General transformations

In this chapter we shall first establish the special form of the transformation formula (11.11). We shall then build up a transformation theory at the heart of which lie problems of existence and uniqueness relating to solutions of the differential equation (Qq), for general differential equations (q), (Q).

22 Establishment of the special form of the transformation formula

22.1 A theorem on transformations of second order differential equations

The special form of the transformation formula (11.11), which is linear with respect to the solutions Y, y of the equations (Q), (q), may perhaps appear to be arbitrary and conditioned by the methods applied to the solution of the transformation problem. The question now arises whether this transformation formula may not be replaced by a more general relationship constructed with an appropriate function f, of the form

$$y(t) = f(t,\ Y[X(t)])$$

We shall show that the answer to this question is, in general, negative.

P. Stäckel [J. reine angew. Math. 111 (1893)], S. Lie (Leipziger Ber. 1894) and E. J. Wilczynski [Amer. J. Math. 23 (1901)] have shown by various methods that generally the linear form of the transformation formula (11.11) is the only one possible.

We can express this result in the following theorem. The formulation of the theorem in a manner suitable for application to the Jacobian form of the differential equation, and also its proof, were kindly supplied by Frau Z. Mikolajska-Młak.

Theorem. Let s, S be three-dimensional spaces with point coordinates (t, y, z), (T, Y, Z), *in which*

$$a < t < b; \qquad -\infty < y, z < \infty; \qquad A < T < B; \qquad -\infty < Y, Z < \infty.$$

Moreover, let

$$\left.\begin{aligned} t &= x(T),\\ y &= f(t,\ Y, Z),\\ z &= g(t,\ Y, Z), \end{aligned}\right\} \qquad (\boldsymbol{t})$$

$$\left.\begin{aligned} T &= X(t),\\ Y &= F(t, y, z),\\ Z &= G(t, y, z) \end{aligned}\right\} \qquad (\boldsymbol{T})$$

be simple, mutually inverse, mappings of the spaces S on s and of the space s on S. We assume:

1°. *The mappings* $\boldsymbol{t}$, $\boldsymbol{T}$ *are of class* C_2 *and the following relations hold for all values* $t \in (a, b)$, $T \in (A, B)$; $-\infty < Y, Z < \infty$

$$\dot{x}(T) \neq 0; \quad f(t, 0, 0) = 0; \quad f_Y(t,\ Y, Z) \neq 0; \quad g_Z(t,\ Y, Z) \neq 0.$$

2°. The mapping $\boldsymbol{t}$ *carries over the solutions of each system of differential equations with continuous coefficients* Q *of the form*

$$\left.\begin{aligned} \dot{Y} &= Z, \\ \dot{Z} &= Q(T)\,Y \end{aligned}\right\} \tag{Q}$$

into solutions of an analogous system

$$\left.\begin{aligned} y' &= z, \\ z' &= q(t)y. \end{aligned}\right\} \tag{q}$$

On these hypotheses, the mapping $\boldsymbol{t}$ *has the following linear form*

$$\begin{aligned} t &= x(T), \\ y &= w(t)\,Y, \\ z &= w'(t)\,Y + w(t)X'(t)Z. \end{aligned}$$

Proof. Consider a system (Q) and one of its solution curves $\mathfrak{K}$: $Y(T)$, $Z(T)$. The image $\mathfrak{K}$ of the latter in the mapping $\boldsymbol{t}$ has the parametric coordinates

$$\mathfrak{k}:\ \ y(t) = f(t,\ Y[X(t)], Z[X(t)]);\quad z(t) = g(t,\ Y[X(t)], Z[X(t)]).$$

From 2°, $\mathfrak{k}$ is a solution curve of the system (q). We have therefore

$$z(t) = y'(t) = f_t(t,\ Y, Z) + f_Y(t,\ Y, Z)\,\dot{Y}[X(t)]X'(t) + f_Z(t,\ Y, Z)\dot{Z}[X(t)]X'(t),$$
$$q(t)\cdot y(t) = z'(t) = g_t(t,\ Y, Z) + g_Y(t,\ Y, Z)\,\dot{Y}[X(t)]X'(t) + g_Z(t,\ Y, Z)\dot{Z}[X(t)]X'(t)$$

and moreover, since Y, Z satisfy the system (Q)

$$z(t) = f_t(t,\ Y, Z) + f_Y(t,\ Y, Z)Z[X(t)]X'(t) + f_Z(t,\ Y, Z)Q[X(t)]\,Y[X(t)]X'(t),$$
$$\begin{aligned} q(t)\cdot y(t) = g_t(t,\ Y, Z) &+ g_Y(t,\ Y, Z)Z[X(t)]X'(t) \\ &+ g_Z(t,\ Y, Z)Q[X(t)]\,Y[X(t)]X'(t). \end{aligned}$$

