9. General (multiple-valued) mappings

9.1. Basic notions and properties

The notion of a mapping of the set G into the set G^* may be generalized by the following definition:

A *general (multiple-valued) mapping of G into G^** is a relation between the elements of both sets by which there corresponds, to every element of G, at least one element of G^*.

Let $\boldsymbol{g}$ be a general mapping of G into G^*. Then each element $a \in G$ has at least one, generally more, maybe even an infinite number of images in G^*; the set of these images is denoted by $\boldsymbol{g}a$.

If every element $a^* \in G^*$ is contained in the set of the images of some element $a \in G$, then $\boldsymbol{g}$ is said to be a general mapping of G *onto* G^*.

In that case $\boldsymbol{g}$ determines a certain general mapping of G^* onto G, called the *inverse of* $\boldsymbol{g}$ and denoted by $\boldsymbol{g}^{-1}$. It is defined in the way that to any element $a^* \in G^*$ there corresponds every element $a \in G$ whose set of images under $\boldsymbol{g}$ contains a^*. By this definition both $a^* \in \boldsymbol{g}a$ and $a \in \boldsymbol{g}^{-1}a^*$ are simultaneously valid, that is to say, if $a^* \in \boldsymbol{g}a$ then $a \in \boldsymbol{g}^{-1}a^*$, and conversely.

It is easy to show that *the mapping inverse of $\boldsymbol{g}^{-1}$ is the original mapping $\boldsymbol{g}$*, i.e., $(\boldsymbol{g}^{-1})^{-1} = \boldsymbol{g}$. Indeed, the relation $a^* \in \boldsymbol{g}a$ yields $a \in \boldsymbol{g}^{-1}a^*$ whence $a^* \in (\boldsymbol{g}^{-1})^{-1}a$, so that $\boldsymbol{g}a \subset (\boldsymbol{g}^{-1})^{-1}a$; vice versa, from $a^* \in (\boldsymbol{g}^{-1})^{-1}a$ follows $a \in \boldsymbol{g}^{-1}a^*$ whence $a^* \in \boldsymbol{g}a$ and so $(\boldsymbol{g}^{-1})^{-1}a \subset \boldsymbol{g}a$. So we have $(\boldsymbol{g}^{-1})^{-1}a = \boldsymbol{g}a$ and the proof is complete.

On taking account of the above property of the inverse mapping, we call both $\boldsymbol{g}$ and $\boldsymbol{g}^{-1}$ inverse without discerning which is inverse of which.

If $\boldsymbol{g}$ is single-valued, then $\boldsymbol{g}^{-1}$ is, as a rule, general. In that case the set of the images $\boldsymbol{g}^{-1}a^* \subset G$ of an element $a^* \in \boldsymbol{g}G$ consists of all the points of G that are, under the function $\boldsymbol{g}$, mapped onto a^*; $\boldsymbol{g}^{-1}a^*$ is therefore an element of the decomposition in G corresponding to $\boldsymbol{g}$.

The concept of a general mapping may serve as the basis of an extensive theory which, naturally, also comprises results concerning the single-valued mappings considered above. From this theory we shall now introduce a few details about the composition of general mappings.

The notion of a composite mapping which we have, in 6.7, introduced for single-valued mappings may be directly extended to general mappings. Let G, H, K denote nonempty sets, $\boldsymbol{g}$ a general mapping of G into H and $\boldsymbol{h}$ a general mapping of H into K. Then the mapping $\boldsymbol{hg}$, composed of the functions $\boldsymbol{g}$ and $\boldsymbol{h}$ (in this order), is defined by associating, with every element $a \in G$, all the $\boldsymbol{h}$-images of the individual elements lying in $\boldsymbol{g}a$. Obviously, there holds $\boldsymbol{hg}a \subset K$. In particular, the set of the images $\boldsymbol{g}^{-1}\boldsymbol{g}a$ of an element $a \in G$ consists of elements $x \in G$ such that $\boldsymbol{g}x \cap \boldsymbol{g}a \neq 0$.

Finally, let us remark that the notion of a general mapping of G into G^* may be extended by assuming that there exist elements of G with which no element of G^* is associated. Such a general mapping is called a *relation of the set G into the set G^*.*

In the following study of general mappings we shall restrict our attention to the case when G and G^* coincide and we have to deal with general mappings of G *onto* itself.

