element of $(\overline{Y}, \mathring{A})$ and, in fact, the element containing a. In a similar way we can verify that the set $s(\bar{a} \sqsubset \overline{Y} \sqcap \bar{x})$ is the element of $(\overline{X}, \mathring{B})$, containing a. From this and from $(\overline{X}, \mathring{B}) = (\overline{Y}, \mathring{A})$ there follows the equality we were to prove.

5.6. Exercises

1. If the decompositions $\bar{A}, \bar{B}$ are complementary, then the formulae $(\mathring{A}, \mathring{B}) = (\overline{X}, \mathring{B}) = (\overline{Y}, \mathring{A}) = ((\overline{X}, \overline{Y}), [\bar{A}, \bar{B}])$, valid for modular decompositions $\overline{X} \geqq \bar{A}$, $\overline{Y} \geqq \bar{B}$ (see 4.3. (2)), may be completed by $(\mathring{A}, \mathring{B}) = [(\overline{X}, \bar{B}), (\overline{Y}, \bar{A})]$. In that case the decompositions $(\overline{X}, \bar{B})$, $(\overline{Y}, \bar{A})$ are complementary as well.
2. Show that in a set of four elements there exist, beside the pairs consisting of a covering and a refinement, only the following pairs of complementary decompositions: a) pairs of decompositions consisting of two elements each of which comprises only two points of the set; b) pairs of disjoint decompositions each of which contains three elements.

6. Mappings of sets

The theory of decompositions in sets considered in the previous chapters is the set-basis of the theory of groupoids and groups we intend to develop. But the results we have hitherto arrived at are only one part of the means necessary to attain our object. The other part consists of the theory of the mappings of sets, dealt with in the following chapters. The reader will certainly welcome the fact that the preceding, at times rather complicated, deliberations will now again be replaced by simpler ones.

6.1. Mappings into a set

In everyday life we often come across phenomena connected with the mathematical concept of mapping. Such phenomena are, in the simplest case, of the following kind: We have two nonempty sets G, G^* and between their elements a certain relation by which there corresponds, to each element of G, exactly one element of G^*. For example:

[1] Between the spectators at a certain performance and the tickets issued for the latter there exists the relation that each of the spectators is present on the ground of exactly one ticket.

[2] Between the pupils of a certain school and its classes there is the relation that each of the pupils belongs to exactly one class.

[3] The number n of certain objects is determined by way of associating each object with exactly one integer $1, 2, \ldots, n$; this is generally done by taking each of the objects, one by one, in hand and marking it, actually or only in mind, with one of the integers $1, 2, \ldots, n$.

Let G, G^* stand for nonempty sets. By a *mapping of the set G into G^** we understand a correspondence between the elements of both sets such that to each element of G there corresponds exactly one element of G^*; in other words, a relation by which each element of G is mapped exactly into one element of G^*.

A mapping of the set G into G^* is also called a *function* the domain of which is the set G and the range a part of G^*.

Consider an arbitrary mapping $\boldsymbol{g}$ of the set G into G^*. The mapping $\boldsymbol{g}$ associates, with each element $a \in G$, a certain element $a^* \in G^*$. The element a is called an *inverse image of a^** and the element a^* the *image of a* under the mapping $\boldsymbol{g}$; we write $a^* = \boldsymbol{g}(a)$ or only $a^* = \boldsymbol{g}a$. Sometimes we also say that a^* is the *value of the function $\boldsymbol{g}$ in a*. Another way of notation is $\begin{pmatrix} a \\ a^* \end{pmatrix}$; the symbol $\begin{pmatrix} a & b & \ldots \\ a^* & b^* & \ldots \end{pmatrix}$ expresses $a^* = \boldsymbol{g}a$, $b^* = \boldsymbol{g}b, \ldots$

If A is a subset of G and A^* the subset of G^* consisting of the images of the individual elements of A, we write $A^* = \boldsymbol{g}(A)$ or only $A^* = \boldsymbol{g}A$. If $A \neq \emptyset$, then we can associate, with every element $a \in A$, the element $\boldsymbol{g}a \in G^*$ and thus obtain a mapping of the set A into G^*. It is called the *partial mapping (function) determined by $\boldsymbol{g}$* and denoted $\boldsymbol{g}_A$.

By the definition of a mapping of G into G^* there corresponds, to an arbitrary element $a \in G$, exactly one image $a^* \in G^*$. Accordingly, such mappings are called *single-valued*.

In our study we shall sometimes meet with several mappings $\boldsymbol{g}, \boldsymbol{h}, \ldots$ simultaneously. In such cases we mark the concepts connected with the single mappings by a prefix, for example: $\boldsymbol{g}$-, $\boldsymbol{h}$-, ... and speak about $\boldsymbol{g}$-images, $\boldsymbol{h}$-inverse images, etc.

If two mappings $\boldsymbol{g}, \boldsymbol{h}$ of the set G into G^* are such that $\boldsymbol{g}a = \boldsymbol{h}a$ for each element $a \in G$, we call them *equal* and write $\boldsymbol{g} = \boldsymbol{h}$. In the opposite case we call them *different* and write $\boldsymbol{g} \neq \boldsymbol{h}$.

