27. Cyclic groups


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27. Cyclic groups

27.1. Definition

A group \( G \) is called cyclic if it contains an element \( a \), called generator of \( G \), such that each element of \( G \) is a power of \( a \). If \( G \) is a cyclic group and \( a \) its generator, then \( G \) is denoted by the symbol \( \langle a \rangle \). From the first formula (1) in 19.3 it follows that every cyclic group is Abelian.

27.2. The order of a cyclic group

Consider a cyclic group \( \langle a \rangle \). If the powers \( a^i, a^j \) of \( a \) with any two different exponents \( i, j \) are different, then the group \( \langle a \rangle \) has the order \( 0 \) because it contains an infinite number of elements

\[
..., a^{-2}, a^{-1}, a^0, a^1, a^2, ...
\]

As each element of \( \langle a \rangle \) is a power of \( a \), the group \( \langle a \rangle \) does not include any other elements but these so that \( \langle a \rangle \) consists of the elements (1). Now suppose that the powers of \( a \) with some different exponents \( i, j \) are equal and so \( a^i = a^j \), \( i \neq j \). Hence \( a^{-i} . a^i = a^{-j} . a^j \), i.e., \( a^{i-j} = 1 \). Since one of the numbers \( i - j, j - i \) is positive and the powers of \( a \) with these exponents equal \( 1 \), we observe that there exist positive integers \( x \) satisfying the equation \( a^x = 1 \). One of them is the least; let us denote it \( n \), thus \( a^n = 1 \). Now consider the following elements of \( \langle a \rangle \):

\[
1, a, a^2, ..., a^{n-1}
\]

First, it is easy to verify that every two of them are different: in fact, if for any of them there holds \( a^i = a^j \), then one of the numbers \( i - j, j - i \) is a positive integer smaller than \( n \) and satisfies the equation \( a^x = 1 \); but that contradicts the definition of \( n \). Consequently, the group \( \langle a \rangle \) comprises at least \( n \) elements (2) and has therefore the order \( 0 \) or \( \geq n \). Moreover, it is easy to show that \( \langle a \rangle \) does not include any other elements, hence its order is \( n \). To that purpose, consider an element \( a^x \) of \( \langle a \rangle \). Dividing \( x \) by \( n \), we obtain a quotient \( q \) and a remainder \( r \) whence \( x = qn + r, 0 \leq r \leq n - 1 \); consequently, \( a^r \) is one of the elements (2). The formulae (1) in 19.3 yield

\[
a^x = a^{qn+r} = a^{qn} . a^r = (a^n)^q . a^r = 1^q . a^r = 1 . a^r = a^r
\]

and we have \( a^z = a^r \). Thus we have verified that the group \( \langle a \rangle \) consists of the elements (2) and therefore has the order \( n \). Furthermore, the product \( a^k . a^l \) of an element \( a^k \) and an element \( a^l \) of \( \langle a \rangle \) is the element \( a^k, k \) being the remainder of
the division of $i + j$ by $n$ because $a^i \cdot a^j = a^{i+j}$. To sum up, we arrive at the following theorem:

The order $n$ of every cyclic group $(a)$ is either 0, in which case $(a)$ consists of the elements (1), or $n > 0$, and then $(a)$ consists of the elements (2). The product $a^i \cdot a^j$ of the elements $a^i$ and $a^j$ of $(a)$ is, in the first case, the element $a^{i+j}$ whereas, in the second case, it is $a^k$, $k$ being the remainder of the division of $i + j$ by $n$. In the latter, $n$ is the least positive integer such that $a^n = 1$.

Note that in both cases $a^{n-1}$ is the inverse of $a^i$.

