29 A survey of recent results in transformation theory

In: Otakar Borůvka (author); Felix M. Arscott (translator): Linear Differential Transformations of the Second Order. (English). London: The English Universities Press, Ltd., 1971. pp. [237]–242.

Persistent URL: http://dml.cz/dmlcz/401701

29 A survey of recent results in transformation theory

In this section we shall survey recent developments in the field of Jacobian differential equations which have reference to the theory of transformations. We shall be particularly concerned with the theory of dispersions and its applications, especially in the field of central dispersions, and with the generalization of parts of the theory of transformations.

29.1 General dispersions

The theory of general dispersions of two differential equations (q), (Q) has been extended in connection with the study of the phase group $\mathfrak{G}$ (§ 10 and § 20.7). In particular a new characterization of such dispersions has been discovered, and all the general dispersions common to two pairs of differential equations (q_1), (Q_1) and (q_2), (Q_2) determined.

We denote by $I(Qq)$ the subset of $\mathfrak{G}$ comprising all general dispersions of (q), (Q). Then, as we know, $I(qq)$ and $I(QQ)$ are the subgroups conjugate with $\mathfrak{E}$ with respect to arbitrary phases α, $\mathbf{A}$ of (q), (Q) respectively. The fresh information relating to the characterization of general dispersions of (q), (Q) lies in the theorem that the elements $\xi \in I(Qq)$ are characterized by the relationship $\xi^{-1}I(QQ)\xi = I(qq)$. [2*]

With regard to common general dispersions of the above-mentioned pairs o equations, such common dispersions have been shown to exist if and only if all phases of the differential equations (Q_{12}), (q_{12}) are contained in one and the same element of the smallest common covering of the two partitions $\mathfrak{G}/_l\mathfrak{E}$, $\mathfrak{G}/_r\mathfrak{E}$; the carriers Q_{12}, q_{12} are determined by the fact that their phases include the functions $\mathbf{A}_1\mathbf{A}_2^{-1}$, $\alpha_1\alpha_2^{-1}$ formed from arbitrary phases α_1, $\mathbf{A}_1$, α_2, $\mathbf{A}_2$ of q_1, Q_1, q_2, Q_2. [3*]

Further problems involving general dispersions arise in connection with properties of the structure of the phase group $\mathfrak{G}$, and are discussed in [4*] and [5*]. In particular, it has been shown that different concepts from the theory of general dispersions, apparently remote from each other, are related to the centre $\mathfrak{Z}$ of the subgroup of $\mathfrak{E}$ comprising the increasing elements of $\mathfrak{G}$. For instance, the subgroup $\mathfrak{H}$ of the elementary phases (§ 10.4) is the normalizer of $\mathfrak{Z}$ in $\mathfrak{G}$. [5*]

A comparison theorem for general dispersions was given in [39*].

29.2 Dispersions of the 1st and 2nd kinds

New advances have been made in [8*] in the theory of dispersions of the 1st and 2nd kinds. The subgroup $\mathfrak{P}$ ($\subset I(qq)$) comprising the increasing dispersions of $I(qq)$ is the union of three disjoint subsets D_1, D_2 and D_3. The first of these, D_1, consists of the single dispersion $\xi(t) \equiv t$ $(t \in j = (-\infty, \infty))$ which, of course, is a dispersion for every oscillatory differential equation (q). The set D_2 comprises those dispersions for which $\xi(t) \neq t$ $(t \in j)$, and the final set, D_3) comprises all other dispersions ξ. The properties of these functions in the various cases are examined; a particular problem studied is to find the conditions under which a dispersion ξ determines the carrier q uniquely. It transpires that this is the case if and only if the dispersion belongs to D_3. Another question considered is the extent to which the subgroup generated by certain dispersions of $\mathfrak{P}$ serves to determine the carrier q. This involves complicated study and can only be answered in special cases; for instance, a continuous one-parameter subgroup in $\mathfrak{P}$ essentially determines q uniquely. These researches can be extended to cover dispersions of the 2nd kind. New problems arise if we consider simultaneous dispersions of the 1st and 2nd kinds; complicated relations occur among these. The only such result to be mentioned here is that the differential equation (q) is determined uniquely "in the general case" by its fundamental dispersions of the 1st and 2nd kinds.