6.2. Mappings onto a set

By the definition of a mapping $\boldsymbol{g}$ of the set G into G^*, each element of G has, under the mapping $\boldsymbol{g}$, an image but, conversely, each element of G^* need not have an inverse image. If each element of G^* has an inverse image, then $\boldsymbol{g}$ is said to be a mapping of G onto G^*; we also say that the function $\boldsymbol{g}$ maps the set G onto G^*. If $\emptyset \neq A \subset G$, then $\boldsymbol{g}_A$ is evidently a mapping of the set A onto the set $\boldsymbol{g}A$.

From the above examples, the second [2] as well as the third [3] is a mapping onto a set: to each class there belongs at least one pupil associated with it under the mentioned mapping; if we have n objects and are to determine their number, then each object is marked by one of the numbers $1, 2, \ldots, n$. Example [1], on the other hand, is a mapping onto a set only if we assume that the house is quite full. In the opposite case, there have still remained some tickets for which there are no spectators.

6.3. Simple (one-to-one) mappings

In the notion of a mapping of the set G into G^* there is a further asymmetry with regard to both sets: Under the mapping $\boldsymbol{g}$ each element of G has exactly one image in G^* whereas, conversely, the same element on G^* may have several, even an infinite number of, inverse images in G.

If each element of G^* has, under $\boldsymbol{g}$, at most one inverse image, then $\boldsymbol{g}$ is called a *simple* or *one-to-one mapping of the set G into G^*.*

It is clear that $\boldsymbol{g}$ is a simple mapping of the set G onto the set G^* if and only if each element of G^* has exactly one inverse image.

The above example [3] is a simple mapping onto a set; [2] is an example of a simple mapping onto a set only if (in theory) each class has only one pupil; [1] is an example of a simple mapping onto a set only if the house is full and no tickets have remained.

6.4. Inverse mappings. Equivalent sets. Ordered finite sets

The concept of a simple mapping of a set onto a set is connected with two important notions: the notion of the inverse mapping and the notion of equivalent sets.

1. *Inverse mapping.* Suppose $\boldsymbol{g}$ is a simple mapping of the set G onto G^*. Then we can define a mapping of G^* onto G, denoted by $\boldsymbol{g}^{-1}$ and called the *inverse mapping* with regard to $\boldsymbol{g}$, in the following way: Each element $a^* \in G^*$ is, under $\boldsymbol{g}^{-1}$, associated with its $\boldsymbol{g}$-inverse image $a \in G$.

In example [1], provided the house is full and no tickets have remained, there corresponds, under $\boldsymbol{g}^{-1}$, to each ticket the spectator who owns it.

Obviously, the inverse mapping is simple and its inverse, $(\boldsymbol{g}^{-1})^{-1}$, is again the mapping $\boldsymbol{g}$, hence $(\boldsymbol{g}^{-1})^{-1} = \boldsymbol{g}$.

2. *Equivalent sets.* Given two nonempty sets G, G^*, there need not exist any mapping of G *onto* G^* as we see, for example, in the case when G consists of one element and G^* of two elements; therefore even a simple mapping of a set onto another does not necessarily exist.

Note that if there exists a simple mapping $\boldsymbol{g}$ of G onto G^*, then there also exists a simple mapping, $\boldsymbol{g}^{-1}$, in the opposite direction, i.e., a mapping of G^* onto G.

If there exists a simple mapping $\boldsymbol{g}$ of G onto G^*, then the set G *is said to be equivalent to* G^*. Then, of course, the set G^* is also equivalent to G. With regard to this symmetry, we speak about equivalent sets G, G^* without differentiating which is equivalent to which. The equivalence of sets is expressed by the formulae: $G^* \simeq G$ or $G \simeq G^*$.

For example, every set A consisting of n (> 0) elements and the set $1, 2, \ldots, n$ are equivalent because, if the elements of A are denoted, let us say, $a_1, a_2, \ldots, a_n$ (it makes no difference for which element each symbol stands), then we have a simple mapping of A onto the set $(1, 2, \ldots, n)$, namely:

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 2 & \ldots & n \end{pmatrix}.$$

3. *Ordered finite sets.* If the set A consists of n (> 0) elements and a simple mapping of A onto the set $\{1, 2, \ldots, n\}$ is given, then A is said to be an *ordered set* and the mapping is called an *ordering of* A. An ordering of A is obtained, for example, by way of ranging its elements in a certain order, i.e., a certain element $a_1 \in A$ is marked as first, the next one as second, etc. and the last: $a_n \in A$ as the n^{th}. Then A is said to be the ordered set of elements $a_1, a_2, \ldots, a_n$. This notion therefore depends on the order in which the names of the individual elements are quoted or written.

By the *inversely ordered set* we mean the ordered set $\{a_1', \ldots, a'_{n-1}, a_n'\}$ where $a_1' = a_n, \ldots, a_n' = a_1$.

