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17 Bunch curves and Radon curves

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# 17 Bunch curves and Radon curves

In our study of polar functions we have met plane curves with special centro-affine properties, among them the Radon curves (§ 6.1). These are curves which are always intersected by the straight lines of a bunch in at least two points, and such that the tangents at these points of intersection are parallel. This section is devoted to an investigation of these curves from the geometrical standpoint. It will appear that these curves stand in a close relationship to the $F$-carriers and $R$-carriers, whose integral curves they in fact are. Our object is to determine the geometrical properties of these curves and to derive finite equations for them. This, as may be expected, involves an application of the analytical apparatus which we have derived above.

## 17.1 Fundamental concepts

We begin with the definition of the curves to be studied, which we shall call bunch curves.

Let $(F)$ be a straight line bunch with centre $O$. By a *bunch curve with respect to* $(F)$ or more shortly a *bunch curve* we mean a plane curve $\mathfrak{F}$ with the following property:

The curve $\mathfrak{F}$ is met by each straight line of the bunch $(F)$ in at least two distinct points, and is such that the tangents to the curve at each of the points of intersection are parallel.

By the *pole* of the curve $\mathfrak{F}$ we mean the centre $O$ of $(F)$. In view of the methods to be applied we wish to restrict ourselves to curves $\mathfrak{F}$ which are regular (i.e. locally convex and without turning points). Moreover we shall assume that every curve $\mathfrak{F}$ can be specified by means of parametric coordinates $U$, $V$ with respect to a coordinate system with origin $O$; the functions $U$, $V$ are defined in an open interval and belong to the class $C_2$.

Obvious examples of such bunch curves are ellipses and logarithmic spirals.

Let $\mathfrak{F}$ be a bunch curve with parametric coordinates $U$, $V$ in the interval $J$. We notice first that from the definition of bunch curves, the functions $U$, $V$ must be linearly independent. It follows that the functions $U$, $V$ are integrals of a linear differential equation (a) with continuous coefficients $\bar{a}$, $\bar{b}$:

$$Y'' + \bar{a}Y' + \bar{b}Y = 0. \qquad \text{(a)}$$

We see further that the properties of the curve $\mathfrak{F}$ which have been described are centro-affine invariant and consequently remain valid under every transformation of parameter (§ 4.1). If we choose, in particular, the curve parameter given by the formula (1.1), then the differential equation (a) takes the Jacobian form (q).

Hence, for appropriate choice of the curve parameter, the bunch curve $\mathfrak{F}$ is an integral curve of a differential equation (q). We call $q$ the *carrier of the bunch curve* $\mathfrak{F}$.

## 17.2 Determination of the carriers of the bunch curves

We now determine the carriers $q$ of the bunch curves.

Since we are only considering regular bunch curves, we can assume from the start that $q(t) \neq 0$ for $t \in j$ (§ 4.6). Let (q) be a differential equation in the interval $j$, $q(t) \neq 0$ for $t \in j$ and $\mathfrak{K}$ an integral curve of (q) with the parametric coordinates $u(t)$, $v(t)$ with respect to a coordinate system with origin $O$. Let $P(t) \in \mathfrak{K}$, $t \in j$ be an arbitrary point and $\tau(t)$ the tangent to $\mathfrak{K}$ at the point $P(t)$. Further, let $p(t)$ be the straight line $OP(t)$.

The following theorem is the foundation of further developments:

*Theorem. Two points $P(t_1)$, $P(t_2) \in \mathfrak{K}$, $t_1 \neq t_2$, lie on a straight line passing through the point $O$ if and only if the numbers $t_1$, $t_2$ are 1-conjugate.*

*Two tangents $\tau(t_1)$, $\tau(t_2)$ to $\mathfrak{K}$, $t_1 \neq t_2$, are parallel if and only if the numbers $t_1$, $t_2$ are 2-conjugate.*

*The tangent $\tau(t_2)$ to $\mathfrak{K}$ and the straight line $p(t_1)$ are parallel if and only if $t_2$ is 3-conjugate with $t_1$ and consequently $t_1$ is 4-conjugate with $t_2$.*

*Proof.* Two points $P(t_1)$, $P(t_2) \in \mathfrak{K}$, $t_1 \neq t_2$, lie on a straight line passing through the point $O$ if and only if $u(t_1)v(t_2) - u(t_2)v(t_1) = 0$.

