16 Differential equations with coincident central dispersions of the x-th and (x + 1)-th kinds (x = 1, 3)


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In this paragraph we shall be concerned with differential equations \((q)\) whose central dispersions of the first and second kinds \(\phi_v\) and \(\psi_v\) or of the third and fourth kinds \(\chi_p\) and \(\omega_p\) coincide \((v = 0, \pm 1, \pm 2, \ldots; p = \pm 1, \pm 2, \ldots)\). Obvious examples of such differential equations are those equations \((q)\) whose carrier \(q\) is a negative constant in the interval \((-\infty, \infty)\). A carrier \(q\) with the property \(\phi_v = \psi_v\) or \(\chi_p = \omega_p\) we shall call, for brevity, an \(F\)-carrier or an \(R\)-carrier respectively.

Consider an oscillatory differential equation \((q)\) in the interval \(j = (a, b)\) and assume that \(q < 0\) for all \(t \in j\). We denote by \(\phi, \psi, \chi, \omega\) the fundamental dispersions of the corresponding kinds; these are thus defined in the entire interval \(j\).

A convenient starting point for the theory of \(F\)- and \(R\)-carriers is provided by the properties of normalized polar functions (§ 6). Let \(\theta(t) = \beta(t) - \alpha(t)\) be a polar function of the carrier \(q\), and \(h(\alpha), -k(\beta), p(\xi)\) be the corresponding 1-, 2-, 3-normalized polar functions. The functions \(h, -k, p\) are therefore defined in the interval \(J = (-\infty, \infty)\), and the following relations hold at every point \(t \in j\)

\[
\begin{align*}
\frac{\beta(t)}{\alpha(t)} &= h(\alpha(t)) + k(\alpha(t)), \\
\frac{\beta(t) - \alpha(t)}{\alpha(t)} &= p(\zeta(t)), \\
\frac{\zeta(t)}{\alpha(t)} &= p(\zeta(t)) < (n + 1)\pi; \quad n \text{ integral}
\end{align*}
\]

1. Theory of \(F\)-Carriers

16.1 Characteristic properties

First we note that from the formulae (12.2), (12.3) it follows that \(q\) is an \(F\)-carrier if and only if its fundamental dispersions of the 1st and 2nd kinds coincide; \(\phi = \psi\) for all \(t \in j\).

In the development of the theory which follows we shall confine ourselves generally to the properties of the 1-normalized polar function \(h\). We can reach the same objective by making use of suitable properties of the 2- or 3-normalized polar functions \(-k, p\), but we shall content ourselves in this respect with a few comments as opportunity offers.
Theorem. The carrier \( q \) is an \( F \)-carrier, if and only if the 1-normalized polar function \( h \) has period \( \pi \).

Proof. (a) Let \( q \) be an \( F \)-carrier. Then in the interval \( j \) we have \( \phi(t) = \psi(t) \). Then, taking account of (1),

\[
\begin{align*}
  h[\alpha(t) + \varepsilon\pi] &= h\alpha\phi(t) = \beta\phi(t) - \alpha\phi(t) = \beta\psi(t) - \alpha\psi(t) \\
  &= (\beta(t) + \varepsilon\pi) - (\alpha(t) + \varepsilon\pi) = h\alpha(t) \\
  (\varepsilon = \text{sgn} \alpha' = \text{sgn} \beta'),
\end{align*}
\]

and consequently \( h(\alpha + \pi) = h(\alpha) \) for \( \alpha \in (-\infty, \infty) \).

(b) Let the polar function \( h \) have period \( \pi \), so \( h(\alpha + \pi) = h(\alpha) \) for \( \alpha \in (-\infty, \infty) \). Then at every point \( t \in j \),

\[
\beta\phi(t) = \alpha\phi(t) + h\alpha\phi(t) = \alpha(t) + \varepsilon\pi + h[\alpha(t) + \varepsilon\pi] = \alpha(t) + \varepsilon\pi + h\alpha(t) = \beta(t) + \varepsilon\pi,
\]

and it follows that \( \psi(t) = \phi(t) \).

We have thus determined all \( F \)-carriers:

The \( F \)-carriers are precisely those which are derived by the formula (6.29) from normalized polar functions \( h \) with period \( \pi \) in the interval \( (-\infty, \infty) \) (\( h \nabla > -1 \)).

Similarly, the \( F \)-carriers can be characterized by periodicity with period \( \pi \) or \( 2\pi \) of the 2- or 3-normalized polar functions \( -k \) or \( p \).