9.2. Congruences

A general mapping $\boldsymbol{g}$ of the set G onto itself is called a *congruence on G* if it has the following properties:

a) For $a \in G$ there holds $a \in \boldsymbol{g}a$;

b) if for $a, b, c \in G$ there holds $b \in \boldsymbol{g}a$, $c \in \boldsymbol{g}b$, then $c \in \boldsymbol{g}a$.

These properties are expressed by saying that $\boldsymbol{g}$ is *reflexive* or *transitive*, respectively.

A congruence on a set is therefore a general mapping of a set onto itself which is both reflexive and transitive.

Suppose $\boldsymbol{g}$ is a congruence.

The relation $b \in \boldsymbol{g}a$ is expressed by saying that b is congruent to a under $\boldsymbol{g}$.

It is easy to realize that *the inverse general mapping $\boldsymbol{g}^{-1}$ is also a congruence.* In fact, the mapping $\boldsymbol{g}^{-1}$ is obviously reflexive. Moreover, from $b \in \boldsymbol{g}^{-1}a$, $c \in \boldsymbol{g}^{-1}b$ there follows $a \in \boldsymbol{g}b$, $b \in \boldsymbol{g}c$, hence $a \in \boldsymbol{g}c$ and we have $c \in \boldsymbol{g}^{-1}a$, so that $\boldsymbol{g}^{-1}$ is even transitive.

The congruence $\boldsymbol{g}^{-1}$ is, of course, called *the inverse of $\boldsymbol{g}$*. The congruence inverse of $\boldsymbol{g}^{-1}$ is $\boldsymbol{g}$.

If we have, for example, arbitrary decompositions $\bar{A}$, $\bar{B}$ on G and associate, with every element $\bar{a} \in \bar{A}$, all the elements of $\bar{A}$ that may be connected with $\bar{a}$ in $\bar{B}$, then we have a congruence $\boldsymbol{g}$ on $\bar{A}$ (see 3.1 a, b). In this case, every element $\bar{b} \in \bar{A}$ that may be connected with $\bar{a}$ in $\bar{B}$ is congruent to $\bar{a}$. In the inverse congruence $\boldsymbol{g}^{-1}$ there correspond, to every element $\bar{a} \in \bar{A}$, all the elements of $\bar{A}$ with which $\bar{a}$ may be connected in $\bar{B}$; the latter are, according to 3.1 c, precisely those elements that can be connected with $\bar{a}$ in $\bar{B}$. Consequently, in this particular case, the inverse congruence $\boldsymbol{g}^{-1}$ is identical with $\boldsymbol{g}$, i.e., $\boldsymbol{g}^{-1} = \boldsymbol{g}$.

Some other examples of congruences: Let us associate, with every decomposition $\bar{A}$ of the set G, all its coverings (refinements). In both cases we obtain a congruence on the set of all decompositions of G (see 2.4 a, b). Every decomposition of G which is a covering (refinement) of $\bar{A}$ is congruent to $\bar{A}$. Either of the congruences is the inverse of the other.

Of particular importance are the symmetric and antisymmetric congruences.

9.3. Symmetric congruences

A congruence $\boldsymbol{g}$ on the set G is called *symmetric* if:

From $b \in \boldsymbol{g}a$ there follows $a \in \boldsymbol{g}b$. (S)

This property expresses the symmetry of the congruence $\boldsymbol{g}$: Of every two elements in G either not one is or both are contained in the set of images of the other. If $b \in \boldsymbol{g}a$, then we write $b \equiv a$ $(\boldsymbol{g})$, briefly $b \equiv a$. Then, of course, we also have $a \equiv b$ and say that *the elements a, b are congruent.*

For instance, the congruence on the dècomposition $\bar{A}$, given in 9.2, first example, is symmetric according to 3.1c.

Let $\boldsymbol{g}$ be an arbitrary symmetric congruence on G.

The congruence $\boldsymbol{g}$ has the remarkable property that *the system of all subsets of G, each of which consists of all the elements congruent to some element of G, is a decomposition of G.* The latter is said to *belong* or to *correspond* to $\boldsymbol{g}$; its elements are called *classes of the congruence $\boldsymbol{g}$.*

The proof of this statement is an easy generalization of the proof in 3.4 where we have shown that the system $\bar{\bar{A}}$ of subsets of $\bar{A}$ is a decomposition on $\bar{A}$; we leave it to the reader to carry it out himself.