Two tangents $\tau(t_1)$, $\tau(t_2)$ to $\mathfrak{K}$, $t_1 \neq t_2$ are parallel if and only if $u'(t_1)v'(t_2) - u'(t_2)v'(t_1) = 0$.

The tangent $\tau(t_2)$ to $\mathfrak{K}$ and the straight line $p(t_1)$ are parallel if and only if $u'(t_2)v(t_1) - v'(t_2)u(t_1) = 0$.

The result now follows immediately from the above facts and the theorem of § 3.12.

We observe also that:

The tangents $\tau(t_1)$, $\tau(t_2)$, $(t_1 \neq t_2)$, to the curve $\mathfrak{K}$, at two points of intersection of the latter with a straight line passing through the point $O$ are parallel if and only if the numbers $t_1$, $t_2$ are both 1-conjugate and 2-conjugate.

The tangents $\tau(t_1)$, $\tau(t_2)$ to the curve $\mathfrak{K}$ and the straight lines $p(t_1)$, $p(t_2)$ passing through the point $O$ and their points of contact $P(t_1)$, $P(t_2)$ with the curve $\mathfrak{K}$ are parallel if and only if each number $t_1$, $t_2$ is both 3- and 4-conjugate with the other.

Now let $q$ be a carrier of the bunch curve $\mathfrak{F}$ and $q \neq 0$ for $t \in j$.

Let $t_1 \in j$ be arbitrary and $P(t_1) \in \mathfrak{F}$ the corresponding point of $\mathfrak{F}$. From the definition of a bunch curve we know that the straight line $OP(t_1)$ passing through the point $O$ must have at least one point $P(t_2)$ $(\neq P(t_1))$ where it meets the curve $\mathfrak{F}$. Obviously, $t_2 \neq t_1$, and by the above theorem $t_2$ is 1-conjugate to $t_1$. Consequently, every number $t_1 \in j$ has 1-conjugate numbers, so we conclude that every integral of (q) vanishes at least twice in the interval $j$.

We note that the differential equation (q) is either of finite type $(m)$, $m \geqslant 3$, or of infinite type. It follows, in the first place, that $q < 0$ for $t \in j$ and further that the differential equation (q) admits of fundamental dispersions of the first and second kinds $\phi$, $\psi$. These are defined in appropriate intervals $i \subset j$, $i' \subset j'$.

Now let $t \in i$ be arbitrary. Since $\phi(t)$ is 1-conjugate with $t$, the points $P(t)$, $P\phi(t)$ lie on a straight line through the point $O$. Since, moreover, $\mathfrak{F}$ is a bunch curve, the tangents $\tau(t)$, $\tau\phi(t)$ are parallel. Taking account of the above theorem, we conclude that $\phi(t)$ is 2-conjugate with $t$, so $\phi(t) = \psi_\nu(t)$ for some integer $\nu$, $\nu \geqslant 1$. There is therefore an integral $y$ of (q), whose derivative $y'$ vanishes at the points $t$, $\phi(t)$ and possibly also at further points $x_1, \ldots, x_{\nu-1}$ lying between $t$ and $\phi(t)$. Now since the function $q < 0$ in $j$, every interval $(t, x_1), \ldots, (x_{\nu-1}, \phi(t))$ contains precisely one zero of $y$, so that between $t$ and $\phi(t)$ there are precisely $\nu$ zeros. Hence $\nu = 1$, and $\phi(t) = \psi(t)$; that is to say $i \subset i'$ and $\phi(t) = \psi(t)$ for $t \in i$.

Now we show that the opposite relation $i' \subset i$ also holds, and $\psi(t') = \phi(t')$ for $t' \in i'$. For let us suppose, if possible, that $t' \in i'$ but $t' \notin i$. Then the number $\bar{t}$ such that $(t' <)\ \bar{t} = \psi(t')\ (\in j)$ is defined, but there exists no number which is 1-conjugate with $t'$ on the right. We know that the straight line $OP(\bar{t})$ cuts the curve $\mathfrak{F}$ at least twice, whence follows the existence of numbers $\phi_\nu(\bar{t})\ (\in j)$, $\nu$ integral, which are 1-conjugate with $\bar{t}$. Since there is no number which is 1-conjugate with $t'$ on the right, and consequently none 1-conjugate with $\bar{t}\ (> t')$, we must have $\nu < 0$, so the number $\phi_{-1}(\bar{t})\ (= t)$ exists. But the tangents to the curve at the points $P(\bar{t})$, $P(t) \in \mathfrak{F}$ are parallel, which shows that $t$ is 2-conjugate with $\bar{t}$. We have therefore $\phi_{-1}(\bar{t}) = \psi_{-n}(\bar{t})$, for some $n \geqslant 1$, and from this we find, as above, that $\phi_{-1}(\bar{t}) = \psi_{-1}(\bar{t})\ (= t')$. We have therefore $t' = \phi_{-1}(\bar{t})$, hence $\bar{t} = \phi(t')$; this contradicts the above assumption that $t' \notin i$, and so establishes the desired result.