We have also the following result (due to M. Laitoch [41].

Theorem. The carrier \( q \) is an \( F \)-carrier if and only if its fundamental dispersion of the first kind, \( \phi \), is linear:

\[
\phi(t) = ct + k \quad (c > 0, k = \text{const}).
\]

This follows immediately from the above results and (13.32).

16.2 Domain of definition of \( F \)-carriers

We now wish to determine the intervals of definition of the \( F \)-carriers.

Let \( q \) be an \( F \)-carrier. The 1-normalized polar function \( h \) is therefore periodic with period \( \pi \), and formula (2) holds. From (13.31) we obtain

\[
c = \exp 2 \int_0^\pi \cot h(d) \, d\rho.
\]

Now the formula (2) gives, for the \( v \)-th central dispersion \( \phi_v(t) \), \( v = 0, \pm 1, \pm 2, \ldots \)

\[
\phi_v(t) = ct + k \frac{c^v - 1}{c - 1} \quad \text{or} \quad q_v(t) + vk,
\]

according as \( c \neq 1 \) or \( c = 1 \).

From the facts that \( \phi_v(t) \to b \) as \( v \to \infty \), and \( \phi_v(t) \to a \) as \( v \to -\infty \), we have (from (4)): in the case \( c > 1 \)

\[
b = \infty, \quad a = -k/(c - 1), \quad \text{hence} \quad j = (a, \infty), \quad a \text{ finite};
\]
in the case $c < 1$

$$b = k/(1 - c), \quad a = -\infty, \quad \text{hence} \quad j = (-\infty, b), \quad b \text{ finite};$$

in the case $c = 1$

$$k > 0, \quad a = -\infty, \quad b = \infty.$$

We have thus determined the intervals of definition of all $F$-carriers:

The interval of definition $j$ of the $F$-carrier $q$ is unbounded on one or both sides according as

$$\int_0^\pi \cot h(\rho) \, d\rho \neq 0 \quad \text{or} \quad = 0.$$

16.3 Elementary carriers

We remind the reader that this term is applied to carriers whose first phases are elementary (§ 8.4). Now we show that:

The carrier $q$ is elementary if and only if the 1-normalized polar function $h$ has period $\pi$ and satisfies the following conditions

$$\int_0^\pi \cot h(\rho) \, d\rho = 0, \quad \int_0^{\varepsilon \pi} \left( \exp 2 \int_0^\sigma \cot h(\rho) \, d\rho \right) d\sigma = \pi \varepsilon a'_0.$$

(16.5)

For, if the carrier $q$ is elementary, then its fundamental dispersion of the 1st kind $\phi(t)$ has the form (2) with $c = 1, k = \pi$. The 1-normalized polar function $h$ has therefore period $\pi$, and from (13.31), (13.30) the relations (5) follow. The second part of the theorem is proved similarly.

We have thus determined all elementary carriers:

The elementary carriers $q$ are precisely those derived by the formula (6.29) from 1-normalized polar functions $h$ defined in the interval $(-\infty, \infty)$, having period $\pi$, and satisfying the conditions (5) ($h' > -1$).

Similarly, the elementary carriers may be expressed in terms of 2- or 3-normalized polar functions $-k$ or $p$, being given explicitly by the formulae (6.36) or (6.41).

16.4 Kinematic properties of $F$-carriers

We now make use of the kinematic significance of integrals of the differential equation (q), described in § 1.5, as applied to an $F$-carrier $q$.

Let $q$ be an $F$-carrier. Consider two points $P, P'$ lying on the oriented straight line $G$, whose motion is given by integrals, $u, v$ of the differential equation (q).

Since the differential equation (q) is oscillatory, the motion of each of these points consists of an oscillation about the fixed point (the origin) $O$ of the straight line $G$.  

We assume that at any instant $t_0$ at which the point $P$ passes through $O$, the point $P'$ does not coincide with $O$ and its velocity is zero. At the instant $t_0$, therefore, the point $P'$ is at a relative maximum distance from $O$. The times at which the point $P$ passes through the origin $O$ are obviously $\psi_r(t_0)$; $r = \ldots, -1, 0, 1, \ldots$. Since $q$ is an $F$-carrier, we have $\phi_r(t_0) = \psi_r(t_0)$.

We see therefore that:

The oscillations of the points $P, P'$ about the origin $O$ are such that the point $P$ passes through the origin $O$ when the point $P'$ is at a relative maximum distance from $O$.

