The growth of mathematical culture in the Lvov area in the autonomy period (1870–1920)

Special cases

In: Stanislaw Domoradzki (author): The growth of mathematical culture in the Lvov area in the autonomy period (1870–1920). (English). Praha: Matfyzpress, 2011. pp. 236–279.

Persistent URL: `http://dml.cz/dmlcz/402159`

CHAPTER V

SPECIAL CASES

5.1. Introduction

This chapter contains considerably more mathematics than the previous ones. Here we present an analysis of Puzyna's work *Teorya funkcyj analitycznych* [Theory of analytic functions], the content of mathematical publications in the journal *Kosmos* (the organ of the Natural Scientists' Society in Lvov) as well as problems of sessions of the Lvov Mathematical School. In this chapter we will also mention the figure of Lucjan Böttcher, a mathematician of the Lvov Polytechnic School. Nowadays he is known as one of the pioneers of the modern theory of iteration and dynamics in the complex plane.

Also, one of the aims of this chapter is to demonstrate the spectrum of mathematical theories that were known to and used by the mathematicians of that time.

5.2. Puzyna's work *Teorya funkcyj analitycznych*: an attempt of discussion

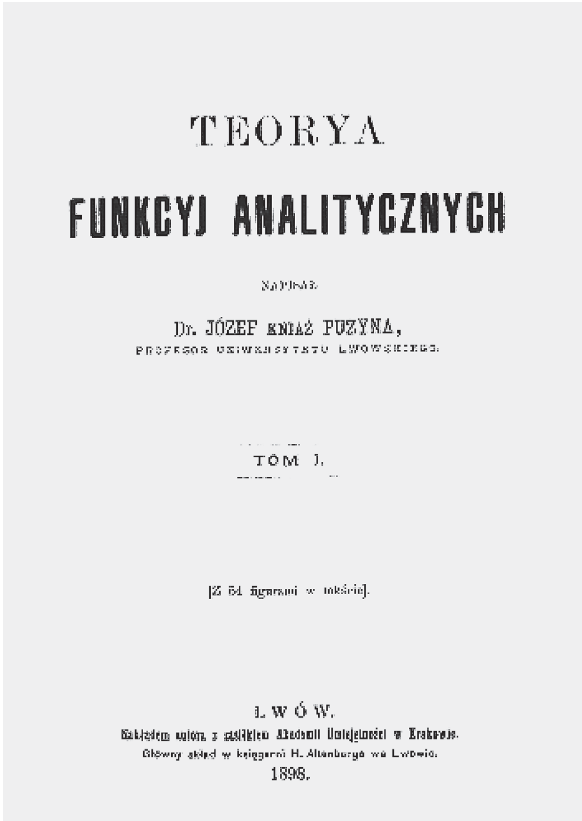
From the modern point of view, the two-volume monograph by J. Puzyna is an outstanding event in the history of mathematics in Lvov, in particular in the period on which the present book is concentrated.

Już tom pierwszy bogactwem swej treści i samodzielnem opracowaniem przedmiotu, zwrócił uwagę zagranicy, wywołując żal, że dzieło napisane w niezrozumiałym (!) języku. „Naturae Novitates“ berlińskie w nr. 8. b. r. piszą o tem dziele na str. 273 dosłownie tak: „Aus einem uns mit diesem umfangreichen und schön ausgestatteten Werke zugesandten französischen Referate ersehen wir, dass wir es hier mit einem gründlich ausholenden und von grossem Fleisse und besonderer Belesenheit zeigendem Werke des Ordinarius der Lemberger-Universität zu thun haben. Keine blosse Übersetzung, sondern vom Anfang bis zum Ende Originalarbeit. Um so bedauerlicher ist es, dass das Buch in einer unverständlichen Sprache geschrieben ist, so dass es für die Welt (?) mit Ausnahme eines ganz kleinen Kreises einfach nicht existirt. Wird diese Übertreibung (!) des Nationalgefühls nicht einmal der besseren Überzeugung weichen, dass die Wissenschaft immer international bleiben wird und dass es nur dem Autor selbst und seiner Nation schadet, wenn sein, noch so verdienstvolles Werk, schon als Makulatur (!?) geboren wird“. Miejmy nadzieję, że z drugim tomem gniew się ułagodzi a może i Niemcy zapoznać się zechcą o tyle z językiem polskim, aby mogli korzystać z dzieła polskiego profesora, który może dokładniej poznał i zrozumiał teoryę Weierstrassa, niż niejeden z uczonych rodaków tego mistrza.

Dr. Placyd Dziwiński.

A fragment of Dziwiński's report in *Kosmos*.

Let us start with a discussion of Puzyna's work on the pages of *Kosmos* by P. Dziwiński, which refers to the German reviews Naturae Novitates. The author expresses regrets of German scholars that this work was published in Polish.


TEORYA

FUNKCYJ ANALITYCZNYCH

NAPISAŁ

Dr. JÓZEF KNIAŹ PUZYNA,

PROFESOR UNIWERSYTETU LWOWSKIEGO.

TOM I.

[Z 54 figurami w tekście].

LWÓW.

Główny skład w księgarni H. Altenberga we Lwowie.

1898.

Title page of the work.

In the literature on history of mathematics, one cannot find many publications devoted to the analysis of Puzyna's monograph. Perhaps the most complete one is that of A. Płoski (Płoski 1998) in Polish. First of all, the place of Puzyna's monograph among the existing textbooks in analysis (at least European ones) is characterized. Since the famous Osgood's monograph on the function theory was published only in 1906 and the previous books (by Cauchy, Jordan, Picard) were rather general courses in analysis, Puzyna's book was the pioneering work in analysis completely devoted to the theory of analytic functions.

Puzyna's monograph is of encyclopedic character. In the Introduction to Vol. I, Puzyna briefly described his motivation to write such a monograph. First of all, lecturing as a professor at the Lvov University, he realized the need of a book that could acquaint the reader with the theory of analytic functions, which forms a basis for understanding of classical and modern works, on one hand, and, on the other hand, could give a possibility of reaching to new, not yet investigated, analytic relations. Therefore Puzyna decided to use his teaching experience and material of the other authors working in this area and to publish Volume I of the *Theory of analytic functions*.

The author decided to place in the Vol. I also the introductory material from higher Algebra and the Function Theory. The reason for this was that some topics of these disciplines, in particular, the set theory, symmetric functions, multi-valued functions, eliminations, groups and the functions of regular (*umiarowe* in Polish) polyhedral forms were completely rewritten by Puzyna and he did not repeat what was published before.

Having in mind the future applications to the expositions of all contemporary theory in two volumes, the author deals in the Vol. I only with rational functions of one and several variables. There, characteristic properties of these functions were developed, in particular, special attention was given to the conditions necessary for determining these new directions of applications.

Two concluding parts were devoted to the theory of power series, which forms the background for the whole theory of analytic functions. In the Vol. I only the most general classification of these functions is presented: single–valued and irrational functions. There was also a mention of functions with gaps and analytic expressions that can be represented by different functions in different domains.

The material of Vol. I does rely neither on the integral equations nor the differential equations. The exposition follows the method of Weierstrass and is also related to mathematicians who worked in this direction: Stolz, Biermann, Pringsheim, Tannery, Poincaré, Borel, Bendixon, Dautheville and others.

Let us describe briefly the content of Volume I.

Part 1. On numbers, variable quantities and sets.

Chapter I. “ From arithmetic’s” consists of the following topics:

Definition of real numbers, rationales irrationales, and their systematic expansions.

Expansion of a rational numbers is infinite periodic form. Infinite non-periodic expansions.

Definition of irrational numbers by means (view) of series of rational numbers tending to infinity.

Arithmetic operations with irrational numbers. Series of positive numbers, their divergence and convergence.

Arithmetic operations for series of positive summands.

Practical conditions of convergence.

Series of positive and negative terms. Definition of unconditional convergence.

Conditional convergence. Examples.
Convergence of series with sign-charging terms.
Oscillating series.
Theorems of Abel and Dirichlet.
Infinite products with real factors. A necessary condition of their convergence.
Unconditionally convergent products.
Coexistence of unconditional and absolute convergence.
Conditionally convergent products.
Expansion of real numbers into chain fraction.
Finite and infinite continuous fractions.
Systematic expansion of Strauss.