6.5. The decompositions of sets, corresponding to mappings

Let $\boldsymbol{g}$ stand for a mapping of the set G onto G^*. We have already noticed that an element $a^* \in G^*$ may have, under $\boldsymbol{g}$, several inverse images.

Consider the system $\overline{G}$ of all subsets $\bar{a}$ of G each of which is formed by all the inverse images under $\boldsymbol{g}$ of an element $a^* \in G^*$. Each element of $\overline{G}$ is therefore a subset of G, consisting of all the points mapped, under $\boldsymbol{g}$, onto the same point of G^*. Since G^* contains at least one element a^*, the system $\overline{G}$ is not empty because it contains the set $\bar{a}$ consisting of the inverse images of a^*. As $\boldsymbol{g}$ is a mapping of G onto G^*, each element of G^* has at least one inverse image, hence the set $\bar{a}$ of the inverse of each element $a^* \in G^*$ is not empty. $\overline{G}$ is therefore a nonempty system of nonempty subsets of G.

Moreover, it is easy to see that the system $\overline{G}$ is disjoint, i.e., every two of its elements are disjoint, and that it covers G (each element $a \in G$ has exactly one image $a^* \in G^*$ and therefore lies in exactly one element $\bar{a} \in \overline{G}$, namely in the set of the inverse images of a^*). Consequently, *the system $\overline{G}$ of all subsets of G, each of* *which is formed by all the inverse images under the mapping* $\boldsymbol{g}$ *of some element of* G^*, *is a decomposition of the set* G. We say that this decomposition *corresponds* or *belongs to the mapping* $\boldsymbol{g}$.

In the above example [2], the corresponding decomposition consists of single sets of pupils belonging to the same class.

Note, in particular, the following extreme cases: If the set G^* consists of one element only, then the corresponding decomposition $\overline{G}$ is $\overline{G}_{\max}$. If $\boldsymbol{g}$ is a simple mapping, then the corresponding decomposition is $\overline{G}_{\min}$.

6.6. Mappings of sets into and onto themselves

The above deliberations do not exclude that G^* may be identical with G. If $G^* = G$, then we speak about a mapping of the set G *into* or *onto itself*.

Associating, for example, with every natural number n the number $n + 1$, we obtain a mapping of the set of all natural numbers into itself.

The simplest mapping of the set G onto itself is obtained by associating, with every element $a \in G$, again the element a; it is the so-called *identical mapping of* G, denoted $\boldsymbol{e}$.

A simple mapping of the set G onto itself is called a *permutation of* G. Permutations of finite sets are the object of a more detailed study in Chapter 8.

6.7. Composition of mappings

The concept of a composite mapping. Let G, H, K stand for arbitrary nonempty sets, $\boldsymbol{g}$ denote a mapping of the set G into H and $\boldsymbol{h}$ a mapping of the set H into K. Then there corresponds, under the mapping $\boldsymbol{g}$, to every element $a \in G$ a certain element $\boldsymbol{g}a \in H$ and to $\boldsymbol{g}a$ there corresponds, under the mapping $\boldsymbol{h}$, an element $\boldsymbol{h}(\boldsymbol{g}a) \in K$. Associating with every element $a \in G$ the element $\boldsymbol{h}(\boldsymbol{g}a) \in K$, we have a mapping of the set G into K. It is called the *composite mapping of* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ (in this order) and is denoted by $\boldsymbol{hg}$. As a mapping of the set G into K, $\boldsymbol{hg}$ has the property that, for $a \in G$, there holds $(\boldsymbol{hg})a = \boldsymbol{h}(\boldsymbol{g}a)$.

Let us note some particular cases. If $\boldsymbol{g}$ maps the set G onto H and $\boldsymbol{h}$ maps the set H onto K, then $\boldsymbol{hg}$ is obviously a mapping of the set G onto K.

If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are simple mappings, then $\boldsymbol{hg}$ is simple as well because, in that case, any two different elements $a, b \in G$ have two different $\boldsymbol{g}$-images: $\boldsymbol{g}a, \boldsymbol{g}b \in H$ and the latter have two different $\boldsymbol{h}$-images: $\boldsymbol{hg}a, \boldsymbol{hg}b \in K$.

Furthermore, it is clear that if the set K is identical with G so that $\boldsymbol{h}$ is a mapping of the set H into G, then $\boldsymbol{hg}$ is a mapping of the set G into itself; if $\boldsymbol{g}$ maps the set G onto H and $\boldsymbol{h}$ the set H onto G, then $\boldsymbol{hg}$ is a mapping of the set G onto itself; in particular, if the mapping $\boldsymbol{g}$ is simple and $\boldsymbol{h} = \boldsymbol{g}^{-1}$, then $\boldsymbol{hg}$ is the identical mapping of the set G.

Note, moreover, that if the sets H and K are both identical with G so that both $\boldsymbol{g}$ and $\boldsymbol{h}$ are mappings of G into itself, then even $\boldsymbol{hg}$ is a mapping of G into itself; if both $\boldsymbol{g}$ and $\boldsymbol{h}$ map G onto itself, then even $\boldsymbol{hg}$ maps G onto itself.