II. Theory of $R$-Carriers

16.5 Characteristic properties of $R$-carriers

From the formulae in § 12.4 we have, for all $t \in j$,

\[
\begin{align*}
\chi^{(0)} &= \psi, & \omega_0 &= \phi, \\
\omega_n &= \phi^{n-1} \omega, & \omega_{-n} &= \phi^{-n-1} \chi^{-1}, \\
\chi_n &= \psi^{n-1} \chi, & \chi_{-n} &= \psi^{-n-1} \omega^{-1} \\
(n = 1, 2, \ldots; \phi_{-1} = \phi^{-1}, \psi_{-1} = \psi^{-1}).
\end{align*}
\]  

(16.6)

Hence, from $\chi = \omega$ it follows that $\phi = \psi$ and $\chi_\rho = \omega_\rho$ for $\rho = \pm 1, \pm 2, \ldots$.

This gives the result:

$q$ is an $R$-carrier if and only if its fundamental dispersions of the third and fourth kinds coincide: $\chi = \omega$ for $t \in j$. An $R$-carrier is always an $F$-carrier.

Theorem. The carrier $q$ is an $R$-carrier if and only if the 1-normalized polar function $h$ satisfies the following relation in the interval $J = (-\infty, \infty)$:

\[
h \alpha + h[\alpha + h \alpha - n \pi] = (2n + 1) \pi.
\]  

(16.7)

Proof. If (7) is satisfied, then on applying it at the point $\alpha + h \alpha - n \pi$, there follows the $\pi$-periodicity of $h$:

\[
h(\alpha + \pi) = h \alpha.
\]  

(16.8)

We shall now give the proof first for the case $n = 0$. We then have $0 < \beta - \alpha < \pi$, so the corresponding Abel functional equations (13.18), (13.20) hold.

(a) Let $q$ be an $R$-carrier, so that $\chi = \omega$. Then, in the interval $j$, we have

\[
\beta \chi(t) = \alpha \chi(t) + h \alpha \chi(t)
\]

and further, from (13.18), (13.20),

\[
\alpha(t) + \pi = \beta(t) + h \left[ \beta(t) + \frac{1}{2} (\varepsilon - 1) \pi \right].
\]
Since however $q$ is an $F$-carrier, the function $h$ has period $\pi$, and on taking account of (1) the last relationship gives the formula (7) for the case $n = 0$.

(b) Now let the relation (7) be satisfied when $n = 0$; then, from (8), the function $h$ is $\pi$-periodic. From (1) and (13.20), we have

$$\beta \omega(t) = \alpha \omega(t) + h \omega(t) = \beta(t) + \frac{1}{2}(\epsilon - 1)\pi + h \left[\beta(t) + \frac{1}{2}(\epsilon - 1)\pi\right]$$

$$= \alpha(t) + \frac{1}{2}(\epsilon + 1)\pi - \pi + h \alpha(t) + h \left[\alpha(t) + h \alpha(t) + \frac{1}{2}(\epsilon - 1)\pi\right].$$

Since the function $h$ satisfies (7) and has period $\pi$, the last expression, in view of (13.18), equal to $\beta \chi(t)$. We have, therefore $\chi = \omega$ for $t \in j$.

The extension of the proof to the general case, in which $n$ is any integer, is simple. We set

$$h = h_0 + n \pi.$$  \hspace{1cm} (16.9)

Then $h_0$ is a 1-normalized polar function of the carrier $q$ with the property $0 < h_0 < \pi$.

If $q$ is an $R$-carrier, then from (a) the function $h_0$ satisfies the condition

$$h_0 \alpha + h_0 \alpha + h_0 \alpha = \pi;$$  \hspace{1cm} (16.10)

and from this and (9) the relation (7) follows.

If, conversely, the condition (7) is satisfied, then (10) holds; from that we deduce (using (b)) that $q$ is an $F$-carrier. This completes the proof.

We have thus determined all the $R$-carriers;

*The R-carriers are precisely those derived by the formula (6.29) from the 1-normalized polar functions $h$ defined in the interval $(-\infty, \infty)$ and satisfying (7) ($h' > -1$).*

Similarly, the $R$-carriers can be determined by means of 2- or 3-normalized polar functions satisfying the conditions

$$k \beta + k[h \beta + k \beta + n \pi] = -(2n + 1)\pi$$  \hspace{1cm} (16.11)

and

$$p \zeta + p(\zeta + \pi) = (2n + 1)\pi,$$  \hspace{1cm} (16.12)

being given by the formulae (6.36) and (6.41).

