BEGINNINGS OF THE CALCULUS
The First Published Books

Witold Więsław

Below I shall present shortly some remarks about the first published books from the calculus. I want to discuss mainly the level of rigour in old mathematical texts. So I shall concentrate myself only on a few selected texts (see References).

1 Isaac Newton

The famous book [5] contains an axiomatic treatment of physics. At the beginning Newton gives a formal definition of the definite integral for monotonous continuous functions, however without using any special terminology. Today we know that there exist continuous nowhere monotonous functions. Nevertheless his proofs are correct for the continuous monotonous functions (see enclosed page 25 from [5]). At first he proves (Lemma II) that if one divides an interval AE into n equal parts, then the lower and upper integral sums tend to the same limit, the area under the function. In the next Lemma III Newton proves, that the same holds in general, when one divides AE into n unequal parts under the assumption that the interval of maximal length tends to zero with n tending to infinity. However, Newton gives no algorithms in [5] for calculating such areas.

2 Guillaume François de l'Hospital

[1, 2] is in fact the first textbook presenting calculus. De l'Hospital applies Newton's method of fluxions and his terminology. The book is divided in two parts:

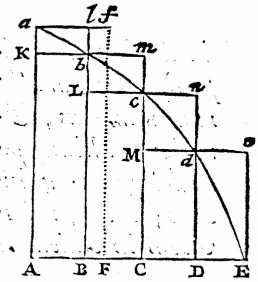
Part I. A Treatise of Fluxions.

Part II. Being an APPENDIX, Containing the Inverse Method of FLUXIONS, with the Application thereof in the Investigation of the Areas of Superficies, Lengths of Curve Lines, Contents of Solids, and the Determination of their Centres of Gravity and Percussion.

LIBER PRIMUS.

LEMMA II.

Si in Figura quavis AacE, *rectis* Aa, AE & *curva* acE *comprehenſa, inſcribantur parallelogramma quotcunque* Ab, Bc, Cd, &c. *ſub baſibus* AB, BC, CD, &c. *æqualibus, & lateribus* Bb, Cc, Dd, &c. *Figuræ lateri* Aa *parallelis contenta; & compleantur parallelogramma* aKbl, bLcm, cMdn, &c. *Dein horum parallelogrammorum latitudo minuatur, & numerus augeatur in infinitum: dico quod ultimæ rationes, quas habent ad ſe invicem Figura inſcripta* AKbLcMdD, *circumſcripta* AalbmcndoE, & *curvilinea* AabcdE, *ſunt rationes æqualitatis.*



Nam Figuræ inſcriptæ & circumſcriptæ differentia eſt ſumma parallelogrammorum *Kl*, *Lm*, *Mn*, *Do*, hoc eſt (ob æquales omnium baſes) rectangulum ſub unius baſi *Kb* & altitudinum ſumma *Aa*, id eſt, rectangulum *ABla*. Sed hoc rectangulum, eo quod latitudo ejus *AB* in infinitum minuitur, fit minus quovis dato. Ergo (per Lemma 1) Figura inſcripta & circumſcripta & multo magis Figura curvilinea intermedia fiunt ultimo æquales. *Q.E.D.*

LEMMA III.

Eædem rationes ultimæ ſunt etiam rationes æqualitatis, ubi parallelogrammorum latitudines AB, BC, CD, &c, *ſunt inæquales, & omnes minuuntur in infinitum.*

Sit enim *AF* æqualis latitudini maximæ, & compleatur parallelogrammum *FAaf*. Hoc erit majus quam differentia Figuræ inſcriptæ & Figuræ circumſcriptæ; at latitudine ſua *AF* in infinitum diminuta, minus fiet quam datum quodvis rectangulum. *Q.E.D.*

Corol. 1. Hinc ſumma ultima parallelogrammorum evaneſcentium coincidit omni ex parte cum Figura curvilinea.