29.3 Central dispersions

In the topic of central dispersions, attention has been directed mainly to central dispersions of the first kind in connection with problems of a different sort. The applicability of central dispersions of all four kinds rests on the fact that, in the first place, they admit of clear geometrical and analytical interpretations, and in the second place that they connect the values taken by integrals and their derivatives (of the relevant Jacobian differential equations) at conjugate and hence distant points; this makes it possible to treat problems of a global character.

This paragraph describes (A) some boundary value problems (B) asymptotic behaviour (particularly boundedness) of integrals and (C) some geometrical studies.

(A) *Boundary value problems*

(i) Consider an oscillatory differential equation $y'' = q(t, \lambda)y$, with $q(t, \lambda) \in C_0$, $t \in j = (-\infty, \infty)$ for every $\lambda > 0$. For such an equation every value λ_0 of λ represents the n-th eigenvalue of the problem

$$y(t_0, \lambda_0) = y(\phi_n(t_0, \lambda_0), \lambda_0) = 0; \qquad (t_0 \in j)$$

here, of course, $\phi_n(t, y)$ denotes the n-th central dispersion of the first kind ($n = 1, 2, \ldots$). [15*]

In the following cases

$$q(t, \lambda) = \lambda q(t), \qquad q(t) < 0,$$
$$q(t, \lambda) = q(t) + \lambda, \qquad q(t) < 0,$$

and in particular for periodic functions q, it has been possible to find bounds for the elements occurring in the boundary value problem, these bounds being partly of a novel structure. [10*], [14*]

(ii) Using phase theory as the basic tool, in [17*] a necessary and sufficient condition is obtained for the existence of a complete generalized Liouville transformation, which takes solutions of

$$\ddot{Y} = (\lambda R(T) + Q(T))Y \qquad \text{on } (A, B)$$

into solutions of

$$y'' = (\lambda r(t) + q(t))y \qquad \text{on } (a, b)$$

over their whole domains of definition.

As a special case of this theory, a study is made in [18*] of differential equations $\ddot{Y} = \lambda R(T)Y$ and $y'' = (q(t) + \lambda)y$, with periodic carriers, in $t \in (-\infty, \infty)$.

(iii) The two-parameter boundary value problem of F. M. Arscott:

$$y'' + [q(t;\lambda, \mu) + r(t)]y = 0, \quad y(a) = y(b) = y(c) = 0$$

and the special case

$$y'' + [\lambda q(t) + \mu s(t) + r(t)]y = 0, \quad y(a) = y(b) = y(c) = 0,$$

with $q(t;\, \lambda, \mu) \in C_0$ and $q(t) \in C_0$ respectively, for $t \in [a, c)$ and $b \in (a, c]$, was examined in [9*]. For given positive integers n_1, n_2 with $n_1 < n_2$, there exist (under fairly general conditions) eigenvalues of λ, μ such that b is the n_1-th and c the n_2-th zero of $y(t)$ subsequent to a.

(B) *Behaviour of integrals*

We consider an oscillatory differential equation (q), $y'' = q(t)y$, $t \in (a, b)$. Using the theory of central dispersions a study has been made of the connection between, on the one hand, the distribution of zeros of solutions as $t \to b-$ and, on the other hand, the boundedness or the asymptotic behaviour of these solutions.

(i) Let ϕ, $\bar{\phi}$ be respectively the fundamental dispersions of the first kind of (q) and ($\bar{\text{q}}$) and let ϕ_n be the n-th iterate of ϕ ($n = 1, 2, \ldots$). The following results have been obtained:

Every solution of (q) is bounded on $[t_0, b)$ if and only if $\phi_n'(t)$ is bounded on $[t_0, \phi(t_0)]$ for all positive integers n. [58], [19*]

Every solution of (q) belongs to $L^2[t_0, b)$ (i.e. is such that $\int_{t_0}^{b} y^2(\sigma)\, d\sigma < \infty$) if and only if [21*]

$$\sum_{n=0}^{\infty} \int_{t_0}^{\phi(t_0)} \phi_n'^2(\sigma)\, d\sigma < \infty.$$