Chapter II. “On complex numbers” contains the following topics:
Definition of complex numbers by means of arithmetic operations.
Absolute value of a complex numbers.
Theorems on absolute value of the sum, difference, product and ratio.
Geometric representation of complex numbers.
Numerical plane. Geometric description of arithmetic operations.
Introduction of the symbol 1_ϕ.
Formulas of Moivre and Euler.
Computing the *n*-th root of a complex numbers, for $n = 2q$.
Algebraic form that provides an approximation for arbitrary *n*.
The *n*-th root of the unit +1. Prime roots of the unit.
Two theorems on prime roots.
Cases of algebraic solutions of the equation $w_n - 1 = 0$.
Geometric construction performed by means of straightedge and compass.
Remark on the general power $(a + b\mathrm{i})^{\alpha + \beta\mathrm{i}}$.
Infinite series with positive terms. Their conditional and unconditional convergence.
Infinite products with complex factors. Various forms of their convergence.

Chapter III of Volume I contains the material from the set theory. It is worth mentioning that it was the very first exposition of the fundamentals of set theory in Polish. (The name “teoria mnogości” for “set theory” belongs to Puzyna; in modern Polish terminology, “zbiór” is used for “set”). Let us describe the content of this chapter. It is important to note that the chapter contains a material which belongs to backgrounds of general topology.

A real variable and real variables. Their bounded and unbounded domains. Loci and neighborhoods.

Domain of one imaginary variable *x*.

Different methods of bounding of the domain of one imaginary variable. Locus of the domain. Neighborhood of the locus.

Point at infinity.

Neighborhood of a point lying in finite (domain) or infinity.

Domains of several imaginary variables.

Infinite set of removed loci. Accumulation points.

Derivative sets. Sets of the first and second type. Examples.

The set of isolated loci. Closed set. (Everywhere) dense set. Relation of the derivatives to the given set.

Transfinite numbers. Derivative sets with transfinite coefficients.

Sets of the first cardinality or countable sets. Examples.

Sets composed of countable sets.

Sets of higher cardinalities. Sets of the second cardinality in a single domain.

Set in an n-tuple domain. Continuous domains (continua).

The first and the second (class) sets in an n-tuple domain.

Domain of the points remaining after removing a countable set.

The notion of upper and lower limit.

Stereographic projection of numerical plane onto the sphere.

As it is seen from the table of contents, in this chapter the author introduces some fundamental notions of the set theory. The exposition does not start with the notion of the set but with informal description of a real variable attaining all the values from $-\infty$ to $+\infty$. The set of values is then called a (single) unbounded domain of this variable. Any interval (a,b) of the real line is a (single) bounded domain.

Similarly, the set of all $x = u + v\mathrm{i}$, where u,v run over unbounded real domains, is an unbounded complex (imaginary) domain. Also, the bounded complex domains are defined as those that lie in rectangles. Also defined are real and complex n-dimensional spaces, neighborhoods and points at infinity.

The derivative of a set in a domain is then defined. A set is of the first type if a finite iteration of the operation of the derivative leads to the empty set.

Next, the author deals with infinite sets of arbitrary nature. The notion of a countable set is defined (the countable sets are called the sets of the first class in the monograph). It is explained in detail that a set is countable if is elements can be exhausted by some counting and that not every ways of counting to exhaust the set.

Some basic properties of countable sets are established:

- any family of segments that lie in a given segment and such that their interiors are disjoint is countable;
- any countable set of points whose coordinates can be expressed by means of finitely many parameters each of which runs over a countable set is countable;
- the set of all proper fractions is countable;
- the union of countable many countable sets is again countable;

- the union of a countable set and a finite set is countable;
- any infinite subset of a countable set is also countable.

These properties are applied to the notion of the derivative of the set. We have:

- every set of countable derivative is countable;
- suppose that some derivative of finite order of the set is empty (i.e. the set is of finite order), then the set itself is countable.

It is proved that, for any countable set in the unit cube, there exists a point in the cube such that every its neighborhood contains a point of the set (the so called accumulation point). In modern terminology, this is precisely the proof of compactness of the cube in the Euclidean space.

Proof of compactness of the unit square. As one can guess, the method of the proof is "dividing by halves".

The concept of cardinality is discussed in details and is supplemented with numerous examples. It is proved, in particular, that the unit interval is not a set of countable cardinality.

The following topological notions are also mentioned: accumulation point, the derivative of the set, closed set, dense set, isolated point. The author does not define compactness and uses the term "zwarty" ("compact") in the sense of "connected" or rather "arcwise connected". The connected domains are considered and it is proved that the complement in a connected domain of two variables to a countable set is again connected. The exposition here is not strict with respect to modern mathematical standards, since some undefined notions are used. As an example, we present here the translation of a definition from this part of the book:

A "continuum" P such that from every its place (x_1', ..., x_n') to every other its place (x_1'', ..., x_n'') one can pass only via places belonging to the same "continuum" is called compact or a compact domain. (See the remark above concerning the terminology.)

Also, defined are the derivatives of the set of transfinite orders.

I.		1	2	...	ν...
II.	ω	$\omega+1$	$\omega+2$	...	$\omega+\nu$...
	2ω	$2\omega+1$	$2\omega+2$	...	$2\omega+\nu$...
	⋮	⋮	⋮	...	⋮
	$\nu_1\omega$	$\nu_1\omega+1$	$\nu_1\omega+2$	...	$\nu_1\omega+\nu$...
	⋮	⋮	⋮	...	⋮
	ω^2	ω^2+1	ω^2+2	...	$\omega^2+\nu$...
	$2\omega^2$	$2\omega^2+1$	$2\omega^2+2$	...	$2\omega^2+\nu$...
	⋮	⋮	⋮	...	⋮
	$\nu_2\omega^2$	$\nu_2\omega^2+1$	$\nu_2\omega^2+2$	...	$\nu_2\omega^2+\nu$...
	⋮	⋮	⋮	...	⋮
	$\nu_2\omega^2+\nu_1\omega$	$\nu_2\omega^2+\nu_1\omega+1$	$\nu_2\omega^2+\nu_1\omega+2$	...	$\nu_2\omega^2+\nu_1\omega+\nu$...
	⋮	⋮	⋮	...	⋮
	$\nu_k\omega^k+\ldots+\nu_1\omega$	...	...	...	$\nu_k\omega^k+\ldots+\nu_1\omega+\nu$...
	⋮	⋮	⋮	...	⋮
	ω^ω	$\omega^\omega+1$	...	...	...

The sets of the second class are then defined as the sets whose derivatives of the infinite (but not finite) transfinite order are empty.

The following is an example of a set of the second class. Let P denote the set consisting of the following points:

$$(1)\ (s_1) = \frac{1}{2} + \frac{1}{2^{1+s_1}};$$

$$(2)\ (s_2) = \frac{1}{2} + \frac{1}{2^{1+s_1}} + \frac{1}{2^{2+s_1+s_2}};$$

$$(\mathrm{r})\ (s_r) = \frac{1}{2} + \frac{1}{2^{1+s_1}} + \frac{1}{2^{2+s_1+s_2}} + \ldots + \frac{1}{2^{2+s_1+s_2+\ldots+s_r}},$$

where $s_1, \ldots, s_r$ run over the set of all natural numbers, are examples of the sets of the 1st, 2nd, …, rth order. Then the rth derivative of the subset (s_r) is $\frac{1}{2^r}$, therefore, the derivative of the order ω is nonempty, $P^{(\omega)} = \{0\}$.

Using similar but more complicated construction from Mittag-Leffler's article, Puzyna provides examples of sets P such that $P^{(\omega+\nu)} = \{0\}$ and $P^{(\omega+\nu+1)} = \varnothing$.

The following question is asked in the Chapter: What can be the cardinality of a subset in the n-dimensional real domain? First, the author considers the (closed) n-dimensional cube. It is interesting to note that the notation for this set quite differs from the modern style and is the following:

$(x_1, \ldots, x_n) = (0\ldots1, 0\ldots1, \ldots, 0\ldots1.)$

Also it is also interesting to note that, in Chapter III, a proof that the n-dimensional cube (in modern terminology) and the unit segment are of equal cardinality is given. This question was later asked by W. Sierpiński.

