7. Mappings of decompositions


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1. Sets

exactly one point in the plane is mapped onto itself. Under the mapping \( g[a; a, b] \) no point in the plane is mapped onto itself unless the numbers \( \alpha, a, b \) are connected by the relation:

\[
a \cdot \cos \frac{1}{2} \alpha + b \cdot \sin \frac{1}{2} \alpha = 0;
\]

in that case all the points in the plane that are mapped onto themselves form a straight line. For the composition of the mappings \( f[a; a, b], g[a; a, b] \) there hold the following formulae:

\[
f[\beta; c, d] \ f[a; a, b] = f[a + \beta; a \cdot \cos \beta + b \cdot \sin \beta + c, -a \cdot \sin \beta + b \cdot \cos \beta + d],
\]

\[
g[\beta; c, d] \ f[a; a, b] = g[a + \beta; a \cdot \cos \beta + b \cdot \sin \beta + c, a \cdot \sin \beta - b \cdot \cos \beta + d],
\]

\[
f[\beta; c, d] \ g[a; a, b] = f[a - \beta; a \cdot \cos \beta + b \cdot \sin \beta + c, -a \cdot \sin \beta + b \cdot \cos \beta + d],
\]

\[
g[\beta; c, d] \ g[a; a, b] = g[a - \beta; a \cdot \cos \beta + b \cdot \sin \beta + c, a \cdot \sin \beta - b \cdot \cos \beta + d].
\]

Remark. The mappings \( f[a; a, b] \) and \( g[a; a, b] \) are called Euclidean motions in a plane.

6. Every \( \alpha \)-membered (infinite) sequence on a set \( A \) is the set formed from the images of the elements of the set \( \{1, \ldots, \alpha\} \) \((1, 2, \ldots)\) onto \( A \) under a convenient mapping of the latter onto the set \( A \) (1.7).

7. For the equivalence of nonempty sets \( A, B, C \) the following statements are correct:

a) \( A \simeq A \) (reflexivity); b) from \( A \simeq B \) there follows \( B \simeq A \) (symmetry); c) from \( A \simeq B, B \simeq C \) there follows \( A \simeq C \) (transitivity) (6.4).

8. Let \( g, h \) denote mappings of the set \( G \) into itself and \( \overline{G}_g, \overline{G}_h, \overline{G}_{hg} \) be decompositions on \( G \), corresponding to the mappings \( g, h, hg \). Show that the following relations apply:

a) \( hgG \subset hg, \overline{G}_{hg} \geq \overline{G}_g \),

b) the equality \( hgG = hG \) yields \( gG \cap \overline{G}_h = \overline{G}_g \) and vice versa,

c) the equality \( \overline{G}_{hg} = \overline{G}_g \) yields \( gG \cap \overline{G}_h = (gG)_{\min} \) and vice versa. \((gG)_{\min} \) is the least decomposition of the set \( gG \).

9. Any two adjacent chains of decompositions in \( G \) have a coupled refinement. (Prove it by means of the construction described in 4.2.)

7. Mappings of decompositions

Let \( g \) denote a mapping of the set \( G \) onto a set \( G^* \). Thus every element \( a \in G \) is, under \( g \), mapped onto a certain element \( a^* \in G^* \); \( a^* \) is the image of the element \( a \) under the mapping \( g \). To the mapping \( g \) there corresponds a certain decomposition \( \overline{G} \) on \( G \); each element of \( \overline{G} \) consists of all \( g \)-inverse images of the same point in \( G^* \). The decomposition \( \overline{G} \) is equivalent to the set \( G^* \).
7. Mappings of decompositions

7.1. Extended mappings

The mapping \( g \) determines a mapping \( \bar{g} \) of the system of all subsets of \( G \) into the system of all subsets of \( G^* \), the so-called extended mapping. \( \bar{g} \) is defined in the way that, for \( \emptyset \neq A \subset G \), \( \bar{g}A \subset G^* \) is the set of the \( g \)-images of all the points lying in \( A \); moreover, we put \( \bar{g}\emptyset = \emptyset \). In particular, for \( a \in \bar{g} \), the set \( \bar{g}a \) consists of a single point of \( G^* \), namely, of the \( g \)-image of the points of \( G \) lying in \( a \).

To simplify the notation, we generally write \( g \) instead of \( \bar{g} \). The symbol \( g \) is thus applied to the points of \( G \), e.g. \( a \in G \), and then the result \( ga \) denotes the image of the point \( a \) under the original mapping \( g \). The symbol \( g \) is also applied to subsets of \( G \), e.g. \( A \subset G \), in which case the result \( gA \) denotes the image of the subset \( A \) under the extended mapping \( \bar{g} \).