If $\phi'(t)$ is bounded above on $[t_0, b)$ away from 1, i.e. $\phi'(t) \leqslant k < 1$, where k is a constant, then $b < \infty$ and every solution y of (q) tends to zero as $t \to b-$; every such solution also belongs to $L^p[t_0, b)$ for every $p > 0$ (i.e. $\left[\int_{t_0}^{b} |y(\sigma)|^p\, d\sigma\right]^{1/p} < \infty$. [19*]

If $\phi'(t) \leqslant 1$ on $[t_0, b)$ then every solution of (q) is bounded on $[t_0, b)$. [19*]

If $\phi'(t) = 1$ on $[t_0, b)$ then $b = \infty$ and every solution of (q) is periodic. [20*]

If $\phi'(t) \geqslant \text{constant} > 1$ on $[t_0, b)$, then $b = \infty$ and every non-trivial solution of (q) is unbounded on $[t_0, b)$. [19*]

(ii) Now let (q) and ($\bar{q}$) have the same dispersion ϕ. Then if any of the following statements is true for (q), the same is true for ($\bar{q}$):

(a) every solution is bounded on $[t_0, b)$.
(b) every solution is periodic on $[t_0, b)$.
(c) every solution belongs to $L^2[t_0, b)$.
(d) one non-trivial solution belongs to $L^2[t_0, b)$ but every non-trivial solution which is not a constant multiple of this solution does not belong to $L^2[t_0, b)$.
(e) no non-trivial solution belongs to $L^2[t_0, b)$.

Moreover, if (d) holds, then every solution of both (q) and ($\bar{q}$) belonging to $L^2[t_0, b)$ has the same zeros [19*] and [21*].

In the papers quoted above there are generalizations of these results to L^p classes and also some comparison theorems.

(iii) In [22*], [56] the theory of phases and central dispersions is used to construct all stable periodic differential equations (q), $q \in C_0$, $t \in (-\infty, \infty)$.

Differential equations (q), $q \in C_0$, $t \in (-\infty, \infty)$ with only periodic solutions are shown in [20*] to be characterized by the relation $\phi_n(t) = t +$ constant. A criterion for the periodicity of all solutions of (q) is given in [57] and an explicit form for all such differential equations can be found in [23*]. In [5*] there is an explicit formula for all differential equations (q) with $\phi_n(t) = t + m\pi$, m, n positive integers.

(iv) The first formula (5.5) can be used to construct all differential equations (q) with $\phi_n(t) = t +$ constant, even for carriers which are defined as generalized derivatives of one-sided continuous functions [14*]. The methods used can be extended to the case of linear differential equations of the n-th order [13*]. The geometrical significance of $r(t)$ can be applied to characterize periodic solutions of $y'' + p(t)y - cy^{-3} = 0$ by means of the Liapunov resolvent of $y'' + p(t)y = 0$ [12*].

(v) Oscillatory differential equations (q) in which the derivative $q'(t)$ is a monotonic function of order n, (i.e. $(-1)^i q^{(i+1)}(t) \geqslant 0$, $i = 0, 1, \ldots, n$; $n \geqslant 1$; $t > 0$) are studied in [37*], [38*]. For a differential equation of this kind the zeros t_k of an integral form a monotonic sequence of order n (i.e. $(-1)^i \Delta^{i+1} t_k \geqslant 0$, $k = 0, 1, 2, \ldots$). Theorems have been discovered governing connections between zeros of the integrals of such differential equations and their derivatives, some of which have applications to Bessel functions.

(C) *Geometric studies*

In [20*] a study is made of the close connection between differential equations (q), $q \in C_0$, $t \in (-\infty, \infty)$ with $\phi_n(t) = t +$ constant, and closed (but not necessarily simple closed) curves. This paper also introduces a generalization of the (centro) affine length of the arc of a curve u between the points with parameters t_0 and t_1; the formula is

$$^{c}\Lambda_{t_0}^{t_1}(u) = \operatorname{sgn}\,[u, u'] \int_{t_0}^{t_1} \frac{|[u', u'']|^c}{|[u, u']|^{3c-1}}\, d\sigma$$

or

$$^{c}_{*}\Lambda_{t_0}^{t_1}(u) = \int_{t_0}^{t_1} \frac{\operatorname{sgn}\,[u', u''] \cdot |[u', u'']|^c}{|[u, u']|^{3c-1}}\, d\sigma$$

where $[u, u'] = u_1(\sigma)u_2'(\sigma) - u_1'(\sigma)u_2(\sigma)$, etc. These formulae generalize the formulae of W. Blaschke, O. Borůvka and L. A. Santaló, for which the parameter c takes the particular values $c = \frac{1}{3}$, $c = \frac{1}{2}$, $c = 1$ respectively.