Corol. 2. Et multo magis Figura rectilinea, quæ chordis evaneſ-

D centium

A page from Newton's *Principia Mathematica*

The text on pages 1–38 is almost identical with the beginning of [6]. Probably
textscde l'Hospital had to know the original NEWTON's manuscript. DE L'HOSPITAL starts with the following definitions:

Definition I. Variable Quantities are those that continually increase or decrease; and constant or standing Quantities, are those that continue the same while others vary. As the Ordinates and Abcisses of a Parabola are variable Quantities, but the Parameter is a constant or standing Quantity.

Definition II. The infinitely small Part whereby a variable Quantity is continually increased or decreased, is called the Fluxion of that Quantity. [...]

For example, in Sections I and II he discusses a problem of tangents, in Section III minima and maxima of functions.

3 Newton posthumous publications [6, 7]

Many historians of mathematics state that NEWTON used in his calculus infinitely small elements. Indeed, he used the term, but not too often.Moreover it is often stated that Newton's rigour is not too exact. So let's try to understand Newton's own presentation.

NEWTON wrote in ([6], pages 19-20) the following:

59. *But whereas we need not consider the Time here, any farther than as it is expounded and measured by an equable local Motion; and besides, whereas only Quantities of the same kind can be compared together, and also their Velocities of Increase or Decrease: Therefore in what follows I shall have no regard to Time formally consider'd, but I shall suppose some one of the Quantities proposed, being of the same kind, to be increased by an equable Fluxion, to which the rest may be reffer'd, as it were to Time; and therefore, by way of Analogy, it may not improperly receive the name of Time. Whenever therefore the word Time occurs in what follows, (which for the sake of perspicuity and distinction I have sometimes used,) by that Word I would not have it understood as if I meant Time in its formal Acceptation, but only that other Quantity, by the equable Increase of Fluxion whereof, Time is expounded and measured.*

60. *Now those Quantities which I consider as gradually and indefinitely increasing, I shall call Fluents, or Flowing Quantities, and shall represent them by the final Letters of the Alphabet v,x,y, and z; that I shall distinguish them from other Quantities, which in Equations are to be consider'd as known and determinate, and which therefore are represented by the initial Letters a,b,c, & c. And the Velocities by which every Fluent is increased by its generating Motion, (which I may call Fluxions, or simply velocities or Celerities,) I shall represent by the same Letters pointed thus v,x,y, and z.*

At other places NEWTON writes rather about indefinitely small interval of Time (i.e. of a variable) than about infinitely little, although the last statement appears sometimes in the text. NEWTON uses power series for representing functions (fluents). He says nothing about their convergence, but he carefully uses suitable expansions of functions into power series.

The text [7] is a version of NEWTON's original Latin manuscript.

Fluat quantitas x uniformiter & invenienda ſit fluxio quantitatis x^n. Quo tempore quantitas x fluendo evadit $x + o$, quantitas x^n evadet $\overline{x+o}|^n$, id eſt per methodum ſerierum infinitarum, $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2}$ + &c. Et augmenta o & $nox^{n-1} + \frac{nn-n}{2}oox^{n-2}$ + &c. ſunt ad ſe invicem ut 1 & $nx^{n-1} + \frac{nn-n}{2}ox^{n-2}$ + &c. Evaneſcant jam augmenta illa, & eorum ratio ultima erit 1 ad nx^{n-1}: ideoque fluxio quantitatis x eſt ad fluxionem quantitatis x^n, ut 1 ad nx^{n-1}.

Newton's proof that $(x^n)' = nx^{n-1}$

The most important role in teaching of the calculus in the XVIII century was played by the textbooks of EULER. However, they are described in detail in many publications, so I am not going to discuss them here.