### 16.6 Further properties of $R$-carriers

The following study takes us further into the properties of $R$-carriers.

Let $q$ be an $R$-carrier in the interval $j (= (a, b))$.

We consider an integral curve $R$ of the differential equation (q) with the parametric co-ordinates $u(t), v(t)$ in which, for precision, we take the Wronskian $w = uv' - u'v < 0$. We denote the origin of the coordinate system by $O$.

Let $P, \bar{P} \in R$ be points determined by the parameters $t, \chi(t)$ where $t \in j$ is arbitrary. *Our interest will centre upon the area $\Delta$ of the triangle $POP$*. Obviously

$$2\Delta = r(t) \cdot r \chi(t) \cdot \sin \theta(t);$$  \hspace{1cm} (16.13)
where \( r(t), r' \) are the lengths of the vectors \( \overrightarrow{OP}, \overrightarrow{OP} \) and \( \theta(t) \) is the angle formed by the latter.

Let \( \alpha \) be a proper first phase of the basis \( (u, v) \). Since \( -w > 0 \) we have \( \alpha' > 0 \); we also have \( \chi(t) > t \), consequently \( \alpha \chi(t) > \alpha(t) \) and since \( 0 \leq \theta(t) < 2\pi \),

\[
\theta(t) = \alpha \chi(t) - \alpha(t) + 2n\pi, \quad 0 \geq n \text{ integral.} \quad (16.14)
\]

Moreover, let \( \beta \) be the proper second phase of \( (u, v) \) neighbouring to \( \alpha \), so that \( 0 < \beta - \alpha < \pi \).

We write the relation (14) as follows

\[
\theta(t) = [\alpha \chi(t) - \beta(t)] + [\beta(t) - \alpha(t)] + 2n\pi
\]

and apply the formulae (13.20). Since \( \varepsilon = 1 \) and \( 0 < \beta - \alpha < \pi \), we have

\[
\theta(t) = \beta(t) - \alpha(t), \quad (16.15)
\]

so \( \theta \) is that polar function of the basis \( (u, v) \) generated by \( \alpha \) and lying between 0 and \( \pi \).

To help on the development of this study, it is convenient to quote here the following formulae:

\[
\theta = -\theta + \pi, \quad \alpha' = \beta' \chi \cdot \chi', \quad \beta' = \alpha' \chi \cdot \chi' \quad [(13.18), (13.20)] \quad (16.16)
\]

\[
r' \chi \cdot \chi = -rr' \quad [(16) and (6.8)] \quad (16.17)
\]

\[
\alpha' = \frac{w \cdot q \chi}{s^2 \chi} \chi', \quad \beta' = \frac{-w}{r^2 \chi} \chi' \quad [(16) and (5.14), (5.23)] \quad (16.18)
\]

Logarithmic differentiation of (13) shows that

\[
\frac{\Delta'}{\Delta} = \frac{r'}{r} + \frac{r' \chi}{r \chi} \chi' + \cot \theta \cdot \theta',
\]

and the formulae (6.8), (5.14) and then (17), (18) give

\[
\cot \theta \cdot \theta' = -\frac{1}{w} rr' (\beta' - \alpha') = -\frac{1}{w} rr' \cdot \beta' - \frac{r'}{r} = -\frac{r' \chi}{r \chi} \chi' - \frac{r'}{r}.
\]

Consequently \( \Delta' = 0 \), and we have the result:

**Theorem.** The area \( \Delta \) of the triangle \( PO \tilde{P} \) is constant throughout the curve \( \mathcal{R} \).

### 16.7 Connection between \textit{R}-carriers and Radon curves

From the relationships [(16) and (5.28)]

\[
rs \cdot \sin \theta = -w, \quad r\chi \cdot s\chi \cdot \sin \theta = -w \quad (16.19)
\]

there follows, when we take account of (13),

\[
r\chi = \frac{2\Delta}{-w} s, \quad s\chi = \frac{-w}{2\Delta} r. \quad (16.20)
\]
Moreover we have from (13.20) and (18)

\[ W\alpha \chi = W\beta, \quad W\beta \chi = W\alpha \pm \pi, \quad (16.21) \]

in which the sign + or − must be taken according as \(0 \leq W\alpha < \pi\) or \(\pi \leq W\alpha < 2\pi\).