To sum up: the carrier $q$ of the bunch curve $\mathfrak{F}$ is always $< 0$; moreover the differential equation (q) is of finite type $(m)$, $m \geqslant 3$, or of infinite type, and its fundamental dispersions of the first and second kinds, $\phi$, $\psi$, coincide.

We leave it to the reader to convince himself of the validity of the converse statement: every continuous function $q$ with these properties is the carrier of a bunch curve.

From this we have:

*Theorem. The carriers of bunch curves are partial functions of F-carriers.*

The carrier $q$ of a bunch curve can for instance be represented in the form (6.29), in which $h$ is an arbitrary function, defined in an interval $J$ of length $> 2\pi$, with the following properties:

$$\left.\begin{array}{l} 1.\ h \in C_1; \\ 2.\ n\pi < h < (n+1)\pi, \quad n \text{ integral}; \\ 3.\ h(\alpha + \pi) = h(\alpha) \quad \text{for} \quad \alpha, \alpha + \pi \in J; \\ 4.\ h^{\diagdown}(\alpha) > -1 \quad \text{for} \quad \alpha \in J. \end{array}\right\} \tag{17.1}$$

## 17.3 Centro-affine length of arcs of bunch curves

Let $\mathfrak{F}$ be a bunch curve with the parametric coordinates $u(t)$, $v(t)$, $t \in j$. We shall continue to employ the notation used above.

First we recall that the centro-affine oriented arc of the curve $\mathfrak{F}$ determined by the points $P(t_1)$, $P(t_2) \in \mathfrak{F}$ is given by the formula (4.14). Its length $s(t_1|t_2)$ is therefore

$$s(t_1|t_2) = \left| \int_{t_1}^{t_2} \sqrt{-q(\sigma)}\, d\sigma \right|. \tag{17.2}$$

We now make use of the formulae (6.29) and (13.31) to obtain, for $t \in i$,

$$-q(\phi(t))\phi'^2(t) = -q(t)$$

and thus, taking account of (2),

$$s(\phi(t_1)|\phi(t_2)) = s(t_1|t_2). \tag{17.3}$$

We know that the points $P(t_1)$, $P\phi(t_1)$ lie on a straight line passing through the point $O$ (§ 17.2); the same holds for $P(t_2)$, $P\phi(t_2)$.

The arcs $P(t_1)P(t_2)$ and $P\phi(t_1)P\phi(t_2)$ of the bunch curve $\mathfrak{F}$, cut off by two arbitrary lines $OP(t_1)$, $OP(t_2)$ of the bunch $F$, have the same centro-affine length.

## 17.4 Finite equations of bunch curves

In this paragraph we shall derive finite equations for bunch curves. Since the bunch curve carriers are partial functions of the $F$-carriers, we shall confine ourselves to bunch curves of $F$-carriers. We shall also assume that the differential equations (q) of the bunch curves under consideration are oscillatory and consequently defined in the intervals given in § 16.2.

Let $\mathfrak{F}$ be a bunch curve and $q$ its carrier in the interval $j = (a, \infty)$ or $(-\infty, b)$ or $(-\infty, \infty)$ where $a$, $b$ are finite. Moreover, let $u$, $v$ be parametric coordinates of the curve $\mathfrak{F}$ with respect to a coordinate system with origin $O$; we assume for definiteness that $(w =)\ uv' - u'v < 0$.

Let $r(t)$ be the amplitude, $\alpha(t)$ be a first phase ($t \in j$) and $h(\alpha)$ ($\alpha \in (-\infty, \infty)$) one of the 1-normalized polar functions of the basis $(u, v)$ generated by the phase $\alpha$. The function $h$ has the above properties 1°–4° and without loss of generality we assume that $n = 0$.