In the footnotes, Puzyna mentions G. Peano's article *Sur une courbe, qui remplit toute un aire plaine*[141]. In this article Peano discusses the problem of existence of continuous maps from the unit segment onto the square.

Some material of Puzyna's monograph is related to the Continuum Hypothesis: there is no cardinality strictly between the countable cardinality and the continuum. To establish whether the Continuum Hypothesis is true was one of the most attractive problems in the set theory. However, at the time of writing the monograph, there were few mathematical publications concerning the Continuum Hypothesis. It was only in 1940 when K. Gödel proved that the Continuum Hypothesis is compatible with the axioms of the set theory. In 1965 P.J. Cohen proved that the Continuum Hypothesis is independent of the axioms of the set theory (he was awarded the Fields medal for this achievement).

Thus, one can see that the Continuum Hypothesis can be solved only in the framework of the axiomatic set theory. In the monograph, Puzyna asserts that the cardinality $P_{\omega\omega}$ distinct of the countable cardinality („the first cardinality",

[141] Mathematische Annalen, 36(1890), p. 157.

in his terminology) appears immediately after all the countable cardinalities. (Actually, he means the ordinal numbers rather than cardinal ones.) In the modern terminology L this is the cardinality ω_1.

Later, it is asserted that the cardinality $P_{\omega\omega}$ is the cardinality of the set of all irrational numbers in the interval (0,1). This means exactly „the Continuum Hypothesis". Most probably, the author did not pay much attention to that topic.

Note that the absence of the topological notion of compactness caused a gap in the proof of the fundamental theorem of algebra.

The chapter devoted to the set theory contains also a description of the stereographic projection map. In modern terminology, this map is a homeomorphism between the plane and the punctured unit sphere. Puzyna uses the term "pokrewienstwo" ("kinship") for this map and speaks on a "circumference kinship" (circumference-preserving homeomorphism) or "isogonal kinship" (conformal map).

Parts II, III and IV of Volume 1 contain the exposition of the necessary material from algebra.

Part II, "On rational functions", consists of the following chapters:

IV: On rational entire functions of one variable.

In particular, this chapter contains the fundamental theorem of algebra. As we already mentioned, the proof contains a gap, since it should be based on some compactness arguments. For this proof, the author cites Cauchy, Gauss as well as more recent publications by Gordan, Holst and Mertens. Then, it is derived that every rational function of degree *m* possesses exactly *m* roots. The case of multiple roots and the roots at 0 and infinity are also considered.

V. On rational fractional functions of one variable.

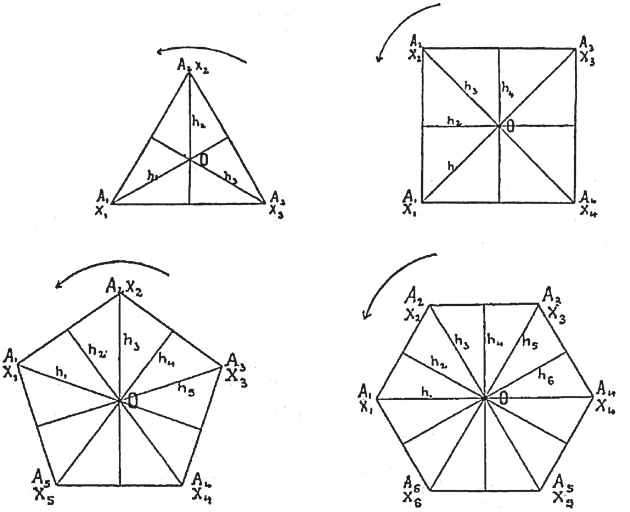
VI. On rational entire and fractional functions of several variables.

Part III is devoted to the symmetric and multiformed functions as well as to the rotations of polyhedra and their functions. The author defines the symmetric functions and proves the fundamental property of symmetric functions, namely that every symmetric function of *n* variables is a rational entire functions $c_1, \ldots, c_n$, where c_i is the sum of all possible products of i variables. Every symmetric function can be represented by means of elementary symmetric functions in a unique manner.

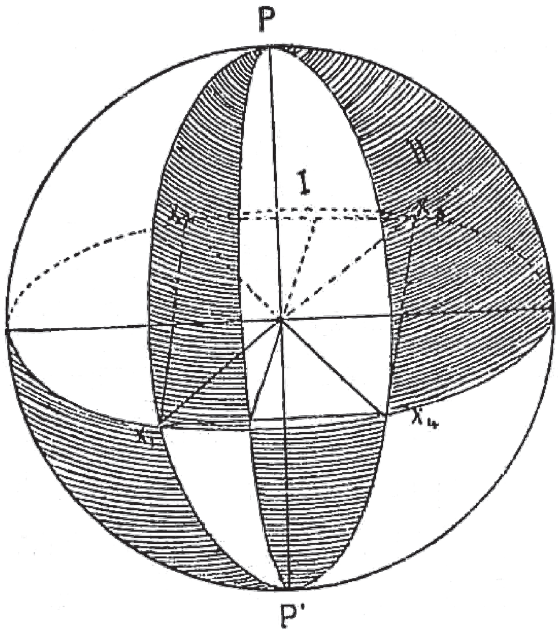
A part of this chapter is devoted to the permutation group. This group is investigated in details. A group is defined as a subgroup of the symmetric (permutation) group. This definition allows us to obtain all the finite groups, because of the Lagrange theorem. As an example, the cyclic groups are considered.

The notion of subgroup, its order and index are discussed.

The group theory is closely connected with the theory of functions of several variables. The group of a function $f(x_1, \ldots, x_n)$ is defined to be the set of all substitutions that do not change the function itself. The Galois type is also defined. In modern language, the fundamental statement of the Galois theory is demonstrated for one extension of $C(x_1, \ldots, x_n)$. The transformation symmetry groups of regular polygons are defined.

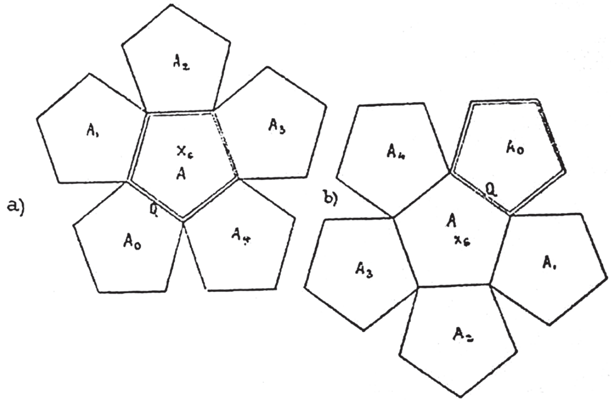


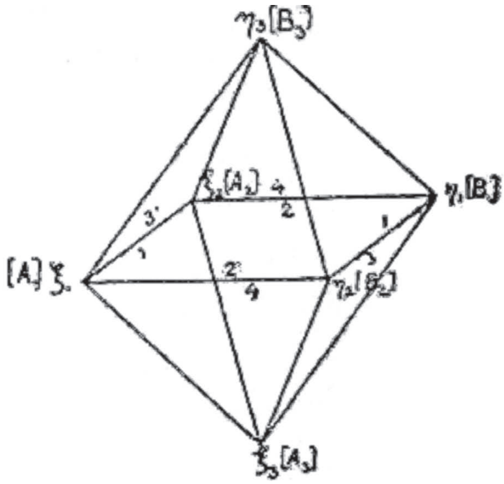
Also, some transformation groups of spheres are considered. For a given transformation group, the notion of fundamental domain is defined.

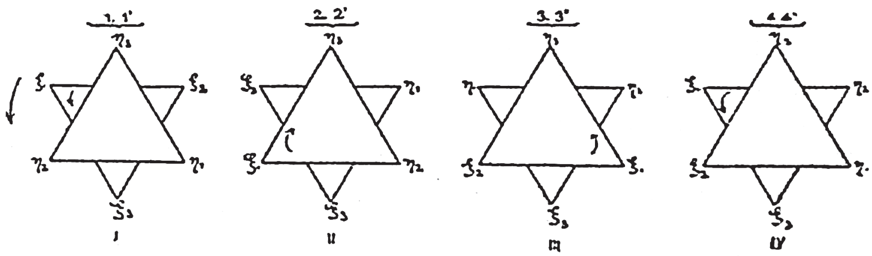


In the picture, a fundamental domain of the action on sphere generated by the action of the rotation group on the equator is a spherical triangle (either black or white).