A simple mapping $\boldsymbol{g}$ of the set G onto itself is called *involutory* if the composite mapping $\boldsymbol{gg}$ is the identical mapping of G: $\boldsymbol{gg} = \boldsymbol{e}$. The inverse mapping $\boldsymbol{g}^{-1}$ of any involutory mapping $\boldsymbol{g}$ obviously equals $\boldsymbol{g}$, i.e., $\boldsymbol{g}^{-1} = \boldsymbol{g}$.

Finally, let us note that for the identical mapping $\boldsymbol{e}$ of the set G and for an arbitrary mapping $\boldsymbol{g}$ of G into itself there holds: $\boldsymbol{eg} = \boldsymbol{ge} = \boldsymbol{g}$.

Example of a composite mapping: If $\boldsymbol{g}$ denotes the mapping considered in the above example [1] and $\boldsymbol{h}$ stands for the mapping of the set of tickets into the set of colours associated with the tickets, then the composite mapping $\boldsymbol{hg}$ associates, with every spectator, a certain colour, namely the colour of his ticket.

The associative law for the composition of mappings. Let us now consider three mappings $\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{k}$, where $\boldsymbol{k}$ stands for a mapping of the set K into some set L (without excluding the case that L is identical with one of the sets G, H, K). An important property of the composition of mappings consists in that there holds:

$$\boldsymbol{k}(\boldsymbol{hg}) = (\boldsymbol{kh})\boldsymbol{g},$$

called the *associative law for the composition of mappings.*

The above equality expresses that every element of G has, under both the mappings $\boldsymbol{k}(\boldsymbol{hg})$ and $(\boldsymbol{kh})\boldsymbol{g}$, the same image lying, of course, in the set L.

To prove this, let us consider the image of an element $a \in G$ under the mapping $\boldsymbol{k}(\boldsymbol{hg})$. The $\boldsymbol{k}(\boldsymbol{hg})$-image of a is the image of the element $(\boldsymbol{hg})a$ under the mapping $\boldsymbol{k}$ and is therefore obtained by associating, with the element $\boldsymbol{g}a \in H$, its $\boldsymbol{h}$-image $\boldsymbol{h}(\boldsymbol{g}a) \in K$ and then, with the latter, its $\boldsymbol{k}$-image $\boldsymbol{k}(\boldsymbol{hg})a \in L$. But the $\boldsymbol{k}$-image of the element $\boldsymbol{h}(\boldsymbol{g}a)$ is, by the definition of the mapping $\boldsymbol{kh}$, the same as the $(\boldsymbol{kh})\boldsymbol{g}$-image of the element a. Consequently, the above equality is true.

Instead of $\boldsymbol{k}(\boldsymbol{hg})$ or $(\boldsymbol{kh})\boldsymbol{g}$ we simply write $\boldsymbol{khg}$.

6.8. The equivalence theorems

Let us now introduce three theorems called equivalence theorems. They can, owing to their simplicity, be justly regarded as describing the properties of certain equivalent sets. Their value is due to the fact that they express the set-structure of important situations connected with the so-called theorems of isomorphism we shall deal with in the theory of groupoids and groups.

1. The first equivalence theorem. *If there exists a mapping of the set G onto the set G^*, then G^* is equivalent to a certain decomposition lying on G and vice versa. The mapping of the decomposition $\bar{G}$ belonging to a mapping $\boldsymbol{g}$ of the set G onto G^* under which there corresponds, to every element $\bar{a} \in \bar{G}$, the $\boldsymbol{g}$-image of the points lying in $\bar{a}$, is simple.*

Indeed, if there exists a mapping $\boldsymbol{g}$ of the set G onto G^*, then the set G^* is equivalent to the decomposition $\bar{G}$ belonging to $\boldsymbol{g}$. A simple mapping of $\bar{G}$ onto G^* is obtained by associating, with every element $\bar{a} \in \bar{G}$, the $\boldsymbol{g}$-image of the points $a \in G$ lying in $\bar{a}$. If, conversely, there exists a simple mapping $\boldsymbol{i}$ of a decomposition $\bar{G}$ of the set G onto the set G^*, then the composite mapping $\boldsymbol{ij}$ maps the set G onto G^*; $\boldsymbol{j}$ denotes the mapping of the set G onto the decomposition $\bar{G}$, associating with each point $a \in G$ that element $\bar{a} \in \bar{G}$ which contains a: $a \in \bar{a} = \boldsymbol{j}a \in \bar{G}$. The decomposition of G belonging to the mapping $\boldsymbol{ij}$ is $\bar{G}$.