This rule is observed even for systems of subsets of \( G \): If \( \bar{A} \) is a nonempty system of subsets of \( G \), then we generally denote the system of the \( \bar{g} \)-images of the individual elements of \( \bar{A} \) by the symbol \( g\bar{A} \).

For instance, if \( \bar{A} \) is a decomposition of \( G \), then \( g\bar{A} \) denotes the system of the \( \bar{g} \)-images of the elements of \( \bar{A} \). If, in particular, \( g\bar{A} \) is a decomposition on \( G^* \), then the extended mapping \( \bar{g} \) defines the partial mapping \( gA \) of the decomposition \( \bar{A} \) onto the decomposition \( g\bar{A} \) under which there corresponds, to every element \( \bar{a} \in \bar{A} \), its image \( g\bar{a} \in g\bar{A} \).

Let \( A \) and \( B \) stand for arbitrary subsets of \( G \).

It is obvious that \( A \subset B \) yields \( gA \subset gB \).

Let us prove the following theorem:

The equality \( gA = gB \) is true if and only if every element of \( \bar{G} \), incident with one of the subsets \( A \), \( B \), is also incident with the other.

Proof. a) Suppose \( gA = gB \). If an element \( \bar{g} \in \bar{G} \) is incident with, for example, \( A \), then there exists an element \( a \in A \) such that \( \bar{g} \) is the set of all the \( g \)-inverse images of \( ga \). Since \( ga \in gA = gB \), there exists an element \( b \in B \) such that \( gb = ga \), so that \( b \in \bar{g} \) and, consequently, \( \bar{g} \) is incident with \( B \).

b) Let every element of \( \bar{G} \), incident with one of the sets \( A \), \( B \), be also incident with the other. Then, e.g., for \( a^* \in gA \), the element \( \bar{g} \in \bar{G} \) which consists of all the \( g \)-inverse images of \( a^* \) is incident with \( A \) and therefore, by the assumption, even with \( B \). Hence there exists an element \( b \in B \) such that \( a^* = gb \in gB \) and we have \( gA \subset gB \). At the same time there holds, of course, the relation \( gB \subset gA \) and we have \( gA = gB \).

The above theorem can, naturally, also be expressed by saying that the equality \( gA = gB \) applies if and only if \( A \subset \bar{G} = B \subset \bar{G} \).

Let \( \bar{A} \) stand for a system of subsets of \( G \).

If all the elements of \( \bar{A} \) have, under the extended mapping \( g \), the same image \( A^* \subset G^* \) so that, for \( \bar{A} \in \bar{A} \), there holds \( gA \subset A^* \), then even the set \( s\bar{A} \) is mapped onto \( A^* \), i.e., \( g(s\bar{A}) = A^* \).
Indeed, first of all, for every element $A \in \tilde{A}$ there holds $A \subset s\tilde{A}$ whence $A^* = gA \subset g(s\tilde{A})$. Moreover, every element $a \in s\tilde{A}$ lies in a certain subset $A \in \tilde{A}$ and we have: $ga \in gA = A^*$ which yields $g(s\tilde{A}) \subset A^*$ and the proof is accomplished.

7.2. Theorems on mappings of decompositions

Let $\tilde{A}$ denote a decomposition on $G$.

The system $g\tilde{A}$ of the subsets of $G^*$ evidently covers the set $G^*$. But this system is not necessarily a decomposition of the set $G^*$ because the $g$-images of two different elements of $\tilde{A}$ may be incident without coinciding.

The following theorem states a necessary and sufficient condition under which the decomposition $\tilde{A}$ is mapped, under $g$, onto a decomposition of $G^*$.

$g\tilde{A}$ is a decomposition of the set $G^*$ if and only if the decompositions $\tilde{A}$, $G$ are complementary.

Proof. a) Suppose $g\tilde{A}$ is a decomposition on $G^*$. Let the elements $\tilde{a} \in \tilde{A}$, $\tilde{g} \in \tilde{G}$ lie in the same element $\tilde{a} \in [\tilde{A}, \tilde{G}]$. We are to show that $\tilde{a} \cap \tilde{g} \neq \emptyset$. Let $\tilde{b} \in \tilde{A}$ stand for an arbitrary element incident with $\tilde{g}$. Then $\tilde{b} \subset \tilde{a}$, hence there exists a binding $\{\tilde{A}, \tilde{B}\}$ from $\tilde{a}$ to $\tilde{b}$:

$$(\tilde{a} =) \quad \tilde{a}_1, \ldots, \tilde{a}_a \quad (= \tilde{b}).$$

By the definition of a binding, every two of its neighbouring elements $\tilde{a}_\beta, \tilde{a}_{\beta+1}$ ($\beta = 1, \ldots, a - 1$) are incident with an element of the decomposition $\tilde{G}$ and thus both images $ga_\beta, ga_{\beta+1}$ are incident. Since $g\tilde{A}$ is a decomposition on $G^*$, we have $ga_\beta = ga_{\beta+1}$ and thus even $ga = gb$. Consequently, $\tilde{a} \subset \tilde{G} = \tilde{b} \subset \tilde{G}$. As $\tilde{g} \in \tilde{b} \subset \tilde{G}$, we have $\tilde{g} \in \tilde{a} \subset \tilde{G}$ so that $\tilde{a} \cap \tilde{g} \neq \emptyset$.