4 Stanislaus Wydra

WYDRA was a professor at Prague University. In his times, in the eighties of the XVIII century there had appeared more than twenty textbooks from the calculus. The textbook [8] was written in the standard terminology and notation of LEIBNIZ. WYDRA applies infinitely small elements in presentation of the calculus. However, despite the state of mathematics in the XVIII century, the book contains interesting examples of application of the calculus to proofs and calculations. For example on the page 66 of his rather short (only 88 pages) textbook WYDRA proves the well-known Archimedes theorem: The area of a circle with a given radius equals to the area of a right triangle with the radius and the perimetr of the circle as its legs.

His proof runs as follows (see the picture below): Let $AC = r$, $AB = x$, $BD = dx$. Then BCD can be treated as an equilateral triangle. Thus the area of the triangle BCD equals $\frac{1}{2}rdx$. It implies that the area of the sector ABC is $\frac{1}{2}\int rdx = \frac{1}{2}rx$. Consequently, if $x =$ perimeter of the circle $= p$, then its area equals $\frac{1}{2}rp$.

77. *Invenire aream circuli.*

Fig. 10. RESOL. Sit ſector circuli ACB finitus, ejus arcus $AB = x$, radius $AC = r$. Sit incrementum infinite parvum BD, arcus AB; erit $BD = dx$. Ducto radio DC. Sector BCD haberi poterit pro triangulo rectilineo; conſequenter area $BDC = \frac{rdx}{2}$, & $\frac{\int rdx}{2} = \frac{rx}{2}$, quæ eſt area ſectoris ABC. Abeat hic ſector in circulum; evadet $x =$ peripheriæ $= p$. Vnde area circuli prodibit $= \frac{rp}{2}$. Vt in *Geometria elementari.*

Extract from the book by Stanisław Wydra

The book [8] contains other interesting examples of applications of the calculus.

5 Simon Lhuilier [3, 4]

I finish my short presentation of textbooks of the calculus with some remarks on LHUILIER's texts. The textbook [4] is a Latin translation of [3]. Originally, the book [3] was written for the competition of the Berlin Academy of Sciences. LHUILIER was awarded a prize for

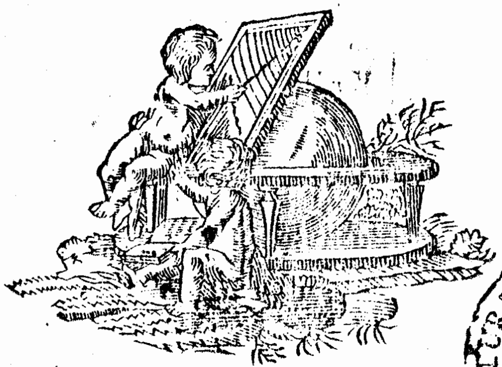
ELEMENTA
CALCVLI DIFFERENTIALIS,
ET
INTEGRALIS.

CONSCRIPTA

A

STANISLAO WYDRA,

IN VNIVERSITATE PRAGENSI MATHESEOS PROFESSORE, ET EXAMINATORE R. P. O.



PRAGAE ET VIENNAE,

apud Ioan. Ferd. Nob. a Schönfeld.

1783.

Title page of *Elementa Calculi Differentialis* by Stanisław Wydra it. He introduces for the first time a notion of the limit and the notation LIM. LHUILIER proves fundamental properties of this notion, presenting proofs of the well-known theorems for the limit of the product and the sum of functions. A derivative of a function is defined there in a standard way. Although his definition of the limit is far from our contemporary definition and is geometric in its nature, it appears for the first time in the literature.

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[2] [G.F. de L'Hospital]. *The method of fluxions both direct and inverse.* The former being a translation from celebrated Marquis De L'Hospital's Analyse des Infiniment Petits: And the latter Supply'd by the translator, E. Stone, F.R.S., London MDCCXXX.

[3] Simon L'huilier *Exposition elementaire des principies des calculs suprieurs, qui a remport le prix propos par l'Academie Royal des Sciences et Belles-Lettres* pour l'annee 1786, Berlin.

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