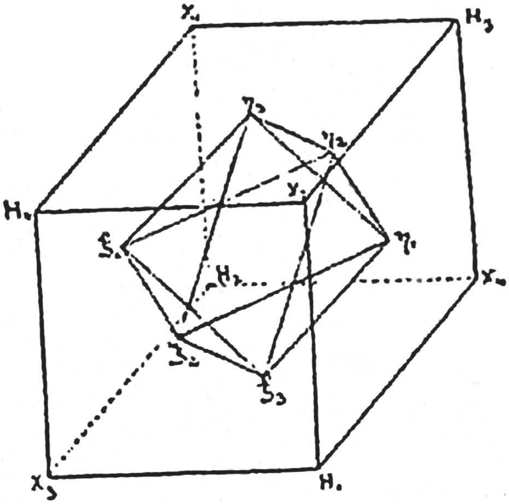
The groups of regular polyhedra are considered; in particular, it is proved that the group of tetrahedron consists of twelve rotations.







The groups of regular polyhedra



Part IV, “On eliminations and theory of binary forms” contains the theory of resultants and elements of the invariant theory.

IX. On eliminations of two equations. Recall that the resultant of two polynomials $P(x)=a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0$
and $Q(x)=b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0$

is the following determinant:

$$\mathrm{R}(P,Q)=\begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_0 & 0 & 0 & \ldots & 0 \\ 0 & a_n & a_{n-1} & \ldots & a_1 & a_0 & 0 & \ldots & 0 \\ 0 & 0 & a_n & \ldots & a_2 & a_1 & a_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \\ b_m & b_{m-1} & b_{m-2} & \ldots & b_0 & 0 & 0 & \ldots & 0 \\ 0 & b_m & b_{m-1} & \ldots & b_1 & b_0 & 0 & \ldots & 0 \\ 0 & 0 & b_m & \ldots & b_2 & b_1 & b_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b_m & b_{m-1} & b_{m-2} & \ldots & b_0 \end{vmatrix}.$$

In the book, Sylvester’s method for forming the resultant of two functions is presented.

It is proved that the resultant of two equations, $f(x,y)=0$ and $g(x,y)=0$ of degrees m,n is of the degree at most mn. The number of common zero sets of two equations of dimensions m,n and the same degrees in two variables is at most mn.

A special case of infinite roots is presented.

Next, the author considers the symmetric functions of all branches of an algebraic function. It is proved here that all the symmetric functions of all the branches of an algebraic function are rational functions of the coefficients of the equation $f(x,y) = 0$ and therefore the rational functions of variable x.

Also, it is shown that the resultant $R(x)$ of two equations, $f(x)=0$, $g(x)=0$, can be represented as a rational symmetric function of all the branches either of the function y of the equation $f=0$ or all the branches of the function g=0.

The case of multiple zeros of the equation of the form $f(x,y) = 0$ is also considered.

Determination of the equation $f(x,y)=0$ from the given zeros of a function f is presented. It is remarked that every algebraic curve of degree m is, in general, completely determined by its $\frac{m}{2}\cdot\frac{m}{3}$ points.

Also, some questions concerning bunches of curves of the m^{th} degree are considered. In particular, the following problem is explicitly formulated: how many ways are there to choose a k^{th} point (x_k,y_k) to given points

$(x_1,y_1),(x_2,y_2),\ldots,(x_{k-1},y_{k-1})$

so that it would form an exclusive system together with these points?

The answer is given by the following statement:

To any $(k-1)$ given points one can find a k^{th} point that forms, together with these points, an exclusive system, in $\frac{(m-1)(m-2)}{2}$ ways.

Finally, the number of points of intersection of two different curves is considered and some estimates are presented.

X. On eliminations of n equations ($n>2$).

It is proved that every symmetric function of the common zeros of two given equations $f(x,y)=0$, $g(x,y)=0$ is a rational function of the coefficients of these two equations. A special case of equations with parameter is considered.

The case of three equations of three variables is considered here, the author introduces the resultant for these data and provides an estimation of the common zeros.

Next, symmetric functions of the common zeros of three equations of three variables are formed.

It is proved that every entire homogeneous and symmetric function of the zeros $(x_1,y_1,z_1),\ldots,(x_n,y_n,z_n)$ of degree m is a rational function of the coefficients of all three equations, is of weight m, and its denominator is a power of the function C_0 and is of weight $=0$.

Finally, the resultant of 4 and more equations is presented. In the subsequent sections, the resultant is presented in the form of a determinant.

The end of this part is devoted to the H. Laurent's interpolation formula. The formula it allows to determine a function of several variables that attains given values at the zeros of given equations.

XI. From the form theory.

Here, the properties of the Hesse and Jacobi determinants are presented. This section also contains the properties of the resultants and discriminants of the homogeneous forms. The invariants of the forms of arbitrary degree are defined and investigated. It is proved, in particular, that the invariant $J(a)$ of the given form f is always a homogeneous function of the coefficients a. The appearance and properties of the invariant of a binary form are presented. A differential equation for the invariant of a binary form is derived. Similarly, covariants and semi-invariants of a binary form are introduced and investigated. The systems of fundamental forms for a cubic and two-quadratic form are presented at the very end of this part.

Part V, "On power series".

XII. On power series and their convergence.

The power series are defined as formal generalizations of polynomials. The main problem considered is that of determination of the domain in which this series is absolutely convergent. The series are classified as those unconditionally convergent on the circle (r) and those that do not display unconditional convergence at any point of this circle.

In addition to these series, the author considers the series $P(x)$ of the argument $1/x$.

Also, the series of several variables, i.e. the series of the form

$$\sum_{(\lambda=0,\, \mu=0,\, \nu=0,\, \ldots)}^{+\infty} a_{\lambda\mu\nu} x^{\lambda} y^{\mu} z^{\nu} \ldots$$

The uniform convergence of series of one and several variables is discussed.

It is proved that any series which vanishes at infinitely many points in its domain of convergence such that this set of points has an accumulation point is identically zero.

XIII. Arithmetic connection of power series.

Here, sums of finitely and infinitely many power series are considered. It is proved that if the sum of countably many series is uniformly convergent then this sum can be replaced by the sum of series of infinitely many variables.

The product and the ratio of series of one and several variables are defined and investigated.

The Newton's series are separately considered.

XIV. Fractional expansions of rational functions.

It is proved that every fractional function can be expanded with respect to increasing powers of its arguments into an inverse series. Conversely, every such series is an expansion of a certain function.

Part VI "Carrying of power series. Definition and the most general subdivision of analytic functions." An algebraic approach to the theory of analytic functions can be seen from this part.

XV. Carrying of power series.

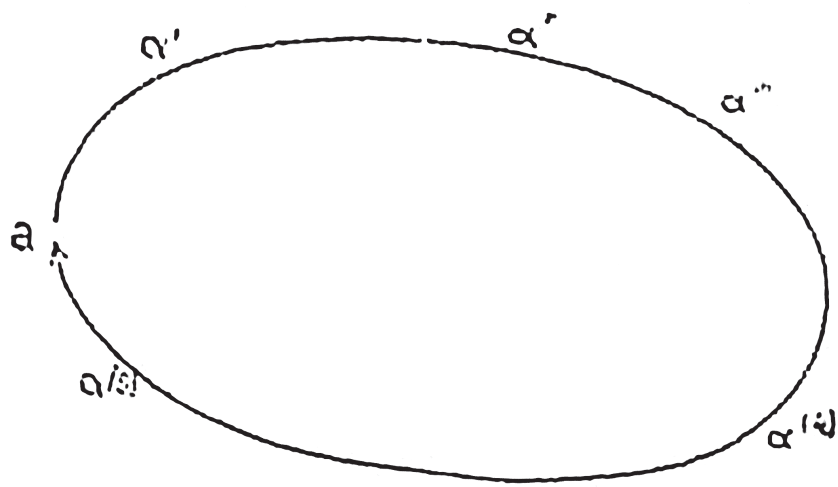
In this section, properties of power series in a complex domain are investigated. The representation of a series $P(x)$ in the form $P_1(x-a)$ in its domain of convergence is presented.