b) Let the decompositions $\tilde{A}$, $\tilde{G}$ be complementary. Our object now is to show that, for $\tilde{a}, \tilde{b} \in \tilde{A}$, the sets $ga, gb$ either are disjoint or coincide. If the sets $ga, gb$ are not disjoint, then there exist points $a \in \tilde{a}, b \in \tilde{b}$ such that $ga = gb \in g\tilde{a} \cap g\tilde{b}$. Then the element $\tilde{g} \in \tilde{G}$, consisting of all the $g$-inverse images of the element $ga$, is incident with both the elements $\tilde{a}, \tilde{b}$ and the latter therefore lie in the same element of the decomposition $[\tilde{A}, \tilde{G}]$. Since the decompositions $\tilde{A}$, $\tilde{G}$ are complementary, there holds $\tilde{a} \subset \tilde{G} = \tilde{b} \subset \tilde{G}$ which yields $g\tilde{a} = g\tilde{b}$.

Let again $\tilde{A}$, $\tilde{G}$ be complementary.

By the above theorem, $g\tilde{A}$ is a decomposition on $G$. The extended mapping $g$ determines the partial mapping of the decomposition $\tilde{A}$ onto $g\tilde{A}$ under which there corresponds, of course, to every element $\tilde{a} \in \tilde{A}$, its image $g\tilde{a} \in g\tilde{A}$. By the mapping $g$ of the decomposition $\tilde{A}$ onto $g\tilde{A}$ we shall, in what follows, understand this partial mapping.
To the mapping \( g \) of \( \bar{A} \) onto \( gA \) there naturally corresponds a certain decomposition \( \bar{A} \) of \( \bar{A} \). Its elements consist of all the elements of \( \bar{A} \) that have, under the extended mapping \( g \), the same image.

We shall show that the covering of the decomposition \( \bar{A} \) enforced by \( \bar{A} \) is the least common covering \([\bar{A}, \bar{G}]\) of the decompositions \( \bar{A}, \bar{G} \).

Indeed, consider an arbitrary element \( \bar{a} \in \bar{A} \). We are to show that the set \( sa \) is an element of the decomposition \([\bar{A}, \bar{G}]\). Let \( \bar{a} \in \bar{A} \) be an arbitrary element and \( \bar{u} \in [\bar{A}, \bar{G}] \) the element of \([\bar{A}, \bar{G}]\), containing \( \bar{a} \); consequently, we have \( \bar{a} \subset sa \cap \bar{u} \).

Every element \( \bar{x} \in \bar{a} \) has, under the extended mapping \( g \), the same image as \( \bar{a} \), hence \( \bar{a} \subset \bar{G} = \bar{x} \subset \bar{G} \); it follows that the element \( \bar{x} \) may be connected with the element \( \bar{a} \) in the decomposition \( \bar{G} \) and therefore lies in the element \( \bar{u} \). Thus we have verified that \( sa \subset \bar{u} \). Conversely, for any element \( \bar{x} \in \bar{A} \) lying in \( \bar{u} \) there holds \( \bar{a} \subset \bar{G} = \bar{x} \subset \bar{G} \); consequently, the element \( \bar{x} \) has, under the extended mapping \( g \), the same image as \( \bar{a} \), thus \( \bar{x} \subset sa \) and we have \( \bar{x} \subset sa \). Hence \( \bar{u} \subset sa \) and the proof is accomplished.