It is also easy to see that *any two points of G lying in the same class of $\boldsymbol{g}$ are congruent, whereas any two that do not lie in the same class are not.* A subset of G which has exactly one point of G in common with each element of the decomposition corresponding to $\boldsymbol{g}$ forms a *system of representatives of the congruence $\boldsymbol{g}$* in the sense that every element of G is congruent to exactly one of the representatives.

Vice versa, *if we have, on the set G, an arbitrary decomposition $\bar{A}$, then there exists a congruence on G such that the corresponding decomposition is $\bar{A}$.* This congruence is defined in the way that each point $a \in G$ is congruent to any point lying in the same element of $\bar{A}$ as a, whereas the other points of G are not congruent to a.

Between the study of symmetric congruences and the study of decompositions of sets there is no essential difference.

Finally, we shall show that *the inverse congruence $\boldsymbol{g}^{-1}$ is equal to $\boldsymbol{g}$*, i.e., $\boldsymbol{g}^{-1} = \boldsymbol{g}$. Indeed, $b \in \boldsymbol{g}^{-1}a$ yields $a \in \boldsymbol{g}b$ and hence, on taking account of (S), there holds $b \in \boldsymbol{g}a$ and we have $\boldsymbol{g}^{-1}a \subset \boldsymbol{g}a$; conversely: by (S), $b \in \boldsymbol{g}a$ yields $a \in \boldsymbol{g}b$ and therefore also $b \in \boldsymbol{g}^{-1}a$ so that $\boldsymbol{g}a \subset \boldsymbol{g}^{-1}a$. Hence $\boldsymbol{g}^{-1}a = \boldsymbol{g}a$, which was to be proved.

Note that every congruence on G that coincides with its own inverse is symmetric.

Symmetric congruences are also called *equivalences.*

9.4. Antisymmetric congruences

1. *Basic concepts and properties.* A congruence $\boldsymbol{g}$ on the set G is called *antisymmetric* if:

$$\text{From } b \in \boldsymbol{g}a,\ a \in \boldsymbol{g}b \text{ there follows } a = b. \tag{AS}$$

This property expresses the antisymmetry of $\boldsymbol{g}$: Of any two different elements of G either not one or exactly one is congruent to the other. If b is congruent to a, i.e., if $b \in \boldsymbol{g}a$, we write $a \leqq b$ $(\boldsymbol{g})$ or $b \geqq a$ $(\boldsymbol{g})$, briefly $a \leqq b$ or $b \geqq a$.

If the congruence $\boldsymbol{g}$ is antisymmetric, then its inverse, $\boldsymbol{g}^{-1}$ is antisymmetric as well, for $b \in \boldsymbol{g}^{-1}a$, $a \in \boldsymbol{g}^{-1}b$ yield $a \in \boldsymbol{g}b$, $b \in \boldsymbol{g}a$ and, consequently, by (AS), there holds $a = b$.

For example, by 2.4c, both congruences on the system of all decompositions on G considered in 9.2 are antisymmetric; as we have already said, either of them is the inverse of the other.

Antisymmetric congruences are also called *partial orderings*; partial orderings which are inverse of each other are called *dual.*

2. *The least upper bound (join) and the greatest lower bound (meet) of two elements.* Remarkable notions based on the concept of antisymmetric congruence are those of the least upper bound and the greatest lower bound of two elements.

Let there be given, on G, an antisymmetric congruence $\boldsymbol{g}$.

The *least upper bound* or the *join of a two-membered sequence of elements $a, b \in G$ with regard to $\boldsymbol{g}$*, briefly, the *least upper bound* or the *join of a, b* is the element $c \in G$ such that $a \leqq c$, $b \leqq c$ and, furthermore, $c \leqq x$ for every $x \in G$ satisfying $a \leqq x$, $b \leqq x$. Any two-membered sequence of elements may have at most one join because, if c, c' are joins, we have $c \leqq c'$ and, simultaneously, $c' \leqq c$ so that, with respect to (AS), there holds $c = c'$. The join of the elements a, b need not exist at all; if it does, it is denoted by $a \cup b$.

