2 Elementary properties of integrals of the differential equation (q)

2.1 Relative positions of zeros of an integral and its derivative

Between two zeros of an integral y of the differential equation (q) there always lies at least one zero of its derivative y'. Between two zeros of the derivative y' there always lies at least one zero of y or one zero of q. It follows that:

Between two neighbouring zeros of an integral y of the differential equation (q) *lies precisely one zero of y', if q does not vanish in this interval. Between two neighbouring zeros of the derivative y' lies precisely one zero of y if $q \neq 0$ in this interval.*

In this statement, the inequality $q \neq 0$ can without loss of generality be replaced by $q < 0$, in consequence of the following theorem.

Theorem. If, between two neighbouring zeros of an integral y of the differential equation (q), *or between two zeros of its derivative y', or between a zero of y and a zero of y', the function q does not vanish then it must be negative, i.e. $q < 0$.*

Proof. Obviously, it is sufficient only to consider the third case. Let $t_1, x_1 \in j$, with $t_1 < x_1$, and assume, for example, that $y(t_1) = y'(x_1) = 0$, while $y(t) > 0$, $y'(t) > 0$ for $t \in (t_1, x_1)$. If possible, let $q > 0$ in the interval (t_1, x_1). Then in this interval $y'' > 0$, the function y' is increasing and since $y'(x_1) = 0$, y' is negative, which contradicts our hypothesis and so proves the theorem.

2.2 Ratios of integrals and their derivatives

For two integrals u, v of the differential equation (q) the following formulae hold in the interval j, with the exception, naturally, of points where the denominators vanish:

$$\left(\frac{u}{v}\right)' = -\frac{w}{v^2}, \quad \left(\frac{u'}{v'}\right)' = \frac{wq}{v'^2}, \quad \left(\frac{uu'}{vv'}\right)' = w\frac{quv - u'v'}{v^2v'^2} \tag{2.1}$$

$$(w = uv' - u'v).$$

We obtain from these the following results:

Let the integrals u, v be linearly independent; then in every interval $i \subset j$ containing no zeros of v, the ratio u/v is either an increasing or a decreasing function, according as $w < 0$ or $w > 0$. On the same assumption, in every interval $i \subset j$ which contains no zeros of v', the ratio u'/v' is an increasing or decreasing function according as $wq > 0$ or $wq < 0$. A similar statement also holds for the function uu'/vv'.

By integration of the above formulae over an interval $(t, x) \subset j$, in which the denominators involved are not zero, we obtain

$$\left.\begin{aligned} \frac{u(x)}{v(x)} - \frac{u(t)}{v(t)} = -w\int_t^x \frac{d\sigma}{v^2}, \quad \frac{u'(x)}{v'(x)} - \frac{u'(t)}{v'(t)} = w\int_t^x \frac{q\,d\sigma}{v'^2},\\ \frac{u(x)u'(x)}{v(x)v'(x)} - \frac{u(t)u'(t)}{v(t)v'(t)} = w\int_t^x \frac{quv - u'v'}{v^2v'^2}\,d\sigma. \end{aligned}\right\} \tag{2.2}$$

If the numbers t, x are zeros of the function u or of the function u', or if one of them is a zero of u and the other a zero of u', then the integral on the right hand side of the corresponding formula is zero.

2.3 The ordering theorems

There are several important laws governing the location of zeros of two independent integrals of the differential equation (q) and of their derivatives. These are described in the following four theorems, the so-called *ordering theorems*. Proofs follow from the formulae (2) above.

Let u, v be independent integrals of the differential equation (q) *and t_1, x_1 be numbers in the interval j with $t_1 < x_1$.*

(1) *Let $u(t_1) = u(x_1) = 0$, $u(t) \neq 0$ for $t \in (t_1, x_1)$, then the integral v has precisely one zero in the interval (t_1, x_1).*

We now make the additional assumption that $q \neq 0$ for $t \in j$.

(2) *Let $u'(t_1) = u'(x_1) = 0$, $u'(t) \neq 0$ for $t \in (t_1, x_1)$, then the function v' has precisely one zero in the interval (t_1, x_1).*

(3) *Let $u'(t_1) = u(x_1) = 0$, $u(t) \neq 0$ for $t \in (t_1, x_1)$. If $t_2 < t_1$ and $v'(t_2) = 0$, then the integral v has a zero $x_2 \in (t_2, x_1)$. If $x_2 > x_1$ and $v(x_2) = 0$, then the function v' has a zero $t_2 \in (t_1, x_2)$.*