We now apply to the integral curve \(\mathcal{R}\) the transformation \(R\) (§ 6.1) which consists of the inversion \(K_{\sqrt{2}A}\), followed by a quarter rotation about \(O\) in the positive sense.

The curve \(\mathcal{R}\) is then transformed into a curve \(\tilde{\mathcal{R}}\): the point \(P \in \mathcal{R}\) goes over into the point \(\tilde{P} \in \tilde{\mathcal{R}}\), while the corresponding amplitudes \(\tilde{r}, s\) and angles \(W\alpha, W\beta; \tilde{\alpha}, \tilde{\beta}\) are transformed as follows [(6.5)]

\[ \tilde{r} = \frac{2\Delta}{-w} s, \quad \tilde{\alpha} = W\beta, \quad \tilde{\beta} = W\alpha \pm \pi, \quad (16.22) \]

in which we take the sign + or − according as \(0 \leq W\alpha < \pi\) or \(\pi \leq W\alpha < 2\pi\).

Comparing this with (20), (21), gives

\[ \tilde{r} = r\chi, \quad \tilde{\alpha} = W\alpha \chi, \quad \tilde{\beta} = W\beta \chi. \quad (16.23) \]

Clearly, the transformation \(R\) takes the curve \(\mathcal{R}\) into itself, so

*The integral curves of an \(R\)-carrier are Radon curves.*

16.8 Connection between \(R\)- and \(F\)-carriers

The second formula (18), taken together with (5.23) gives the following formula holding in the interval \(j\)

\[ \frac{\chi'}{r^2 \chi} = -\frac{q}{s^2} \quad (16.24) \]

and moreover, using (20),

\[ \chi' = -d^2 q \left( d = \frac{2\Delta}{-w} \right). \quad (16.25) \]

This formula is due to E. Barvínk ([2]). It follows that for \(t_0, t \in j\),

\[ \chi(t) = \chi(t_0) - d^2 \int_{t_0}^{t} q(\sigma) d\sigma. \quad (16.26) \]

Similarly the first formula (18) and (5.14) show that

\[ \chi' = -\frac{1}{d^2} \frac{1}{q \chi}; \quad (16.27) \]

thus for \(t_0, t \in j\),

\[ \chi(t) = \chi(t_0) - \frac{1}{d^2} \int_{t_0}^{t} d\sigma \frac{d\sigma}{q \chi(\sigma)}. \quad (16.28) \]
From (25) and (27) we see that the product of the values of the $R$-carrier $q$ at any two points $t$, $\chi(t) \in j$ is constant:

$$q(t)q(\chi(t)) = \frac{1}{d^2}.$$  \hspace{1cm} (16.29)

From the formula (26) we have

$$\chi(t) = \chi(t_0) - d^2 \int_{t_0}^{\chi(t_0)} q(\sigma) \, d\sigma - d^2 \int_{\chi(t_0)}^{\chi(t)} q(\sigma) \, d\sigma.$$  \hspace{1cm} (16.30)

If the last integral is transformed by means of the substitution $\sigma = \chi(\tau)$ and we apply formulae (25) and (29) then we obtain

$$\phi(t) = \chi(t) = t + k$$

with a determinate constant $k (> 0)$. Since $\chi_X = \phi$, this formula shows that every $R$-carrier belongs to the set of $F$-carriers defined in the interval $j = (-\infty, \infty), (\S\ 16.2, \ c = 1)$.

16.9 Kinematic properties of $R$-carriers

Let $q$ be an $R$-carrier.

We consider two points $P, P'$ lying on the oriented straight line $G$, whose motions follow the integrals $u, v$ of the differential equation (q). Let the positions of the points $P, P'$ at an instant $t_0$ be such that the point $P$ passes through the origin $O$ when $P'$ is at a relative maximum distance from $O$. Since $q$ is an $R$-carrier, and consequently also an $F$-carrier, we have the situation described in $\S\ 16.4$. Now the instants at which the point $P$ is at its greatest distance from $O$ are $\chi_p(t_0)$ and those at which the point $P'$ passes through the origin $O$ are $\omega_q(t_0)$: $\rho = \ldots, -1, 1, \ldots$. But since $q$ is an $R$-carrier, we have $\chi_p(t_0) = \omega_q(t_0)$. Thus:

The oscillations of the points $P, P'$ about the origin $O$ are such that each of these passes through the origin at the instant when the other is at a relative maximum distance from the origin.