We now start from the formula (6.25), taking $\alpha_0 = 0$:

$$r(t) = r_0 \cdot \exp \int_0^{\alpha} \cot h(\rho)\, d\rho. \tag{17.4}$$

From (4) and the fact that $\operatorname{sgn}(-w) = \operatorname{sgn} \alpha' = 1$, we have in the interval $j$

$$r\phi(t) = r_0 \cdot \exp \int_0^{\alpha+\pi} \cot h(\rho)\, d\rho$$

and moreover,

$$r\phi(t) = \sqrt{c} \cdot r(t);$$

in this $c$ ($> 0$) is the coefficient of $t$ in the fundamental dispersion $\phi(t) = ct + k$ of the differential equation (q), and the formula (16.3) holds.

We now see that the following function, which is defined for $t \in j$,

$$g(t) = c^{\frac{1}{2\pi}\alpha(t)} \tag{17.5}$$

is transformed by the substitution $t \to \phi(t)$ in the same manner as the function $r(t)$. Consequently, the function

$$f(t) = r(t)/g(t) \tag{17.6}$$

is invariant under this substitution, i.e.

$$f\phi(t) = f(t). \tag{17.7}$$

From (5), (6) it follows that

$$r(t) = C^{\alpha(t)} \cdot f(t) \quad \left(C = c^{\frac{1}{2\pi}}\right). \tag{17.8}$$

Let $F(\alpha)$ ($> 0$) be the function defined in the interval $(-\infty, \infty)$ by means of the relationship

$$f(t) = F(\alpha);$$

in this, $t = \alpha^{-1}(\alpha) \in j$ and $\alpha = \alpha(t) \in (-\infty, \infty)$ are two homologous values.

If we compare the formulae (4) and (8), we obtain

$$F(\alpha) = r_0 \cdot C^{-\alpha} \cdot \exp \int_0^\alpha \cot h(\rho)\, d\rho.$$

Clearly, the function $F$ belongs to the class $C_2$ and has period $\pi$. We have moreover

$$h = \operatorname{arcot}\left(\frac{F^\backprime}{F} + \log C\right), \tag{17.9}$$

$$h^\backprime = -\frac{FF^{\backprime\backprime} - F^{\backprime 2}}{F^2 + (F^\backprime + F \cdot \log C)^2}, \tag{17.10}$$

where the symbol arccot denotes the branch of the above function lying between 0 and $\pi$.

The function $F$ has therefore the following properties in its interval of definition $(-\infty, \infty)$

$$\left.\begin{array}{l} 1.\ F > 0, \\ 2.\ F \in C_2, \\ 3.\ F(\alpha + \pi) = F(\alpha), \\ 4.\ \dfrac{F^{\backprime\backprime}}{F} < \dfrac{F^{\backprime 2}}{F^2} + \left(\dfrac{F^\backprime}{F} + \log C\right)^2 + 1. \end{array}\right\} \tag{17.11}$$

Clearly, the equation of the bunch curve $\mathfrak{Y}$ in polar coordinates is

$$r = C^\alpha \cdot F(\alpha), \tag{17.12}$$

where $C$ ($> 0$) is a constant and $F$ is a function defined in the interval $(-\infty, \infty)$ with the properties (11).

Conversely, it can be shown without difficulty that: if we construct a function $h$ according to the formula (9) using an arbitrary constant $C$ ($> 0$) and a function $F$ defined in the interval $(-\infty, \infty)$ with the properties (11), then this function $h$ represents a 1-normalized polar function of an $F$-carrier $q$, satisfying the relations (1). Moreover, this carrier is specified by means of a formula such as (6.29).

We have thus proved the following theorem:

*Theorem. All bunch curves with F-carriers are given in polar coordinates by the equation* (12), *in which* $C$ $(> 0)$ *is a constant and* $F$ *is a function defined in the interval* $(-\infty, \infty)$ *with the properties* (11).

## 17.5 Radon curves

Let $q$ be the carrier of a bunch curve $\mathfrak{F}$ in an interval $j$; moreover let $O$ be the pole of $\mathfrak{F}$ and $(F)$ the bunch of straight lines with the centre $O$. We shall use the symbols $P(t), p(t), \tau(t)$ $(t \in j)$ in the same sense as in § 17.2.

The curve $\mathfrak{F}$ determines a simple mapping $\boldsymbol{F}$ of the bunch $(F)$ into itself which is defined as follows: corresponding to every straight line $p \in (F)$, $\boldsymbol{F}p \in (F)$ is the line parallel to the tangents to the curve at their points of intersection with $p$.

We call $\mathfrak{F}$ a *Radon curve* when this mapping $\boldsymbol{F}$ is involutory, that is when the combined mapping $\boldsymbol{FF}$ is the identity which maps $(F)$ onto itself.