2. Second theorem. *Every two coupled decompositions $\bar{A}$, $\bar{B}$ in G are equivalent,* i.e., $\bar{A} \simeq \bar{B}$. *The mapping of the decomposition $\bar{A}$ onto $\bar{B}$ under which there corresponds, to every element $\bar{a} \in \bar{A}$, the element $\bar{b} \in \bar{B}$ incident with $\bar{a}$, is simple.*

An important case of this theorem (see 4.1) concerns the equivalence of the closure and the intersection of a subset $X \subset G$ and a decomposition $\bar{Y}$ in G: *If $X \cap \mathbf{s}\bar{Y} \neq \emptyset$, then there holds $X \sqsubset \bar{Y} \simeq \bar{Y} \sqcap X$. The mapping given by the incidence of the elements is simple.*

3. Third theorem. *A decomposition $\bar{\bar{B}}$ of some decomposition $\bar{B}$ of the set G and the covering $\bar{A}$ of $\bar{B}$, enforced by $\bar{\bar{B}}$, are equivalent sets,* i.e., $\bar{\bar{B}} \simeq \bar{A}$. *The mapping of the decomposition $\bar{\bar{B}}$ onto $\bar{A}$ under which there corresponds, to every element $\bar{\bar{b}} \in \bar{\bar{B}}$, the sum $\bar{a} \in \bar{A}$ of the elements of $\bar{B}$ lying in $\bar{\bar{b}}$, is simple.*

6.9. Mappings of sequences and α-grade structures

In this chapter we shall deal with some more complicated notions based on the concept of the equivalence of sets.

1. *Mappings of sequences.* Let α ($\geqq 1$) be a positive integer. Consider wo arbitrary α-membered sequences:

$$(a) = (a_1, \ldots, a_\alpha), \quad (b) = (b_1, \ldots, b_\alpha).$$

a) By a *mapping $\boldsymbol{a}$ of the sequence* (a) *onto the sequence* (b) we naturally understand a simple mapping (6.10.2) of the set formed by the members of (a) onto the set of the members of (b). Under any mapping $\boldsymbol{a}$ of the sequence (a) onto the sequence (b) there corresponds, therefore, to each member a_γ of (a), exactly one member $b_\delta = \boldsymbol{a}a_\gamma$ of (b) and, simultaneously, to members of different indices there correspond, in (b), members with different indices as well. Every mapping $\boldsymbol{a}$ of the sequence (a) onto (b) is uniquely determined by a certain permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ in the sense of the formula: $\boldsymbol{a}a_\gamma = b_{\boldsymbol{p}\gamma}$ $(\gamma = 1, \ldots, \alpha)$. The function inverse of a mapping of the sequence (a) onto (b) is, of course, a mapping of the sequence (b) onto (a).

It is clear that there exist (even in the number $\alpha!$) mappings of the sequence (a) onto (b) as well as mappings in the opposite direction. We see that the sequences (a) and (b) are equivalent. Every two finite sequences of the same length are equivalent.

b) Suppose the members $a_1, \ldots, a_\alpha$ of the sequence (a) as well as the members $b_1, \ldots, b_\alpha$ of the sequence (b) are nonempty sets.

The *sequence* (b) *is* said to be *strongly equivalent to the sequence* (a) if the following situation occurs: There exists a mapping $\boldsymbol{a}$ of the sequence (a) onto (b) such that, to each member a_γ of (a), there corresponds a simple mapping $\boldsymbol{a}_\gamma$ of a_γ onto the member $b_\delta = \boldsymbol{a}a_\gamma$ of (b).

If (b) is strongly equivalent to (a), then (a) has, obviously, the same property with regard to (b). On taking account of this symmetry, we call the sequences (a) and (b) strongly equivalent.

c) Assume the members $a_1, \ldots, a_\alpha$ of the sequence (a) as well as the members $b_1, \ldots, b_\alpha$ of (b) to be decompositions in the set G.

The *sequence* (b) *is* said to be *semi-coupled* or *loosely coupled* (*coupled*) *with* (a) in the following situation: There exists a mapping $\boldsymbol{a}$ of the sequence (a) onto (b) such that every member a_γ of (a) is semi-coupled (coupled) with its $\boldsymbol{a}$-image $b_\delta = \boldsymbol{a}a_\gamma$ in (b).

If (b) is semi-coupled (coupled) with (a), then the sequence (a) has evidently the same property with regard to (b). In that case, on taking account of this symmetry, we call the sequences (a) and (b) semi-coupled (coupled).

Suppose (b) is semi-coupled with (a) and let $\boldsymbol{a}$ stand for a mapping of the sequence (a) onto (b), determining the pairs of semi-coupled (coupled) members. Consider a member a_γ of (a) and its $\boldsymbol{a}$-image $b_\delta = \boldsymbol{a}a_\gamma$ in (b). Then the closures $\mathrm{H}a_\gamma = b_\delta \sqsubset a_\gamma$, $\mathrm{H}b_\delta = a_\gamma \sqsubset b_\delta$ are nonempty and coupled (4.1). According to the second equivalence theorem (6.8), the mapping $\boldsymbol{a}_\gamma$ of the closure $\mathrm{H}a_\gamma$ onto the closure $\mathrm{H}b_\delta$, determined by the incidence of the elements, is simple. In particular, if the sequence (b) is coupled with (a), we have $\mathrm{H}a_\gamma = a_\gamma$, $\mathrm{H}b_\delta = b_\delta$. We see that *two coupled sequences are always strongly equivalent.*

2. *Mappings of α-grade structures.* Let α $(\geqq 1)$ be a positive integer and $\big((A) =\big)$ $(A_1, \ldots, A_\alpha)$, $\big((B) =\big)$ $(B_1, \ldots, B_\alpha)$ stand for arbitrary sequences of nonempty sets. Let, moreover, $\tilde{\boldsymbol{A}}$ be an α-grade structure with regard to the sequence (A) and $\tilde{\boldsymbol{B}}$ a structure of the same kind with regard to (B) (1.9).