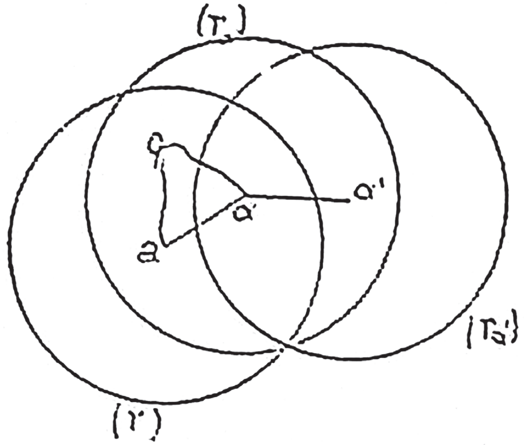
Also, the derivative and primitive series are defined and the following characteristic of the power series in its disc of convergence is proved: for any c, the series attains the value c at infinitely many places if and only if it represents a constant function.

XVI. Definitions and general properties of analytic functions as well as sums of such functions. Here, the author mostly follows the Weierstrass approach to the notion of the analytic function. The Cauchy integral is not used in this chapter.

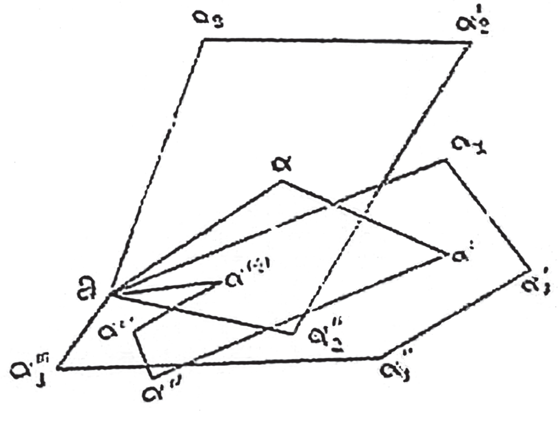
First, the extensions of the series $y+P(x)$ over its domain of convergence is defined.

The analytic and monogenic functions of one or several complex variables are defined, their elements and limits are investigated.





In the picture, we see elements of an analytic function.



XVII. Laurent theorem. The most general subdivision of the univalent analytic functions.

Here, the following famous Laurent theorem is proved. Any analytic function $f(x)$ regular inside an annulus can be expanded in a power series convergent in this annulus.

The classification of singular points of an analytic function of one variable is presented. It is shown that, for any univalent function $f(x)$ which is regular in a closest neighborhood of a point c but is not regular at c, the point c is either inessential or essential singularity. The behavior of a function in a neighborhood of an essentially singular point is described. In particular, it is shown that any function $f(x)$ in a small enough neighborhood of an essential singularity approximates any chosen finite value. The special case $x = \infty$ is also considered by using the transformation $z \to \frac{1}{z}$. The author attributes the results concerning the essential singularities to K. Weierstrass. Also, he quotes Hölder's paper published in Mathematishe Annalen.

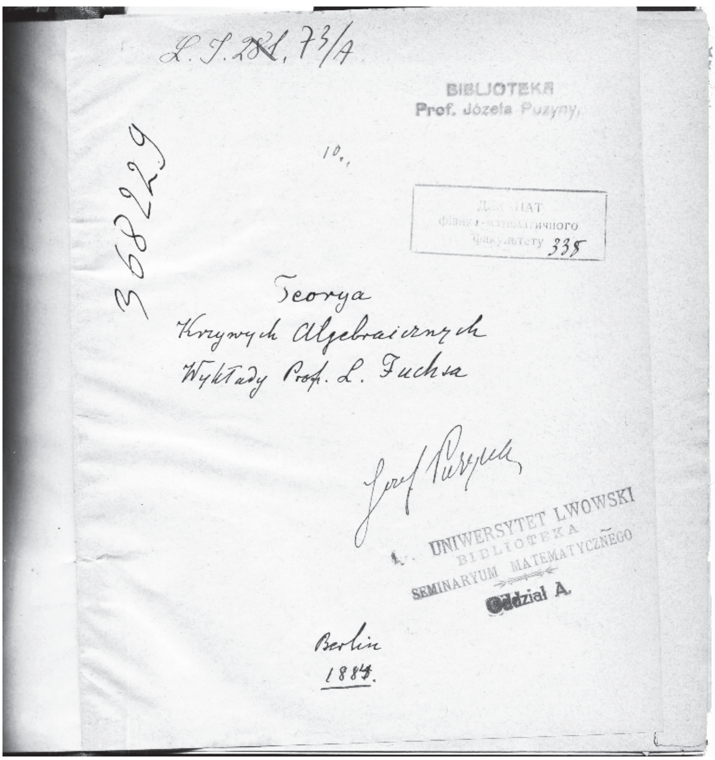
The main result here is the following: If $x=\infty$ is neither a regular point of a univalent function $f(x)$ nor an isolated point, then it is either an infinite point or an essential singularity. As a consequence, the following statement is obtained: Every analytic function that contains no singular point is a constant.

And also: Any analytic function which is constant on an arbitrarily small connected subset (even at every point of an arbitrarily small segment) is constant.

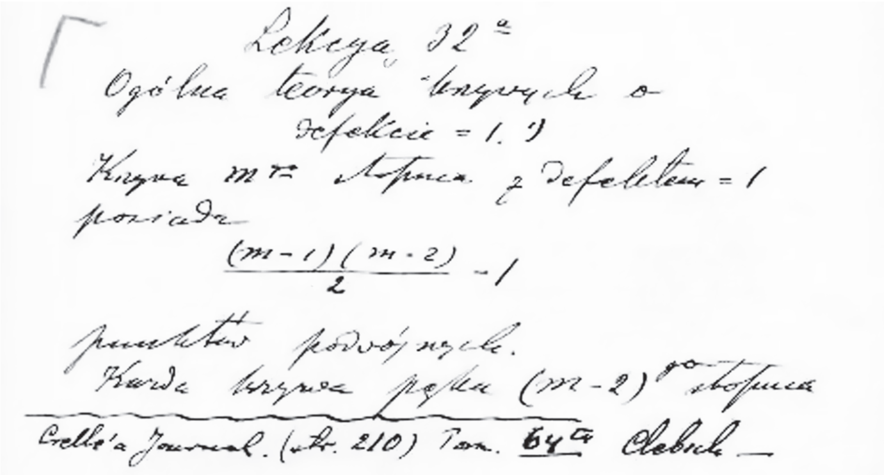
Next, the author describes characteristic properties of the rational functions. It is proven that these functions are characterized by the one of the following properties:

1. Possessing only one essential infinite point at $x=\infty$.
2. Possessing an essentially singular point (for a univalent function).

Any function which possesses the only essentially singular point at $x=\infty$ is an entire transcendental function.



Theory of algebraic curves. Lectures by Prof. L. Fuchs. Berlin 1884.


Lekcya 32a
Ogólna teorya krzywych o defekcie = 1.
Krzywa m-go stopnia z defektem = 1 posiada
$$\frac{(m-1)(m-2)}{2} - 1$$
punktów podwójnych.
Każda krzywa pęku $(m-2)$go stopnia
Crelle'a Journal. (str. 210) Tom. 64 Clebsch

The beginning of one of the lectures on algebraic curves, “The general theory of curves of deficiency = 1”.

The second volume, published in 1900, contains eight parts.

Part I is devoted to the elementary functions as well as the entire transcendental function without zeros. The exponential function, the trigonometric functions and logarithms are defined, their basic properties are proved.

Part II contains the material on univalent functions with finite or infinite number of singularities.

Here, it is shown that any univalent transcendental function with a singularity at infinity with finite number of poles and zeros can be represented by the product with one factor equal to $e^{g(x)}$ and another a rational function $R(x)$ with prescribed zeros and infinite values. If $R(x)$ is entire rational, the function does not contain infinite values and is entire transcendental.

Part III deals with series of several variables and algebraic function of one variable.

Part IV. Rational functions $R(x,y)$ of variables (x,y) of algebraic domain.

Part V deals with Riemann surfaces. The exposition starts with the definition of closed surface and studying topological properties of surfaces by means of their sections by connected simple curves. However, the definition of surface is necessarily not strict as the author avoids using charts, i.e. homeomorphisms onto domains of euclidean spaces.