(4) *Let $u(t_1) = u'(x_1) = 0$, $u(t) \neq 0$ for $t \in (t_1, x_1)$. If $t_2 < t_1$ and $v(t_2) = 0$, then the function v' has a zero $x_2 \in (t_2, x_1)$. If $x_2 > x_1$ and $v'(x_2) = 0$, then the function v has a zero $t_2 \in (t_1, x_2)$.*

Proof. We shall give only the proof of the first part of (3). Assuming the contrary, we suppose that $v(t) \neq 0$ for $t \in (t_2, x_1)$. Then $v(t)v'(t) \neq 0$ for $t \in (t_1, x_1)$ and, indeed, even for $t \in [t_1, x_1]$. We can thus apply the last formula (2) to the integrals u, v in the interval $[t_1, x_1]$ from which it follows that

$$\int_{t_1}^{x_1} \frac{quv - u'v'}{v^2v'^2}\,d\sigma = 0.$$

Obviously we can assume that $v'(t_1) < 0$, $u'(x_1) < 0$. Then in the interval (t_1, x_1) we have $u > 0$, $u' < 0$, $v > 0$, $v^1 < 0$. This, however, is inconsistent with the above integral relationship, so our assumption is false and the proof is completed.

2.4 The (Riemann) integrals $\int_{x_0}^{x_1} \frac{d\sigma}{y^2(\sigma)}, \int_{x_0}^{x_1} \frac{q(\sigma)}{y'^2(\sigma)}\, d\sigma$ in the neighbourhood of a singular point

Let y be an integral of the differential equation (q) with a zero at the point c. We consider a left or right neighbourhood j_{-1} or j_0 of c, in which the integral y does not vanish, and choose first a number $x_0 \in j_{-1}$. We wish to study the behaviour of the integral $\int_{x_0}^{t} d\sigma/y^2(\sigma)$, $t \in j_{-1}$, in the neighbourhood of the singular point c.

Obviously, for $\sigma \in j_{-1}$ we have the formula

$$y(\sigma) = y'(c)\,(\sigma - c) + \frac{(\sigma - c)^2}{2}\, y''(\tau),$$

where $\sigma < \tau < c$. From this it follows that

$$y^2(\sigma) = y'^2(c)\,(\sigma - c)^2 \left[1 + \frac{\sigma - c}{2} \cdot \frac{y''(\tau)}{y'(c)}\right]^2,$$

hence

$$\frac{1}{y^2(\sigma)} = \frac{1}{y'^2(c)\,(\sigma - c)^2} \cdot \frac{1}{\left[1 + \frac{\sigma - c}{2} \cdot \frac{y''(\tau)}{y'(c)}\right]^2}.$$

Now we apply the Taylor expansion formula to obtain

$$\frac{1}{\left[1 + \frac{\sigma - c}{2} \cdot \frac{y''(\tau)}{y'(c)}\right]^2} = 1 - (\sigma - c)\frac{y''(\tau)}{y'(c)} + \frac{(\sigma - c)^2}{4} \cdot \frac{y''^2(\tau)}{y'^2(c)} \cdot \frac{3}{\left[1 + \Theta\frac{\sigma - c}{2} \cdot \frac{y''(\tau)}{y'(c)}\right]^4}$$

with $0 < \Theta < 1$. Consequently

$$\frac{1}{y^2(\sigma)} = \frac{1}{y'^2(c)}\left[\frac{1}{(\sigma - c)^2} - \frac{q(\tau)}{y'(c)} \cdot \frac{\tau - c}{\sigma - c} \cdot \frac{y(\tau) - y(c)}{\tau - c}\right] + O(1),$$

in which the symbol O naturally relates to the left neighbourhood of c. For $\sigma \in j_{-1}$, let

$$g(\sigma) = \frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c)} \cdot \frac{1}{(\sigma - c)^2}; \tag{2.3}$$

we then have

$$g(\sigma) = -\frac{q(\tau)}{y'^3(c)} \cdot \frac{\tau - c}{\sigma - c} \cdot \frac{y(\tau) - y(c)}{\tau - c} + O(1). \tag{2.4}$$

By the formula (3) the function g is continuous in the interval j_{-1}, while (4) shows that it is bounded there. From this follows the existence of the Riemann integral

$\int_{x_5}^{c} g(\sigma)\, d\sigma$. We now make use of the formula (3) to extend the definition of the function g over the interval j_0. An argument similar to that used above shows that for every $x_1 \in j_0$ the integral $\int_{c}^{x_1} g(\sigma)\, d\sigma$ exists.