Note that any element $\bar{\bar{a}} \in \tilde{\boldsymbol{A}}$ $(\bar{\bar{b}} \in \tilde{\boldsymbol{B}})$ is an α-membered sequence $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha)$ $\big(\bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha)\big)$ every member $\bar{a}_\gamma$ $(\bar{b}_\gamma)$ of which is a nonempty part of the set A_γ (B_γ); $(\gamma = 1, \ldots, \alpha)$.

Suppose there is a simple mapping $\boldsymbol{f}$ of the structure $\tilde{\boldsymbol{A}}$ onto $\tilde{\boldsymbol{B}}$.

a) The *mapping* $\boldsymbol{f}$ *is* called *a strong equivalence-mapping of the structure* $\tilde{\boldsymbol{A}}$ *onto* $\tilde{\boldsymbol{B}}$, briefly, a *strong equivalence of* $\tilde{\boldsymbol{A}}$ *onto* $\tilde{\boldsymbol{B}}$ in the following situation:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect: To every member $\bar{a}_\gamma$ of any element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{A}$ there exists a simple mapping $\boldsymbol{a}_\gamma$ of $\bar{a}_\gamma$ onto the member $\bar{b}_{\boldsymbol{p}\gamma}$ of the sequence $\boldsymbol{f}\bar{\bar{a}} = \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{B}$; $(\gamma = 1, \ldots, \alpha)$.

We observe that the inverse function $\boldsymbol{f}^{-1}$ of an arbitrary strong equivalence $\boldsymbol{f}$ of the structure $\tilde{A}$ onto $\tilde{B}$ is a strong equivalence in the opposite direction, i.e., a strong equivalence of $\tilde{B}$ onto $\tilde{A}$.

If there exists a strong equivalence of $\tilde{A}$ onto $\tilde{B}$, then we say that the *structure $\tilde{B}$ is strongly equivalent to $\tilde{A}$*. The notion of strong equivalence is, of course, symmetric with regard to both structures; therefore we also speak about strongly equivalent structures $\tilde{A}$, $\tilde{B}$.

b) Let us now assume that the sequences (A), (B) consist of the decompositions $\bar{A}_1, \ldots, \bar{A}_\alpha$ and $\bar{B}_1, \ldots, \bar{B}_\alpha$ in the set G. Then every element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{A}$ $(\bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{B})$ is an α-membered sequence every member of which, $\bar{a}_\gamma$ $(\bar{b}_\gamma)$, is a decomposition in G, namely, a part of the decomposition $\bar{A}_\gamma$ $(\bar{B}_\gamma)$; $(\gamma = 1, \ldots, \alpha)$.

The mapping $\boldsymbol{f}$ of the structure $\tilde{A}$ onto $\tilde{B}$ is called *equivalence-mapping connected with semi-coupling* or *equivalence-mapping connected with loose coupling* (*equivalence-mapping connected with coupling*), briefly *equivalence connected with semi-coupling* or *with loose coupling* (*with coupling*) if the following situation occurs:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect: Every member $\bar{a}_\gamma$ of any element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{A}$ is semi-coupled (coupled) with the member $\bar{b}_{\boldsymbol{p}\gamma}$ of the element $\boldsymbol{f}\bar{\bar{a}} = \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{B}$ $(\gamma = 1, \ldots, \alpha)$.

It is easy to see that the inverse function $\boldsymbol{f}^{-1}$ of an arbitrary equivalence connected with semi-coupling (coupling) of the structure $\tilde{A}$ onto the structure $\tilde{B}$ is an equivalence-mapping of $\tilde{B}$ onto $\tilde{A}$ which is of the same type.

Let $\boldsymbol{f}$ denote an equivalence connected with a loose coupling (coupling) of the structure $\tilde{A}$ onto $\tilde{B}$. Consider arbitrary members $\bar{a}_\gamma$, $\bar{b}_\delta$ $(\delta = \boldsymbol{p}\gamma)$ which are in the above relation so that $\bar{a}_\gamma$ is in $\bar{\bar{a}}$, $\bar{b}_\delta$ is in $\boldsymbol{f}\bar{\bar{a}} = \bar{\bar{b}}$ and the decompositions $\bar{a}_\gamma$, $\bar{b}_\delta$ are semi-coupled (coupled). Then the closures $\mathrm{H}\bar{a}_\gamma = \bar{b}_\delta \sqsubset \bar{a}_\gamma$, $\mathrm{H}\bar{b}_\delta = \bar{a}_\gamma \sqsubset \bar{b}_\delta$ are non-empty and coupled (4.1). By the second equivalence theorem (6.8), the mapping $\boldsymbol{a}_\gamma$ of the closure $\mathrm{H}\bar{a}_\gamma$ onto $\mathrm{H}\bar{b}_\delta$, given by the incidence of the elements, is simple. In particular, if $\boldsymbol{f}$ is an equivalence connected with coupling, we have $\mathrm{H}\bar{a}_\gamma = \bar{a}_\gamma$, $\mathrm{H}\bar{b}_\delta = \bar{b}_\delta$. We observe that *every equivalence of the structure $\tilde{A}$ onto $\tilde{B}$, connected with coupling, is a strong equivalence.*