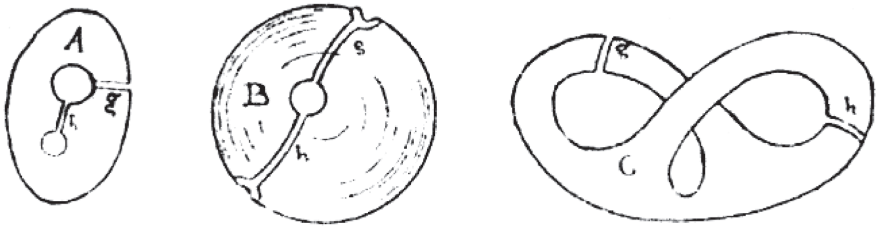
Simple connected surfaces are introduced by means of intuitive definition. These are the surfaces that satisfy the following properties:

1) Every curve connecting two points of the surface can be transformed into another one so that it does not leave the surface in the process of transformation. The endpoints of the curve are either the same or change.

2) Every connected curve contained in the surface can be shrunk to an arbitrary point, while remaining on the surface in the process of shrinking.
3) If the surface possesses the boundary, then every simple (non-self-intersecting) curve that connects two distinct points of the boundary divides the surface into two separate parts.

In modern terms, the author implicitly uses the notion of homotopy (isotopy) of continuous maps in this definition.

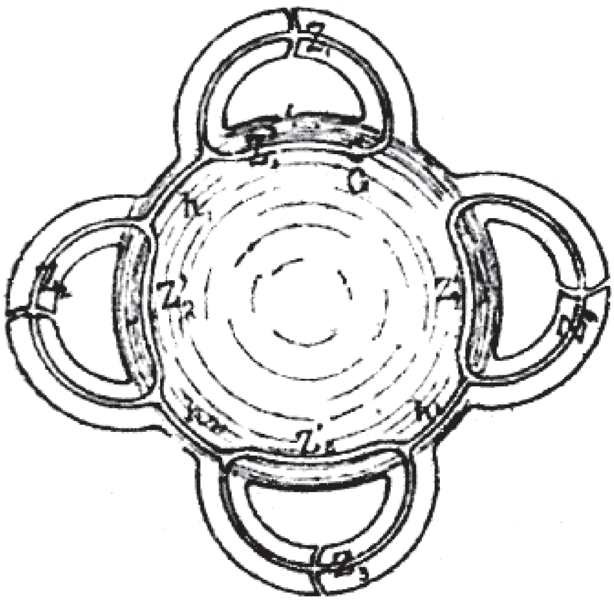
Then n-connected surfaces are introduced. These are the surfaces in which one can make $n-1$ cuts such that the result of cutting is a simply connected surface.



Examples of 2-connected surfaces are: a planar annulus, sphere with two holes etc. The third figure is an example of a 3-connected surface.

The proofs of statements on the surfaces are based on intuitive approach.

Next, the notion of genus of surface is defined.

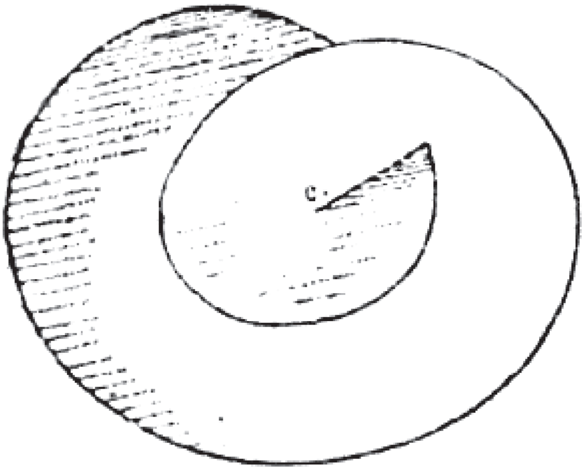


A figure from Puzyna's monograph: Sphere with handles.

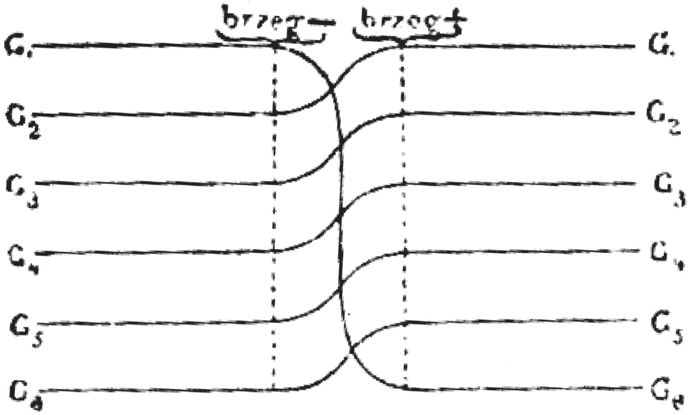
Also, maps (transformations) of surfaces are described and a classification theorem (i.e. that every (oriented) surface is homeomorphic to a sphere with handles) for them is presented. The exposition of this proof is again based on an intuitive approach.

A generalization of Euler's theorem onto (triangulable) surfaces is also given. This allows the author to consider the Euler characteristic of a surface.

The following section contains a description of the construction of the Riemann surfaces, first, at a neighborhood of a branching point. This construction is illustrated by the following pictures.



Neighborhood of a branching point



It is proved that the algebraic notion of genus of any Riemann surface can be also described in topological terms. Actually, the genus is a topological invariant of a surface.

The material also contains various information on algebraic curves. In particular, an analysis of singularities of the algebraic curves by means of the quadratic maps is given.

It is interesting to look on Puzyna's book from the point of view of the unity of mathematics. The introductory parts contain material from the set theory and geometry, as well as algebra, in particular, group theory.

The exposition of the material is rigorous throughout the book. However in some places the style becomes rather narrative when the author deals, e.g., with topology of the plane.

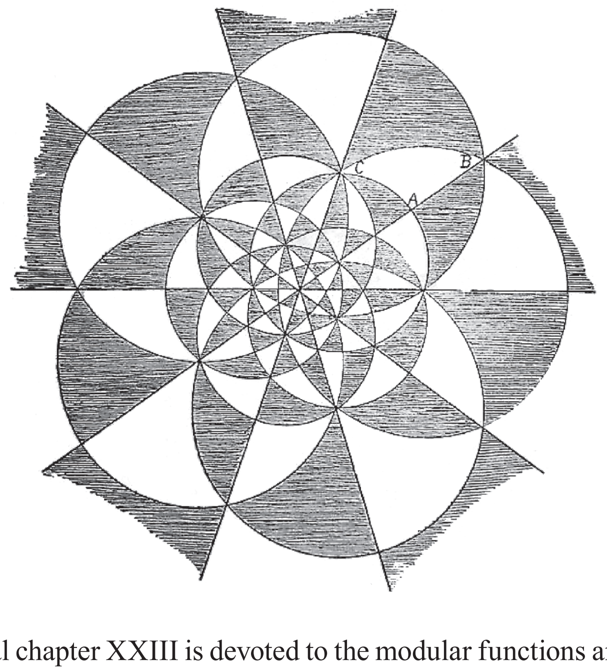
Note that even simply formulated and intuitively evident statement of the planar topology can have complicated proofs, and the famous Jordan theorem is a good example supporting this statement.

Part VI. Integrals of imaginary arguments and residues (Cauchy theory). Periods of Abel integrals. Inversions of algebraic integral.

Part VII. Harmonic functions and their applications. In particular, the harmonic functions on closed Riemann surfaces are considered. Among the applications, there are those to the mappings of two simple connected domains.

Part VIII Schwarz derivative and triangle functions.

— 664 —



The final chapter XXIII is devoted to the modular functions and the modular group.

The following picture illustrates the subdivision of the upper half-plane by means of a modular group.