27.3. Subgroups of cyclic groups

Let us now consider a subgroup $H$ of a cyclic group $(a)$. If $H$ consists of a single element 1, then it is cyclic and its generator is 1. Suppose that $H$ contains besides 1 an element $a^i$ where $i \neq 0$. As $H$ comprises with $a^i$ simultaneously its inverse $a^{-i}$ and as one of the numbers $i$, $-i$ is positive, we see that $H$ includes powers of $a$ with positive exponents. One of the latter is the least; let us denote it $m$, hence $a^m \in H$. $H$ does not contain any powers of $a$ with positive exponents smaller than $m$. Let $a^x$ be an arbitrary element of $H$. Dividing $x$ by $m$, we obtain a quotient $q$ and a remainder $r$, hence $x = qm + r$, $0 \leq r \leq m - 1$. In accordance with the formulæ (1) in 19.3, there follows: $a^x = a^{qm+r} = a^q a^r$. Consequently, $a^r$ is the product of $a^{-qm}$ and $a^x$. Since $a^{-qm}$ is the inverse of the element $(a^m)^q$ which is, as the $q$th power of the element $a^m \in H$, also included in $H$, we see that $a^{-qm}$ is an element of $H$. As even $a^x$ is an element of $H$, the product $q^{-qm} a^x$, namely, the element $a^r$ is included in $H$. Consequently, with regard to the inequalities $0 \leq r \leq m - 1$ and to the definition of $m$, there follows $r = 0$. So we have $a^x = (a^m)^q$. Every element of $H$ is therefore a power of $a^m$, hence $H$ is cyclic with the generator $a^m$. Thus we have arrived at the result that every subgroup of a cyclic group $(a)$ is cyclic.

Since the cyclic group $(a)$ is Abelian, each of its subgroups is invariant in $(a)$.

27.4. Generators

Do there exist, in the cyclic group $(a)$, any other generators besides $a$? Let, again, $n$ denote the order of $(a)$ and suppose that some element $a^r$ of $(a)$ is a generator of $(a)$. Then, in particular, the element $a$ is a power of $a^r$, hence $a = a^q$, $q$ being an integer. If $n = 0$, then $a = a^q$ yields $vq = 1$ because, in that case, any two powers of $a$ with different exponents are different; hence $v = q = 1$ or $v = q = -1$. Consequently, besides $a$, only $a^{-1}$ can be a generator of $(a)$ and, in fact, each element $a^i$ of $(a)$ is the $-i$th power of $a^{-1}$.

If $n = 0$, then the group $(a)$ has exactly two generators: $a$, $a^{-1}$. Note that they are the only two elements of $(a)$ whose exponents are relatively prime to $n$ (= 0).
Let us now consider the case when \( n > 0 \). The cyclic group \((a)\) consists of the elements 1, \( a, a^2, \ldots, a^{n-1} \). If \( r \) is the remainder of the division of \( vq \) by \( n \) so that \( vq = nq' + r \) where \( q' \) is the quotient and \( 0 \leq r \leq n - 1 \), then we have \( a^{vq} = a^{vq'} = a^r \). Consequently, \( r = 1 \) because \( a, a^r \) belong to the sequence 1, \( a, a^2, \ldots, a^{n-1} \) where any two elements with different exponents are different. So we have \( vq - nq' = 1 \) and therefore \( v, n \) are prime to each other. If, conversely, \( v \) is an integer relatively prime to \( n \), then there exist integers \( q, q' \) such that \( vq - nq' = 1 \) and there follows, for every integer \( i \), the relation \( i = v(qi) - n(q'i) \). Consequently, we have \( a^i = (a^r)^i \) and so \( a^r \) is a generator of the group \((a)\). If \( n > 0 \), then the generators of \((a)\) are the powers of \( a \) whose exponents are relatively prime to \( n \).

We saw that the same applies even if \( n = 0 \) and can therefore sum up the above results in the following theorem:

"The generators of the cyclic group \((a)\) of order \( n \geq 0 \) are exactly the powers of \( a \) with exponents relatively prime to \( n \)."

If \( n = 0 \), then \( (a) \) has precisely two generators whereas, if \( n > 0 \), then the number of the generators equals the number of the positive integers not greater than \( n \) and relatively prime to it.