These relations hold for every solution curve $\mathfrak{K}$ of the system (Q). Since one solution curve of (Q) passes through each point $(T,\ Y, Z) \in S$, we have for $t \in (a, b)$ and for all Y, Z:

$$g(t,\ Y, Z) = f_t(t,\ Y, Z) + f_Y(t,\ Y, Z)Z\cdot X'(t) + f_Z(t,\ Y, Z)Q[X(t)]\,Y\cdot X'(t), \tag{22.1}$$

$$q(t)\cdot f(t,\ Y, Z) = g_t(t,\ Y, Z) + g_Y(t,\ Y, Z)Z\cdot X'(t) + g_Z(t,\ Y, Z)Q[X(t)]\,Y\cdot X'(t). \tag{22.2}$$

In these formulae the continuous function Q can (from 2°) be chosen arbitrarily. It follows that

$$f_Z(t,\ Y, Z)\,Y\cdot X'(t) = 0$$

and moreover, since (from 1°) $X'(t) \neq 0$,

$$f_Z(t,\ Y, Z) = 0.$$

The function f is therefore independent of Z:

$$f(t, Y, Z) = h(t, Y). \tag{22.3}$$

We now have, from (1) and (3),

$$g(t, Y, Z) = h_t(t, Y) + h_Y(t, Y)Z \cdot X'(t)$$

and moreover

$$\begin{aligned} g_t(t, Y, Z) &= h_{tt}(t, Y) + h_{tY}(t, Y)Z \cdot X'(t) + h_Y(t, Y)Z \cdot X''(t), \\ g_Y(t, Y, Z) &= h_{tY}(t, Y) + h_{YY}(t, Y)Z \cdot X'(t), \\ g_Z(t, Y, Z) &= h_Y(t, Y) \cdot X'(t). \end{aligned}$$

These formulae show, on taking account of (2), that

$$\begin{aligned} q(t) \cdot h(t, Y) = h_{tt}(t, Y) &+ 2h_{tY}(t, Y)Z \cdot X'(t) + h_{YY}(t, Y)Z^2 \cdot X'^2(t) \\ &+ h_Y(t, Y)[Z \cdot X''(t) + X'^2(t) Y \cdot Q[X(t)]]. \end{aligned}$$

Thence we obtain, by differentiating twice partially with respect to Z,

$$2h_{tY}(t, Y) \cdot X'(t) + 2h_{YY}(t, Y)Z \cdot X'^2(t) + h_Y(t, Y) \cdot X''(t) = 0,$$
$$h_{YY}(t, Y) = 0.$$

This shows that the function h is linear with respect to Y:

$$h(t, Y) = w(t) Y + a(t).$$

From 1° it follows that $a(t) = 0$, and hence we obtain the linear form of the function f:

$$f(t, Y, Z) = w(t) \cdot Y.$$

Finally, from (1) we obtain

$$g(t, Y, Z) = w'(t) Y + w(t)X'(t)Z,$$

and this completes the proof.

22.2 Introduction of the differential equation (Qq)

We consider two differential equations (q), (Q) over arbitrary intervals of definition $j = (a, b)$, $J = (A, B)$:

$$y'' = q(t)y; \tag{q}$$
$$\ddot{Y} = Q(T) Y. \tag{Q}$$

Let ($[w, X] =$) $w(t)$, $X(t)$ be a transformation of the differential equation (Q) into the differential equation (q). The functions w, X are therefore defined in a sub-interval

$i \subset j$ and have the properties 1–3 set out in § 11.2. In particular, the range of X, $I = X(i)$ is therefore a sub-interval of J: $I \subset J$.

We know (§ 11.2) that the transforming function X satisfies the non-linear differential equation of the third order:

$$-\{X, t\} + Q(X)X'^2 = q(t) \qquad \text{(Qq)}$$

in its interval of definition i. We also know that the multiplier w of $[w, X]$ is determined from the function X uniquely except for a multiplicative constant k ($\neq 0$), by means of the formula (11.12).