2 \right) = \\
= \frac{\pi h}{3} \cdot \left(3R^2 - \left(R^2 - r^2\right)\right) = \frac{\pi h}{3} \cdot \left(2R^2 + r^2\right) = \\
= \frac{\pi h}{15} \cdot \left(10R^2 + 5r^2\right) = \frac{h}{C} \cdot \left(10D^2 + 5d^2\right),
\end{aligned}
$$

with the constant C as above.

Thus John Newton uses the correct formula in Case I.

Case II. In this case John Newton assumes, that the barrel is parabolic i. e. it is constructed by turning an arc of the parabola intersecting the points $(0, R)$, $\left(\frac{h}{2}, r\right)$ and $\left(-\frac{h}{2}, r\right)$. He states, that in this case one can use the formula

$$
V = \frac{h}{C} \cdot \left(8D^2 + 7d^2\right) \tag{9}
$$

which is experimental, but very exact. He gives no proofs and no essential comments. I do not believe that he had applied calculus, although we cannot exclude it. The parabola of the barrel has the equation $y = px^2 + q$. In our case (see the figure) $p = (r - R) \cdot \left(\frac{h}{2}\right)^{-2}$, and $q = R$. Thus the volume of the cask is equal to $\frac{\pi}{15} h \cdot \left(3(R - r)^2 + 10(r - R)R + 15R^2\right) = \frac{\pi}{15} \cdot \left(8R^2 + 4Rr + 3r^2\right)$. It implies that $V = \frac{h}{C} \cdot \left(8D^2 + 4Dd + 3d^2\right)$. Since $D > d$, thus the exact volume is greater than (9). John Newton gives also a formula for the volume of an elliptic cask (loc. cit., p. 30).

9 XVIII century

It is rather strange that in no calculus textbook from XVIII century starting from the treatises of de l'Hospital [8], Isaac Newton ([13]-[15]), S. Wydra [23], and finishing on Simon Lhuilier books ([9]-[10]) there is no word how to calculate the volume of a barrel. However the problem was discussed by Christian Wolff [22]. His solution however is much more elementary than in John Newton's treatise [16]. Indeed, if the cask has the height h, the area of the head is p and area of the bung is P, then the first algorithm gives $\tilde{V} = h \cdot \dfrac{p + P}{2}$, i.e.

$$\tilde{V} = \frac{1}{2} \cdot h \cdot \pi \cdot \left(r^2 + R^2\right), \tag{10}$$

where r and R are radii of the head and the bung. By the second Wolff's algorithm the barrel consists from two truncated cones with radii a and b respectively, giving the formula (4). A simple calculation shows that $V - \tilde{V} = \frac{1}{6}\pi h \cdot (R - r)^2$.

Measures and weights were standardized in many countries. Barrels were also standardized. For example, two standardized kinds of barrels in Poland and in Lithuania are described in details in [6]. It was accepted by Sejm (Polish Parlament). Similar regulations held also in other countries. It was an important knowledge for wine merchants to know necessary laws in Europe and outside.

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