If there exists an equivalence connected with semi-coupling (coupling) of the structure $\tilde{A}$ onto $\tilde{B}$, we say that *$\tilde{B}$ is equivalent to and semi-coupled* or *loosely coupled* (*coupled*) *with $\tilde{A}$*. These notions are obviously symmetric with regard to both structures; for that reason we speak about equivalent and semi-coupled or equivalent and loosly coupled (equivalent and coupled) structures $\tilde{A}$, $\tilde{B}$. Especially, *every two equivalent and coupled structures $\tilde{A}$, $\tilde{B}$ are strongly equivalent.*

6.10. Exercises

1. Consider some simple real functions (for example $y = ax + b$ or $y = x^2$ and similar) as particular cases of the above concept of a function.

2. If the sets G and G^* are finite and of the same order, then: a) every mapping of G onto G^* is simple; b) every simple mapping of G into G^* is a mapping *onto* G^*.

3. Assume $A \subset G$ and let $\boldsymbol{g}[A]$ denote the mapping of G into the set $\{0, 1\}$, defined als follows: For $a \in G$ there is $\boldsymbol{g}[A]a = 1$ or 0 according as a lies or does not lie in A. Prove that the following relations are true:
   a) $\boldsymbol{g}[A \cap B]a = (\boldsymbol{g}[A]a) \cdot (\boldsymbol{g}[B]a) =$ the least of the numbers $\boldsymbol{g}[A]a, \boldsymbol{g}[B]a$;
   b) $\boldsymbol{g}[A \cup B]a =$ the greatest of the numbers $\boldsymbol{g}[A]a, \boldsymbol{g}[B]a$;
   c) if $A \cap B = \emptyset$, then $\boldsymbol{g}[A \cup B]a = \boldsymbol{g}[A]a + \boldsymbol{g}[B]a$.

4. Let $\boldsymbol{f}[a]$ denote the mapping of a straight line onto itself, defined as follows: to every point of the straight line with the coordinate x there corresponds the point with the coordinate $x' = x + a$, a standing for a real number. Similarly, let $\boldsymbol{g}[a]$ be the mapping of the straight line onto itself, given by the formula $x' = -x + a$. The distance between two arbitrary points x_1, x_2 of the straight line, i.e., the number[1]) $|x_1 - x_2|$ and the distance between their images under both mappings $\boldsymbol{f}[a]$ and $\boldsymbol{g}[a]$ are equal. Under the mapping $\boldsymbol{f}[a]$, no point of the straight line is mapped onto itself unless $a = 0$ and then we have the identical mapping of the straight line onto itself; under the mapping $\boldsymbol{g}[a]$ exactly one point is mapped onto itself. For the composition of the mappings $\boldsymbol{f}[a]$ and $\boldsymbol{g}[a]$ there hold the following formulae:

$$\boldsymbol{f}[b]\boldsymbol{f}[a] = \boldsymbol{f}[a+b], \quad \boldsymbol{g}[b]\boldsymbol{f}[a] = \boldsymbol{g}[-a+b],$$
$$\boldsymbol{f}[b]\boldsymbol{g}[a] = \boldsymbol{g}[a+b], \quad \boldsymbol{g}[b]\boldsymbol{g}[a] = \boldsymbol{f}[-a+b].$$

   Remark. The mappings $\boldsymbol{f}[a]$ and $\boldsymbol{g}[a]$ are called *Euclidean motions on a straight line.*

5. Let $\boldsymbol{f}[\alpha; a, b]$ denote the mapping of a plane onto itself, defined in the following way: to every point in the plane, with the coordinates x, y, there corresponds the point with the coordinates x', y', where

$$x' = x \,.\, \cos\alpha + y \,.\, \sin\alpha + a,$$
$$y' = -x \,.\, \sin\alpha + y \,.\, \cos\alpha + b,$$

   α, a, b denoting real numbers. Similarly, let $\boldsymbol{g}[\alpha; a, b]$ be the mapping of the plane onto itself, given by the formulae

$$x' = x \cdot \cos\alpha + y \cdot \sin\alpha + a,$$
$$y' = x \cdot \sin\alpha - y \cdot \cos\alpha + b.$$

   The distance between two arbitrary points x_1, y_1 and x_2, y_2 in the plane, namely, the number $\sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}$ and the distance between their images in both mappings $\boldsymbol{f}[\alpha; a, b]$ and $\boldsymbol{g}[\alpha; a, b]$ are equal. Under the mapping $\boldsymbol{f}[\alpha; a, b]$, where α is a multiple of 2π, no point in the plane is mapped onto itself except in the case: $a = b = 0$ and then we have the identical mapping of the plane onto itself. If α is not a multiple of 2π, then