For every two numbers $x_0 \in j_{-1}$, $x_1 \in j_0$, the integral

$$\int_{x_0}^{x_1} g(\sigma)\, d\sigma = \int_{x_0}^{x_1} \left[\frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c)} \cdot \frac{1}{(\sigma - c)^2}\right] d\sigma$$

exists. Now let x_0, x_m ($x_0 < x_m$) be arbitrary numbers in the interval j, which are not zeros of y and between which lie precisely m (≥ 1) zeros $c_1, \ldots, c_m$ of y, ordered so that $x_0 < c_1 < \ldots < c_m < x_m$.

Now we define the function g_m as follows:

$$g_m(\sigma) = \frac{1}{y^2(\sigma)} - \sum_{\mu=1}^{m} \frac{1}{y'^2(c_\mu)} \cdot \frac{1}{(\sigma - c_\mu)^2},$$

this definition being valid in the interval $[x_0, x_m]$ with the exception of the points c_μ. We choose a number x_μ in each interval $(c_\mu, c_{\mu+1})$, $\mu = 1, \ldots, m-1$. From the above result, the following integral exists for $\nu = 1, \ldots, m$:

$$\int_{x_{\nu-1}}^{x_\nu} g_m(\sigma)\, d\sigma = \int_{x_{\nu-1}}^{x_\nu} \left[\frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c_\nu)} \cdot \frac{1}{(\sigma - c_\nu)^2}\right] d\sigma + \sum_{\nu \neq \mu = 1}^{m} \frac{1}{y'^2(c_\mu)} \left[\frac{1}{x_\nu - c_\mu} - \frac{1}{x_{\nu-1} - c_\mu}\right].$$

Then by summation we obtain the formula

$$\begin{aligned}\int_{x_0}^{x_m} g_m(\sigma)\, d\sigma = &\sum_{\nu=1}^{m} \int_{x_{\nu-1}}^{x_\nu} \left[\frac{1}{y^2(\sigma)} - \frac{1}{y'^2(c_\nu)} \cdot \frac{1}{(\sigma - c_\nu)^2}\right] d\sigma \\ &- \sum_{\mu=1}^{m} \frac{1}{y'^2(c_\mu)} \left[\frac{1}{c_\mu - x_{\mu-1}} + \frac{1}{x_\mu - c_\mu}\right] \\ &+ \sum_{\mu=1}^{m} \frac{1}{y'^2(c_\mu)} \left[\frac{1}{c_\mu - x_0} + \frac{1}{x_m - c_\mu}\right].\end{aligned}$$

2.5 Application to the associated equation

Now we assume that $q \in C_2$ and does not vanish in the interval j. Then we can apply the above results to the first associated differential equation ($\hat{q}_1$) of (q) (§ 1.9).

Let y be an integral of (q) and $e \in j$ a zero of its derivative y'. We define the function $h(\sigma)$ in a neighbourhood of e, $\sigma \neq e$, by

$$h(\sigma) = \frac{q(\sigma)}{y'^2(\sigma)} - \frac{1}{q(e)y^2(e)} \cdot \frac{1}{(\sigma - e)^2};$$

then for every two numbers x_0, $x_1 \in j$, which are not zeros of the derivative y' and between which lies precisely the one zero e of y', there exists the integral

$$\int_{x_0}^{x_1} h(\sigma)\,d\sigma = \int_{x_0}^{x_1} \left[\frac{q(\sigma)}{y'^2(\sigma)} - \frac{1}{q(e)y^2(e)} \cdot \frac{1}{(\sigma - e)^2}\right] d\sigma.$$

More generally; let x_0, x_m $(x_0 < x_m)$ be arbitrary numbers in the interval j which are not zeros of the derivative y' and between which lie precisely m $(\geqslant 1)$ zeros $e_1, \ldots, e_m$ of y' ordered such that $x_0 < e_1 < \ldots < e_m < x_m$. In the interval $[x_0, x_m]$ with the exception of the numbers e_μ we define the function h_m as:

$$h_m(\sigma) = \frac{q(\sigma)}{y'^2(\sigma)} - \sum_{\mu=1}^{m} \frac{1}{q(e_\mu)y^2(e_\mu)} \cdot \frac{1}{(\sigma - e_\mu)^2},$$

and in every interval $(e_\mu, e_{\mu+1})$ we choose a number x_μ, $\mu = 1, \ldots, m - 1$. Then the integral of the function h_m exists between the limits x_0, x_m, and we have the following formula:

$$\int_{x_0}^{x_m} h_m(\sigma)\,d\sigma = \sum_{\nu=1}^{m} \int_{x_{\nu-1}}^{x_\nu} \left[\frac{q(\sigma)}{y'^2(\sigma)} - \frac{1}{q(e_\nu)y^2(e_\nu)} \cdot \frac{1}{(\sigma - e_\nu)^2}\right] d\sigma$$
$$- \sum_{\mu=1}^{m} \frac{1}{q(e_\mu)y^2(e_\mu)} \left[\frac{1}{e_\mu - x_{\mu-1}} + \frac{1}{x_\mu - e_\mu}\right]$$
$$+ \sum_{\mu=1}^{m} \frac{1}{q(e_\mu)y^2(e_\mu)} \left[\frac{1}{e_\mu - x_0} + \frac{1}{x_m - e_\mu}\right].$$