Associating, with every element \( \bar{u} \in [\bar{A}, \bar{G}] \), the element \( \bar{a} \in \bar{A} \) which contains all the elements of \( \bar{A} \) lying in \( \bar{u} \), we obtain a simple mapping of the decomposition \([\bar{A}, \bar{G}]\) onto \( \bar{A} \); associating, with every element \( \bar{a} \in \bar{A} \), the element \( \bar{a}^* \in gA \) which is the image of every element \( \bar{a} \in \bar{A} \) lying in \( \bar{u} \), we obtain a simple mapping of the decomposition \( \bar{A} \) onto \( g\bar{A} \). Composing these simple mappings, we get a simple mapping of the decomposition \([\bar{A}, \bar{G}]\) onto \( g\bar{A} \). Under this mapping there corresponds, to every element \( \bar{u} \in [\bar{A}, \bar{G}] \), a certain element \( \bar{a}^* \in g\bar{A} \); the element \( \bar{a}^* \) is the image, under the extended mapping \( g \), of every element of \( \bar{A} \) lying in the element \( \bar{a} \in \bar{A} \) which contains all the elements of \( \bar{A} \) lying in \( \bar{u} \). Since \( \bar{u} = sa \) and for \( \bar{a} \in \bar{a} \) we have \( g\bar{a} = \bar{a}^* \), we conclude, with respect to the last theorem in 7.1, that the element \( \bar{a} \) has, under the extended mapping \( g \), the image \( \bar{a}^* \), i.e., \( g\bar{a} = \bar{a}^* \).

Thus we have the following result:

If a decomposition \( \bar{A} \) on \( G \) is mapped, under \( g \), onto some decomposition \( \bar{A}^* \) on \( G \), then the decompositions \([\bar{A}, \bar{G}]\) and \( \bar{A}^* \) are equivalent, i.e., \([\bar{A}, \bar{G}] \simeq \bar{A}^* \); a simple mapping of the decomposition \([\bar{A}, \bar{G}]\) onto \( \bar{A}^* \) is obtained by associating, with each element of \([\bar{A}, \bar{G}]\), its image under the extended mapping \( g \).

Consequently, every covering of the decomposition \( \bar{G} \) is equivalent to its image under \( g \); the mapping under which every element of the covering is associated with its own image is simple.

### 7.3. Exercises

1. Let \( g \) be a mapping of the set \( G \) onto \( G^* \) and \( A, B \) stand for arbitrary subsets of \( G \). Show that the following relations are true: \( g(A \cup B) = gA \cup gB \); \( g(A \cap B) \subseteq gA \cap gB \).
2. Assuming the situation described in exercise 1., let \( G \) be the decomposition on \( G \) corresponding to the mapping \( g \). Show that the equality \( g(A \cap B) = gA \cap gB \) applies if and only if there holds \((A \cap B) \cap \overline{G} = (A \cap \overline{G}) \cap (B \cap \overline{G})\).

3. Let \( g \) be a mapping of the set \( G \) onto \( G^* \) and \( \{a, b, \ldots\} \) stand for a decomposition on \( G \). Then \( \{g\alpha, g\beta, \ldots\} \) is a decomposition on \( G^* \) if and only if \( \{a, b, \ldots\} \) is a covering of the decomposition corresponding to \( g \).

4. Suppose \( g \) is a simple mapping of the set \( G \) onto \( G^* \). Let, moreover, \( A \subset G \) be a non-empty subset and \( \overline{A}, \overline{B} \) stand for decompositions in (on) \( G \). In this situation there holds:
   a) the extended mapping \( \overline{g} \) of the system of all the nonempty parts of \( G \) onto the system of all the nonempty parts of \( G^* \) is simple;
   b) the sets \( A, gA \) are equivalent, i.e., \( A \simeq gA \);
   c) \( g\overline{A} \) is a decomposition in (on) the set \( G^* \);
   d) the decompositions \( \overline{A}, g\overline{A} \) are equivalent, i.e., \( \overline{A} \simeq g\overline{A} \);
   e) if the decompositions \( A, B \) are equivalent or loosely coupled or coupled, then the decompositions \( gA, gB \) have, in each case, the same property.

8. **Permutations**

In this chapter we shall deal with simple mappings of finite sets onto themselves; they play an important role in algebra, particularly, in the theory of groups.

8.1. **Definition**

By a *permutation of the set* \( G \) we mean a simple mapping of the set \( G \) onto itself (6.6).

In this section we shall restrict our considerations to permutations of *finite* sets. Let \( G \) denote an arbitrary set consisting of a finite number \( n(\geq 1) \) of elements. From the assumption that \( G \) is finite it follows that every simple mapping \( p \) of the set \( G \) into itself is a permutation of \( G \) (6.10.2).

Let the elements of \( G \) be denoted by the letters \( a, b, \ldots, m \). Then we can uniquely associate, with every permutation \( p \) of the set \( G \), a symbol of the form:

\[
\begin{pmatrix}
  a & b & \ldots & m \\
  a^* & b^* & \ldots & m^*
\end{pmatrix}
\]

where \( a^*, b^*, \ldots, m^* \) are the letters denoting the elements \( pa, pb, \ldots, pm \). Since \( pG = G \), the letters \( a^*, b^*, \ldots, m^* \) are again \( a, b, \ldots, m \) written in a certain order.