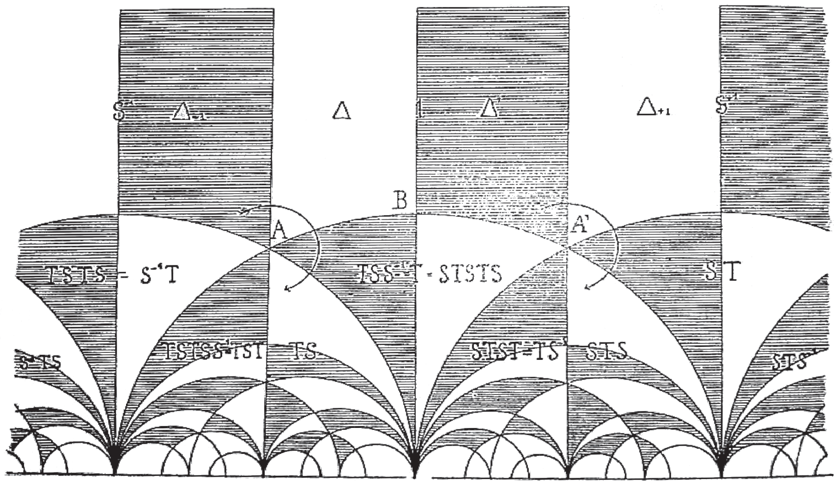


Fig. 121.

In connection to the theory of automorpic functions, Puzyna mentions these and works by Fuchs, Poincaré, Klein, Rausenberg, Hurwitz etc.

We could not find many citations of Puzyna's book in the literature. Perhaps the most important reason for this was that, after its publication, new monographs, which were based on completely new principles, appeared.

In the paper „Topological and metric properties of sets of real numbers with conditions on their expansions in Ostrogradskii series" O. M. Baranovs'kyi, M.V. Prats'ovytyi, and H. M. Torbin (2007)[142] considered the series of the form:

$$\sum_k \frac{(-1)^{k-1}}{q_1 q_2 \ldots q_k} \quad (1)$$

That we cannot find modern citations of Puzyna's monograph does not diminish its importance for the development of complex analysis.

First of all, let us be aware that it has an indirect impact, in particular through later books.

The above mentioned authors remarked that in 1911,[143] considered several expansions of real numbers in series, including expansions (1) and (2). Sierpiński

[142] Ukrainian Mathematical Journal, 59(2007), no. 9.

[143] W. Sierpiński, *Sur quelques algorithmes pour developper les nombres reels en series*, in: *Oeuvres Choisies*, vol. 1, PWN, Warsaw (1974), pp. 236–254.

noted that an expansion of the form (1) were encountered in the monograph *Teorya Funkcyj Analitycznych* (1898) by J. Puzyna.

References of the monograph deserve a special attention. The list includes both classical books (*Cours d'analyse* by A. Cauchy, *Cours d'analyse* by C. Jordan, *Traite d'Analyse* by E. Picard), papers and books of Riemann, Abel, Euler, Gauss, Klein, Mittag-Leffler, Schwarz, J.L. Raabe, F. Franklin, Hermite, Weierstrass, Hilbert, Hurwitz, Gordon, Bachman, Weber, Lipschitz, Gegenbauer, Casorati, Laguerre, Forsyth etc. etc.

Among the cited authors there are also Polish names: W. Żmurko, K. Żorawski, W. Zajączkowski, S. Dickstein, H. Wroński.

Some cited positions appeared few years before the publication of Puzyna's monograph. This definitely witnesses for the fact that Puzyna.

Summing up, we can say that Puzyna's monograph is a clear indication of high mathematical culture at that time in Lvov and one of the greatest scientific phenomena of the period that preceded the activity of the Lvov school of mathematics.

5.3. Mathematical articles, which appeared in Kosmos, the magazine of the (Polish) Copernicus Society of Naturalists

The journal Kosmos was founded by the Polish Society of Naturalists in Lvov in 1876, it led chronicle of scientific societies, occasional articles, current news. The following mathematicians were presented in Kosmos *Sprawozdanie z polskich prac matematycznych* [The reports of the Polish mathematical works]: S. Dickstein, S. Kepiński, Z. Krygowski, S. Zaremba, K. Żórawski, W. Burtan. The authors of the publication were, inter alia, W. Żmurko, J. Puzyna, P. K. Skibinski, A. Raciborski, F. Rauch, W. Gosiewski, W. Zajączkowski, S. Dickstein.

The number of these reports was not too large, taking into account the years of existence of *Kosmos*, specifically for the following three years: 1901, 1902, 1904. Below we place the complete bibliography of mathematical works in Kosmos:

- D. Zbrożek, *O Koperniku* [On Copernicus], r. 1, 1876, pp. 45–54, 160–167.
- Discussion on the paper by W. Zajączkowski from the Academy of Sciences edition: *Teoryja ogólna rozwiązań osobliwych równań różniczkowych zwyczajnych*, 1(1876), p. 350 [General theory of singular solutions of ordinary differential equations].
- Discussion on the paper by W. Żmurko, *O ważności i zastosowaniu funkcji oskulacyjnej w rachunku przemienności, oraz odpowiedź na uwagi dr. Mertensa dotyczące tego przedmiotu*, 1(1876), pp. 355–356. [The importance and application of osculate function in calculus of commutativity, and response to comments by dr. Mertens about this topic].

- W. Żmurko, *O niektórych przyrządach wykreślających*, 5(1880), pp. 44– 52 [On some plotting instruments].
- P. K. Skibiński, *O integratorze dra Żmurki*, 9(1884), pp. 185–189 [On Dr. Żmurko's integrator].
- A. Raciborski, *Znaczenie pojęcia przestrzeni w stosunku do praw matematyki*, 10(1885), pp. 493–535 [Significance of the notion of space in relation to the laws of mathematics].
- R. P., *Pojęcie przestrzeni i zasady geometrii*, 11(1886), pp. 530–547 [Concept of space and principles of geometry].
- A.Raciborski, *Odpowiedź na artykuł Pana R. P.*, 12(1887), pp. 27–37 [Response to the article of Mr. R. P.].
- J. Puzyna, *Prof. Wawrzyniec Żmurko; jego życie i dzieła*. *Kosmos*. XIV [Professor Wawrzyniec Żmurko, his life and works].
- F. Rauch, *O podziale danego kąta*, 20(1895), pp. 71–81 [On division of a given angle].
- S. Dickstein, *O liczbach e i π*, 20(1895), pp. 359–365 [On the numbers e and π].
- W. Gosiewski, *Wywód elementarnej metody najmniejszych kwadratów*, 20(1895), pp. 366–368 [Derivation of the elementary least squares method].
- S. Dickstein, *O najnowszych badaniach nad podstawami matematyki*, 30(1905), pp. 107–129 [On the latest research on the foundations of mathematics].

Partial bibliography of works from the journal of mathematical Kosmos is contained in: *Polskie Towarzystwo Przyrodników im. Kopernika, 1875–1975*.[144]

The subject of works connected to mathematics in the broad sense and contained in Kosmos is popularization of mathematics at a level higher than secondary school course. The subject is fairly broad, in the interest of the scientific community. Some of the topics are conducive to physics, for example, those concerning differential equations. Of some practical significance are articles on plotting devices, integrators, or close to statistical method of least squares. One should note the cultural significance of general topics concerning the history of mathematics or its rules (including the rules of geometry).

In his interesting paper *O liczbach e i π* [On the numbers e and π] S. Dickstein explains the difference between the algebraic and irrational numbers. He remarks that the very first investigations of e and p concern irrationality of these numbers. He also remarks that existence of transcendental numbers was first proved by Liouville by using the properties of continuous fractions and

[144] See: Kosmos, 20(1985), pp. 353–358; *Polskie Towarzystwo Przyrodników im. Kopernika, 1875–1975*, PWN, Warsaw, 1981.

recalls an (inconstructive) proof based on ideas of G. Cantor. Namely, since the set of all algebraic numbers is countable and the set of all real numbers is uncountable, one can deduce that there exist transcendental numbers. However, this reasoning does not provide any information concerning concrete numbers.

Liouville could only prove that neither *e* nor e^2 satisfy any quadratic equation with rational coefficients. This proof is presented in the article. Dickstein remarks that the method used by Liouville does not work for the proof that neither *e* or e^2 are roots of equation of degree grater then 2 with rational coefficients.

About two decades before Dickstein's article, Hermite, in this famous thesis *On the exponential function*, obtained a complete solution. His methods were, however, complicated and required deep knowledge of higher analysis. Using Hermite's formulas Lindemann proved that there is no relationship of the from $N_0 + N_1 \sum e^{z_i} = 0$, where N_0 and N_1 are non zero integers, real or complex and $z_1, z_2, ..., z_n$ are non zero roots of an algebraic equation, then e^z cannot be a rational number.