27.5. Determination of all cyclic groups

1. An important example of a cyclic group of order 0 is the group \( \mathbb{Z} \). Evidently, \( \mathbb{Z} = (1) \). All subgroups of \( \mathbb{Z} \) consist, as we know, of all multiples of a non-negative integer \( n \), hence they are cyclic groups \((n)\). Let \( n \geq 0 \) and consider the factor group \( \mathbb{Z}/(n) \). We know that, for \( n = 0 \), \( \mathbb{Z}/(n) \) consists of the sets \( \bar{i} = \{i\} \) where \( i = \ldots, -2, -1, 0, 1, 2, \ldots \) and, for \( n > 0 \), it consists of the elements \( \bar{a}_0, \ldots, \bar{a}_{n-1} \) where \( \bar{a}_j \) denotes the set of all the elements of \( \mathbb{Z} \) that differ from \( j \) only by a multiple of \( n \); the factor group \( \mathbb{Z}/(n) \) has, in both cases, the order \( n \). It is easy to show that the factor group \( \mathbb{Z}/(n) \) is cyclic with the generator \( \bar{a}_1 \). In fact, by the definition of the multiplication in \( \mathbb{Z}/(n) \), any \( i \)th power of an element \( \bar{a}_k \in \mathbb{Z}/(n) \) is that element of \( \mathbb{Z}/(n) \) which contains the number \( ik \); hence, in particular, \( \bar{a}_j = \bar{a}_1^i \), which proves the above assertion. Thus we have simultaneously verified that there exist cyclic groups of an arbitrary order \( n \geq 0 \).

Now we shall show that, conversely, every cyclic group is isomorphic with a factor group of \( \mathbb{Z} \). Consider a cyclic group \((a)\). To each element \( x \in (a) \) there exists at least one integer \( \xi \) such that \( a^\xi = x \) and, of course, vice versa, for every integer \( \xi \), \( a^\xi \) is an element of \( (a) \). Associating with each element \( \xi \in \mathbb{Z} \) the element \( a^\xi \in (a) \), we obtain a mapping \( d \) of \( \mathbb{Z} \) onto \( (a) \). If \( \xi \) and \( \eta \) are arbitrary elements of \( \mathbb{Z} \) and \( d\xi = x, d\eta = y \), then we have \( x = a^\xi, y = a^\eta \) and therefore \( xy = a^\xi a^\eta = a^{\xi + \eta} \), hence \( d(\xi + \eta) = xy = d\xi d\eta \). Consequently, the mapping \( d \) preserves the multiplications in both groups \( \mathbb{Z}, (a) \) and therefore is a homomorphism. We
observe, first, that \((a)\) is homomorphic with \(\mathbb{Z}\). By the first isomorphism theorem for groups (26.3.1), the set of all \(d\)-inverse images of the unit of \((a)\) is an invariant subgroup \(\mathfrak{A}\) of \(\mathbb{Z}\) and the factor group on \(\mathfrak{A}\), generated by \(\mathfrak{A}\), is isomorphic with \((a)\), i.e., \(\mathbb{Z}/\mathfrak{A} \simeq (a)\). Let \(n (\geq 0)\) be the order of the cyclic group \((a)\). Then even \(\mathbb{Z}/\mathfrak{A}\) has the order \(n\) and so \(\mathfrak{A}\) consists of all multiples of \(n\). Consequently, the cyclic group \((a)\) of order \(n\) is isomorphic with the factor group \(\mathbb{Z}/(n)\) generated by the subgroup \((n)\) of \(\mathbb{Z}\). In particular, every cyclic group of order 0 is isomorphic with \(\mathbb{Z}/(0)\), hence even with \(\mathbb{Z}\).

It is easy to see that any group isomorphic with a cyclic group of order \(n (\geq 0)\) is also cyclic and of order \(n\).

The result:

All cyclic groups of order \(n \geq 0\) are represented by the factor group \(\mathbb{Z}/(n)\) on \(\mathbb{Z}\) in the sense that any cyclic group of order \(n\) is isomorphic with \(\mathbb{Z}/(n)\) and, conversely, any group isomorphic with \(\mathbb{Z}/(n)\) is cyclic and of order \(n\).

2. Example. As an example of a cyclic group of order \(n > 0\) we may introduce the group consisting of the \(n\)th roots of unity with multiplication in the arithmetic sense.