Every Radon curve therefore has the ellipse property (§ 6.1): from $p' = \boldsymbol{F}p$ it follows that $\boldsymbol{F}p' = p$.

Now let $\mathfrak{R}$ be a Radon curve. Then the fundamental dispersions of the first and second kinds, $\phi$, $\psi$ of the differential equation (q) are defined in some interval $i \subset j$ and from this we deduce that the fundamental dispersions of the third and fourth kinds of (q), $\chi$ and $\omega$, also exist in certain intervals $k, k' \subset i$. Clearly, $t < \chi(t) < \phi(t)$ for $t \in k$ and $t < \omega(t) < \psi(t)$ $(= \phi(t))$ for $t \in k'$.

We show first that $k = k'$ and $\chi(t) = \omega(t)$ for $t \in k$.

Let $t \in k$ be arbitrary. Since $\chi(t)$ is 3-conjugate with $t$, the tangent to the curve $\tau\chi(t)$ is parallel to the line $p(t)$. Since $\mathfrak{R}$ is a Radon curve, $\tau(t)$ is parallel to $p\chi(t)$; consequently (§ 17.2) $t$ is 3-conjugate with $\chi(t)$ and also $\chi(t)$ is 4-conjugate with $t$. We have therefore $\chi(t) = \omega_\rho(t)$, $\rho$ being an integer. From $\chi(t) > t$ it follows that $\rho \geqslant 1$; if $\rho \geqslant 2$, then $t < \chi(t) < \phi(t) < \omega_\rho(t)$, but this is impossible. Consequently $\rho = 1$, so $\chi(t) = \omega(t)$, thus $k \subset k'$ and $\chi(t) = \omega(t)$ for $t \in k$. In a similar manner, we find that $k' \subset k$ and $\omega(t) = \chi(t)$ for $t \in k'$. This completes the proof.

We leave it to the reader to convince himself of the truth of the converse theorem: every bunch curve carrier $q$ with the property $\chi = \omega$ is the carrier of a Radon curve.

We thus have the following result:

*Theorem. The carriers of Radon curves are partial functions of R-carriers.*

In what follows, we shall consider only Radon curves with $R$-carriers.

Let $\mathfrak{R}$ be a Radon curve with the $R$-carrier $q$. From § 16.8 the function $q$ is defined in the interval $j = (-\infty, \infty)$ and the fundamental dispersion $\phi$ of the first kind of (q) is given by the formula $\phi(t) = t + k$ $(c = 1; (16.30))$.

From (12) there follows the equation of the curve $\mathfrak{R}$ in polar coordinates:

$$r = F(\alpha); \tag{17.13}$$

where $F$ is a function defined in the interval $(-\infty, \infty)$ with the properties (11).

Since the function $F$ is, from (11), 3°, periodic with period $\pi$ and every two points $P(\alpha)$, $P(\alpha + \pi) \in \mathfrak{R}$ lie on the same straight line passing through $O$, we have:

*The curve $\mathfrak{R}$ is closed and has central symmetry.*

We know (§ 16.7) that the function $h$ defined by means of the formula (9), (with $C = 1$), which we may assume to lie in the interval $(0, \pi)$, satisfies the relation $h\alpha + h[\alpha + h\alpha] = \pi$.

It follows that

$$\frac{F^{\backslash}\alpha}{F\alpha} + \frac{F^{\backslash}\left[\alpha + \operatorname{arccot}\dfrac{F^{\backslash}\alpha}{F\alpha}\right]}{F\left[\alpha + \operatorname{arccot}\dfrac{F^{\backslash}\alpha}{F\alpha}\right]} = 0. \tag{17.14}$$

The equation of the Radon curve $\mathfrak{R}$ in polar coordinates has therefore the form (13); $F$ is a function defined in the interval $(-\infty, \infty)$ with the properties (11) and (14), taking $C = 1$.

Conversely, it can easily be shown that if any function $h$ is constructed by means of (9) using a function $F$ defined in the interval $(-\infty, \infty)$ with the above properties, then this function $h$ represents a 1-normalized polar function of an $R$-carrier, satisfying the conditions (1) and (16.7) with $n = 0$, and this carrier is specified by means of a formula such as (6.29).

To sum up:

*Theorem. All Radon curves with R-carriers are given in polar coordinates by the equation* (13) *in which F denotes a function defined in the interval* $(-\infty, \infty)$ *with the properties* (11) *and* (14).