Using integral inequalities for coefficients of differential equations (q) with periodic solutions, derived in [24*] and [25*], the following generalization of the isoperimetric theorem of W. Blaschke and L. A. Santaló was obtained in [25*]:

Let u be a closed plane curve of class C_2 with index n. Let there exist a straight line p through the origin which intersects u in two points symmetrically located with respect to the origin. Let the areas bounded by all simple arcs of u and the line p be the same. Let ${}^c\Lambda(u)$ or ${}_*^c\Lambda(u)$ denote the length of u and let $V(u)$ be the area enclosed by u (the index of u being taken into account). Then

$$ {}^c\Lambda(u) = {}_*^c\Lambda(u) \leqslant 2[V(u)]^{1-2c}(n\pi)^{2c} $$

for $c \in (0, 1]$. If c, $V(u)$ and the index of u are fixed, the maximum length is attained precisely for ellipses with centre at the origin, these being considered as curves of index n.

Further results in centroaffine geometry of plane curves can be found in [6*] and [20*]; results on periodic curvature and closed curves are in [26*], and the impossibility of extending L. A. Santaló's theorem to general simple closed curves is shown in [27*]. A survey of these and further results on differential equations and plane curves is to be found in the lecture notes [28*].

A theorem of W. Blaschke states that a C_2 closed curve has at least three distinct pairs of points with parallel tangents and equal radii of curvature. In connection with this, the following theorem was proved in [15*]:—

Let the central dispersion ϕ_2 of (q) be linear and of the form $\phi_2(t) = t + T$ (hence $q(t + T) = q(t))$, $t \in (-\infty, \infty)$. Further, let $f(t)$ be continuous in an interval $[\tau, \phi_2(\tau)]$. If $f(\tau) = f(\phi_2(\tau))$, and for every integral y of (q) the property

$$ \int_\tau^{\phi_2(\tau)} f(\sigma)y(\sigma)\, d\sigma = 0 $$

holds, then in the interval $[\tau, \phi(\tau))$ there exist at least three distinct values t_i ($i = 1, 2, 3$) such that

$$ f(t_i)\phi'(t_i)^{-\frac{3}{2}} = f(\phi(t_i))\phi'(\phi(t_i))^{-\frac{3}{2}} \quad (\phi = \phi_1). $$

If $\phi(t) = t + T$ and hence $\phi_2(t) = t + 2T$, then this relation implies the existence of at least three distinct values $t_i \in (\tau, \tau + T)$ for which $f(t_i) = f(t_i + T)$.

We have also the further result: let the functions $f(t)$, $g(t)$ be continuous for $t \in (-\infty, \infty)$, $g(t) > 0$, and periodic with period T, where $T \leqslant \phi_2(\tau)$, ϕ_2 being the (general) central dispersion of (q). Also, let every integral y of (q) satisfy the condition

$$ \int_\tau^{\tau+T} f(\sigma)y(\sigma)\, d\sigma = \int_\tau^{\tau+T} g(\sigma)y(\sigma)\, d\sigma = 0. $$

Then the function $f(t)/g(t)$ has at least four relative extrema in the interval $[\tau, \tau + T)$.

A geometrical interpretation of the result of § 15.6 is given in [10*].

29.4 Generalizations

In [14*], [15*] some results from transformation theory of Jacobian differential equations (q) are generalized to differential equations of the form $y'' + Q'(t)y = 0$, $t \in [a, b]$, where Q' is the generalized derivative of a function Q in the sense of distributions.

The integrals of a differential equation (q) form a two-dimensional linear space R of continuous functions. Consequently the Kummer transformation can be regarded as a transformation of one space R_1 into another space R_2. The theory developed in [29*]–[36*] includes the question of how far the transformation theory of differential equations (q) can be taken over to two-dimensional spaces of continuous functions.

A study of the concept and the properties of central dispersions in an abstract group is to be found in [1*].