The roots in question are:

\[
\varepsilon_0 = 1, \quad \varepsilon_1 = e^{2\pi i/n}, \quad \varepsilon_2 = e^{4\pi i/n}, \ldots, \varepsilon_{n-1} = e^{2(n-1)\pi i/n}
\]

and therefore form the cyclic group \((e^{2\pi i/n})\). The points whose coordinates are real and imaginary parts of these roots are the vertices of a regular \(n\)-gon. For \(n = 6\), for example, we have the vertices of a regular hexagon. The generators of this group of order 6 are \(e^{2\pi i/6}, e^{10\pi i/6}\).

27.6. **Fermat's theorem for groups**

The notion of a cyclic group is important even for groups that are not necessarily cyclic. Consider a group \(\mathcal{G}\). Let \(a\) be an arbitrary element of \(\mathcal{G}\). The individual powers of \(a\) form a cyclic subgroup \((a)\) of \(\mathcal{G}\).

By the order of the element \(a\) we mean the order of the cyclic subgroup \((a)\). The order \(n\) of \(a\) is therefore either 0 or the least positive integer \(x\) for which \(a^x = 1\); in any case there holds \(a^n = 1\).

Furthermore, it is easy to verify that the order \(n\) of each element \(a \in \mathcal{G}\) is a divisor of the order \(N\) of \(\mathcal{G}\), i.e., \(N = nd, d\) integer. For \(N = 0\) this statement is obvious. In case of \(N > 0\) it is true because the order of any subgroup of \(\mathcal{G}\) is a divisor of the order of \(\mathcal{G}\). From the equality \(N = nd\) there follows: \(a^N = a^{nd} = (a^n)^d = 1^d = 1\). Thus we have arrived at Fermat's theorem for groups:

The \(N\)th power of any element of a group of order \(N\) is the unit of the group.
Let us conclude our study with a remark concerning the generating of, for example, the left translations of a finite group by pure cyclic permutations.

Assume $G$ to be a finite group and $a$ an element of $G$. As we saw in 26.2.1, the left translation $a^t$ of $G$ is a permutation of $G$ and is therefore generated by a finite number of pure cyclic permutations; that is to say, there exists a decomposition $G = \{a, \ldots, m\}$ of $G$ such that each element $a, \ldots, m$ is invariant under $a^t$ and the partial permutations $a^t_a, \ldots, a^t_m$ are pure cyclic permutations of the elements $a, \ldots, m$. Any element $x$ of $G$ consists of the elements of the cycle: $x, a^tx, (a^t)^2x, \ldots, (a^t)^{k-1}x$, with $x$ denoting an arbitrary element of $x$ and $k$ being the least positive integer such that $(a^t)^kx = x$. Taking account of the definition of the left translation $a^t$, we have

$$a^tx = ax, (a^t)^2x = a^2x, \ldots, (a^t)^{k-1}x = a^{k-1}x$$

and from $(a^t)^kx = a^kx = x$ there follows $a^k = 1$. We observe that the cycle in question is $x, ax, a^2x, \ldots, a^{k-1}x$ and, furthermore, that the set $\{1, a, a^2, \ldots, a^{k-1}\}$ is the field of the cyclic subgroup $(a)$ of $G$. The element $x$ is therefore the right coset of $x$ with respect to $(a)$. Consequently, $G$ is the right decomposition of $G$ generated by $(a)$.

To sum up:

*The cycles of pure cyclic permutations generating a left translation $a^t$ of a finite group $G$ consist of the same elements as the right cosets with regard to the cyclic subgroup $(a)$ of $G$.*

27.8. **Exercises**

1. An element $a \neq 1$ of a group $G$ has the order 2 if and only if it is inverse of itself.

2. In every finite group of an even order there exist elements of the order 2.

3. If an element $a$ of a group $G$ is of the order $n$, then the order of each element of the cyclic subgroup $(a)$ of $G$ is a divisor of $n$.

4. Every group whose order is a prime number is cyclic.

5. The order of each element $a$ of any factor group on a finite group $G$ is a divisor of the order of each element of $G$ contained in $a$. If the order of $a$ is a power of a prime number $p$, then there exists in $a$ an element $a$ whose order is also a power of $p$. 