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[1]) If x is an arbitrary number, then $|x|$ denotes the absolute value of x, namely, the non-negative number of both x and $-x$.

exactly one point in the plane is mapped onto itself. Under the mapping $\boldsymbol{g}[\alpha; a, b]$ no point in the plane is mapped onto itself unless the numbers α, a, b are connected by the relation:

$$a \,.\, \cos\frac{1}{2}\,\alpha + b \,.\, \sin\frac{1}{2}\,\alpha = 0;$$

in that case all the points in the plane that are mapped onto themselves form a straight line. For the composition of the mappings $\boldsymbol{f}[\alpha; a, b]$, $\boldsymbol{g}[\alpha; a, b]$ there hold the following formulae:

$$\begin{aligned}
\boldsymbol{f}[\beta; c, d]\ \ \boldsymbol{f}[\alpha; a, b] &= \boldsymbol{f}[\alpha+\beta;\ \ a\,.\cos\beta + b\,.\sin\beta + c,\\
&\qquad\qquad\quad -a\,.\sin\beta + b\,.\cos\beta + d],\\
\boldsymbol{g}[\beta; c, d]\ \ \boldsymbol{f}[\alpha; a, b] &= \boldsymbol{g}[\alpha+\beta;\ \ a\,.\cos\beta + b\,.\sin\beta + c,\\
&\qquad\qquad\quad a\,.\sin\beta - b\,.\cos\beta + d],\\
\boldsymbol{f}[\beta; c, d]\ \ \boldsymbol{g}[\alpha; a, b] &= \boldsymbol{g}[\alpha-\beta;\ \ a\,.\cos\beta + b\,.\sin\beta + c,\\
&\qquad\qquad\quad -a\,.\sin\beta + b\,.\cos\beta + d],\\
\boldsymbol{g}[\beta; c, d]\ \ \boldsymbol{g}[\alpha; a, b] &= \boldsymbol{f}[\alpha-\beta;\ \ a\,.\cos\beta + b\,.\sin\beta + c,\\
&\qquad\qquad\quad a\,.\sin\beta - b\,.\cos\beta + d].
\end{aligned}$$

Remark. The mappings $\boldsymbol{f}[\alpha; a, b]$ and $\boldsymbol{g}[\alpha; a, b]$ are called *Euclidean motions in a plane.*

6. Every α-membered (infinite) sequence on a set A is the set formed from the images of the elements of the set $\{1, \ldots, \alpha\}$ ($\{1, 2, \ldots\}$) onto A under a convenient mapping of the latter onto the set A (1.7).
7. For the equivalence of nonempty sets A, B, C the following statements are correct: a) $A \simeq A$ (reflexivity); b) from $A \simeq B$ there follows $B \simeq A$ (symmetry); c) from $A \simeq B$, $B \simeq C$ there follows $A \simeq C$ (transitivity) (6.4).
8. Let $\boldsymbol{g}$, $\boldsymbol{h}$ denote mappings of the set G into itself and $\overline{G}_{\boldsymbol{g}}$, $\overline{G}_{\boldsymbol{h}}$, $\overline{G}_{\boldsymbol{hg}}$ be decompositions on G, corresponding to the mappings $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{hg}$. Show that the following relations apply:
   a) $\boldsymbol{hg}G \subset \boldsymbol{h}G$, $\overline{G}_{\boldsymbol{hg}} \geqq \overline{G}_{\boldsymbol{g}}$,
   b) the equality $\boldsymbol{hg}G = \boldsymbol{h}G$ yields $\boldsymbol{g}G \sqsubset \overline{G}_{\boldsymbol{h}} = \overline{G}_{\boldsymbol{h}}$ and vice versa,
   c) the equality $\overline{G}_{\boldsymbol{hg}} = \overline{G}_{\boldsymbol{g}}$ yields $\boldsymbol{g}G \sqcap \overline{G}_{\boldsymbol{h}} = (\overline{\boldsymbol{g}G})_{\min}$ and vice versa. ($(\overline{\boldsymbol{g}G})_{\min}$ is the least decomposition of the set $\boldsymbol{g}G$.)
9. Any two adjoint chains of decompositions in G have a coupled refinement. (Prove it by means of the construction described in 4.2.)

7. Mappings of decompositions

Let $\boldsymbol{g}$ denote a mapping of the set G onto a set G^*. Thus every element $\boldsymbol{a} \in G$ is, under $\boldsymbol{g}$, mapped onto a certain element $\boldsymbol{a}^* \in G^*$; $\boldsymbol{a}^*$ is the image of the element $\boldsymbol{a}$ under the mapping $\boldsymbol{g}$. To the mapping $\boldsymbol{g}$ there corresponds a certain decomposition $\overline{G}$ on G; each element of $\overline{G}$ consists of all $\boldsymbol{g}$-inverse images of the same point in G^*. The decomposition $\overline{G}$ is equivalent to the set G^*.