Analogously we define the *greatest lower bound* or the *meet of a two-membered sequence of elements $a, b \in G$ with regard to $\boldsymbol{g}$*, briefly, the *greatest lower bound* or the *meet of $a, b \in G$*; it is the element $c \in G$ such that $c \leqq a$, $c \leqq b$ and, furthermore $x \leqq c$ for every $x \in G$ satisfying $x \leqq a$, $x \leqq b$. There may exist at most one meet b; if it does, it is denoted by $a \cap b$.

Comparing the definitions of the join and the meet we observe that *the join (meet) of a, b with regard to $\boldsymbol{g}$, if it exists, is the meet (join) of a, b with regard to $\boldsymbol{g}^{-1}$.*

We leave it to the reader to verify that, *for every three elements $a, b, c \in G$, the following formulae are true whenever the included joins and meets exist*:

a) $a \cup b = b \cup a$, a′) $a \cap b = b \cap a$,

b) $a \cup a = a$, b′) $a \cap a = a$,

c) $a \cup (b \cup c) = (a \cup b) \cup c$, c′) $a \cap (b \cap c) = (a \cap b) \cap c$,

d) $a \cup (a \cap b) = a$, d′) $a \cap (a \cup b) = a$.

Since there hold a) and a′) we generally speak about the join and the meet of two elements without drawing any distinction as to their arrangement.

To give an example of a join and a meet, let us note the antisymmetric congruence on the system of all decompositions of G under which there correspond, to each decomposition of G, all its coverings or refinements. Every two decompositions $\bar{A}$, $\bar{B}$ of G have the join $[\bar{A}, \bar{B}]$ or $(\bar{A}, \bar{B})$ and the meet $(\bar{A}, \bar{B})$ or $[\bar{A}, \bar{B}]$.

9.5. Exercises

1. Let the set G be mapped, under the single-valued functions $\boldsymbol{a}, \boldsymbol{b}$, onto the set A or B, respectively, and let its decompositions corresponding to these mappings be equal. Show that, in that case, $\boldsymbol{f} = \boldsymbol{b}\boldsymbol{a}^{-1}$ is a single-valued and simple mapping of A onto B and $\boldsymbol{f}^{-1} = \boldsymbol{a}\boldsymbol{b}^{-1}$ the inverse mapping of B onto A. Hence, in this case the sets A, B are equivalent.
2. Let n denote an arbitrary positive integer. Associating, with every integer a, each number $a + \nu n$ where $\nu = \ldots, -2, -1, 0, 1, 2, \ldots$, we obtain a symmetric congruence on the set of all integers. The corresponding decomposition consists of n classes; the numbers $0, 1, \ldots, n-1$ form a system of representatives of the congruence.
3. Associating, with every positive integer, each of its positive multiples (each of its positive divisors), we obtain an antisymmetric congruence on the set of all positive integers. Every two positive integers have, with regard to this congruence, a join formed by their least common multiple (greatest common divisor) and a meet formed by their greatest common divisor (least common multiple). Either of the congruences is the inverse of the other.
4. Associating, with every part of G, each of its supersets (subsets), we obtain an antisymmetric congruence on the set of all parts of G. Every two parts of G have, with regard to this congruence, a join formed by their sum (intersection) and a meet formed by their intersection (sum). Either of the congruences is the inverse of the other.
5. If $\boldsymbol{g}$ is an antisymmetric congruence on G and some elements $a, b \in G$ have the join $a \cup b$, then:

   a) $\boldsymbol{g}(a \cup b) = \boldsymbol{g}a \cap \boldsymbol{g}b$ (the right-hand side denotes, of course, the intersection of $\boldsymbol{g}a, \boldsymbol{g}b$),
   b) $\boldsymbol{g}^{-1}(a \cup b) \supset \boldsymbol{g}^{-1}a \cup \boldsymbol{g}^{-1}b$.

10. Series of decompositions of sets

In this chapter we shall develop a theory of the so-called series of decompositions of sets. We shall make use of many results arrived at in the previous considerations and concerning decompositions and mappings of sets. The mentioned theory describes the set-structure of the appropriate sections of the theory of groupoids