2.6 Basis functions

We now consider two differential equations

$$y'' = q(t)y, \tag{q}$$
$$\ddot{Y} = Q(T)Y \tag{Q}$$

on the intervals j, J, i.e. $t \in j$, $T \in J$. We do not exclude the possibility that these two differential equations coincide.

Let (u, v), (U, V) be an ordered pair of arbitrary bases for (q), (Q) respectively. By a *basis function* belonging to this ordered pair of bases we mean a function $F(t, T)$ defined on the region $j \times J$ by one of the following four formulae:

1. $u(t)V(T) - v(t)U(T)$,
2. $u'(t)\dot{V}(T) - v'(t)\dot{U}(T)$,
3. $u(t)\dot{V}(T) - v(t)\dot{U}(T)$,
4. $u'(t)V(T) - v'(t)U(T)$.

Thus there are four basis functions corresponding to the above basis pair for the differential equations (q), (Q) (and consequently to every such basis pair). If the differential equations (q). (Q) coincide, then we speak of *basis functions of the differential equation* (q).

We consider a basis function $F(t, T)$. Let $t_0 \in j$, $X_0 \in J$ be arbitrary numbers for which $F(t_0, X_0) = 0$ and in the cases 2 and 3 assume also that $Q(X_0) \neq 0$. We wish to show that *there is precisely one function $X(t)$ defined in a neighbourhood of t_0, which takes the value X_0 at the point t_0, is continuous in its interval of definition, and satisfies the equation $F[t, X(t)] = 0$. This function X has moreover, in its interval of definition, the continuous derivative*

$$X'(t) = -\frac{F'[t, X(t)]}{\dot{F}[t, X(t)]}.$$

In the individual cases the derivative $X'(t)$ is therefore given by the following expressions

$$1.\quad -\frac{u'(t)V[X(t)] - v'(t)U[X(t)]}{u(t)\dot{V}[X(t)] - v(t)\dot{U}[X(t)]},$$

$$2.\quad -\frac{q(t)}{Q[X(t)]}\cdot\frac{u(t)\dot{V}[X(t)] - v(t)\dot{U}[X(t)]}{u'(t)V[X(t)] - v'(t)U[X(t)]},$$

$$3.\quad -\frac{1}{Q[X(t)]}\cdot\frac{u'(t)\dot{V}[X(t)] - v'(t)\dot{U}[X(t)]}{u(t)V[X(t)] - v(t)U[X(t)]},$$

$$4.\quad -q(t)\,\frac{u(t)V[X(t)] - v(t)U[X(t)]}{u'(t)\dot{V}[X(t)] - v'(t)\dot{U}[X(t)]}.$$

To illustrate the method of proof, take the function

$$F(t, T) = u(t)V(T) - v(t)U(T).$$

According to our assumption we have $F(t_0, X_0) = 0$ and the function F obviously possesses continuous partial derivatives

$$F'(t, T) = u'(t)V(T) - v'(t)U(T),$$
$$\dot{F}(t, T) = u(t)\dot{V}(T) - v(t)\dot{U}(T).$$

at every point $(t, T) \in j \times J$.

Further, $\dot{F}(t_0, X_0) \neq 0$, for otherwise we would have

$$(F(t_0, X_0) =)\quad u(t_0)V(X_0) - v(t_0)U(X_0) = 0,$$
$$(\dot{F}(t_0, X_0) =)\quad u(t_0)\dot{V}(X_0) - v(t_0)\dot{U}(X_0) = 0,$$

and these two relations (when we recall that $u^2(t_0) + v^2(t_0) \neq 0$) contradict the linear independence of the integrals U, V of (Q). Now we only need to apply the classic implicit function theorem, and the proof is complete.

We observe that: if two functions $(z =)\ x$, X are continuous in an interval $i \subset j$, take the same value at a point of the interval i, and satisfy in this interval the equation $F(t, z) = 0$, then they coincide in the interval i. For, if this were not so, there would be numbers $t_1 < t_2$ in the interval i such that, for instance, $x(t_1) = X(t_1)$ and $x(t) \neq X(t)$ for $t_1 < t \leqslant t_2$. This, however, contradicts the above theorem.