Because of the famous formula

$e^{\pi i} = -1$

this proves the irrationality of π (here Dickstein remarks that the relationship between the transcendentality of π and *e* was already know to Wroński.)

Finally, 1885 Weierstrass cosiderably simplified Lindemann's proof and obtained another one of transcendentality of π and *e.*

The article dedicated to the life and work of Prof. Żmurko by J. Puzyna contains biographical information as well as detailed description of some of Żmurko's achievements

It should be emphasized that the author wrote about Żmurko with great reliability and respect, paying tribute to his contribution to the formation of mathematical culture. In 1864 Żmurko published the book *Wykład matematyki na podstawie ilości o dowolnych kierunkach* [Exposition of mathematics on the base of value in arbitrary direction], in two volumes. The book is based on methodology developed by the author as a result of his work at the Technical school. Żmurko came to the conclusion that fundamentals of mathematics which follow from abstract considerations are neither natural nor reliable and therefore they should be replaced by rules based on our three-dimensional space, our experience. The complex plane can serve as an example. The complex numbers are interpreted as the "measuring numbers" in plane. This approach to complex numbers historically differs from that widely accepted, namely, the ways of solving all quadratic equations.

Simultaneously, Żmurko considered spatial numbers.

Żmurko wrote three theses dealing with numerical equations, two of them in German and one in Polish. These are investigations that belong to the theory of functions of one complex variable. As Puzyna remarks, Żmurko tried to find spatial considerations in almost all areas of mathematics. Therefore, the equation

$$f(u) = A_n u^n + A_{n-1} u^{n-1} + \ldots + A_1 u + A_0 = 0$$

with either real or complex coefficients leads him, after putting $u = x + yi$, to the form

$$F(x, y) + iF'(x, y) = 0.$$

Considering the real and imaginary parts separately he obtains two algebraic surfaces. The geometric investigations of these surfaces are essentially investigations of the initial equation. By using this method Żmurko determines a domain to which the roots of the equation belong. Then he shrinks the boundaries of every root and finds it by numerical methods.

These considerations are extended later to systems of equations.

Written in German *Beitrag zur Theorie …* is devoted to synthetic investigations of some types of numerical equations of higher degrees. Since the formulas for the equations of degree 3 and 4 are known, the author looks for the equations of higher degrees that can be reduced to those of lower degrees and having algebraic solutions.

The results of Żmurko's work were highly evaluated by the Academy of Skills in Cracow and the Academy published the thesis *Beitrag zur Theorie …* as volume XLIV of its *Pamiętniki* [Journals].

Żmurko emphasizes that "constructive solution of these equations puts the constructive analysis on the same level with the algebraic analysis".

Puzyna also mentions some tools for drawing algebraic curves. One of them is the Żmurko's ellipsograph which allows one to draw ellipses with arbitrary ratios of axes. The integration of second-order differential equations with linear coefficients was the subject of Żmurko's lectures at the University. In these investigations, Żmurko starts with research of series that are solutions of the normal type of the mentioned equations.

In his paper *Significance of the notion of space in relation to the laws of mathematics*[145] the author provides a philosophical analysis of the notion of space in geometry. The exposition contains some information concerning spherical and hyperbolic geometry, as well as geometry of higher dimensions.

5.4. Sessions of the Mathematical Society in Lvov

The list of talks delivered on the sessions of the Mathematical Society in Lvov witnesses that the level of mathematical knowledge as well as mathematical culture was comparable with those of other European mathematical centres. We provide below some comments that concern the talks.

[145] A. Raciborski, *Znaczenie pojęcia przestrzeni w stosunku do praw matematyki*, (10)1885, pp. 493–535.

List of meetings of the Mathematical Society in Lvov**:**

1. **1917**
- Dr H. Steinhaus, Solved and unsolved problems in the theory of Fourier series
- Prof. I. Grabowski, The harmonic analyzer of Henrici"

2. **1918**
- Prof. J. Puzyna, On the zero traces of power series
- Dr A. Maksymowicz, On Cesàro's series
- Prof. Z. Krygowski, On Tschirnhausen maps in algebra
- Prof. W. Sierpiński, Recent studies on measurable functions
- Dr H. Steinhaus, On linear and continuous operations in a function field
- Prof. W. Sierpiński „On the continuum conjecture
- Prof. W. Sierpiński,Definition of the Lebesgue integral without the measure theory
- Dr H. Steinhaus, Power series in the disk of convergence

3. **1919**
- Dr A. Łomnicki, On the operations of completeness axiom
- Dr S. Ruziewicz, On functions that have equal derivatives everywhere, but not differing by a constant quantity

4. **1920**
- January 22 S. Ruziewicz, Obituary of late Prof. Józef Puzyna
- February 5 Dr H. Steinhaus, About life and merits of late dr. Zygmunt Janiszewski
- Dr S. Ruziewicz, Some examples of functions with equal derivatives everywhere whose difference is not constant
- February 26 Dr H. Steinhaus, More recent results in the field of orthogonal functions and Fourier series
- Talk: E. Żyliński, One result of the group theory
- March 11 Prof. Żyliński,nOn the principles of the theory of ideals
- May 20 Dr A. Maksymowicz, Research in the field of harmonic functions
- Dr S. Ruziewicz, communicates Sierpiński's proof of the theorem that there are infinitely many primes of the form $4k+1$.
- June 10 Prof. Dr Ernst, Sundman's research on the problem of three bodies.

Without claiming completeness, below we present brief information that will enable the reader to understand the subjects of reports of the Society.
An ideal (more precisely, a two-sided ideal) is a subset I of a ring R if $(I, +)$ is a subgroup of $(R, +)$ and for all x in I and for all r in R, the elements $x \cdot r$ and $r \cdot x$ are in I. The notion of ideal allows us to construct the quotient rings.

Fourier series

Fourier series were introduced in 1807 by J. Fourier (1768–1830). In order to find a trigonometric expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for a given 2p-periodic function $f(x)$, one has to find the coefficients $a_0, a_1, \ldots, a_n, b_n$. If the function $f(x)$ is integrable on $<-p, p>$, in both proper or improper sense (in the case of improper integrability it should also be assumed that the function is absolutely integrable) we obtain the following *Euler-Fourier* formulas for the coefficients:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx \quad (\mathrm{n} = 1, 2, 3, \ldots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mxdx \quad (\mathrm{n} = 1, 2, 3, \ldots)$$

These coefficients are called the Fourier coefficients, and the series created with their help is called the trigonometric Fourier series (see Fichtenholtz, 1974, p. 349). The Fourier series was one of the areas of Steinhaus's research. In the years 1913–1930, he published the following works in this area:

1. *O rozwinięciu na szereg Fouriera iloczynu dwóch funkcji* [Sur de développement du produit de deux fonctions en une série de Fourier], *Bulletin International de l'Academie des Sciences de Cracovie* (1913), pp. 113–116.
2. *O niejednostajnej zbieżności szeregów Fouriera* [Sur la convergence non-uniforme des séries de Fourier], ibidem, pp. 145–160.
3. *O pewnej szczególnej funkcji, którą można przedstawić w postaci szeregu Fouriera* [Sur une fonction remarquable répresentée par une série de Fourier], ibidem, pp. 291–304.
4. *Niektóre własności szeregów trygonometrycznych i szeregów Fouriera*, Rozprawy Akademii Umiejętności (1916), pp. 176–225.
5. *On Fourier's coefficients of bounded functions*, Proceedings of the London Mathematical Society, Series 2, 20(1922), pp. 273–275.
6. *Sur quelques propriétés des séries trigonométriques et de celles de Fourier*, Rozprawy Akademii Umiejętności 56(1925), pp. 175–225
7. *Szeregi Fouriera*, Lvov 1930, Komisja Senatu Akademickiego U. J. K., p. 138[146].

[146] Wiadomości Matematyczne, 17(1973), list of publications by H. Steinhaus.

Cesàro series

The divergence of the product of two convergent series leads to the question whether or not you can not sum up the divergent series. It turns out that this is possible, and one such method is the method of Cesàro, called the method of arithmetic averages. Briefly, it can be summarized as follows:

Consider the series:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots + a_n + \ldots$$

Define the sequence of partial sums for it:

$A_0 = a_0$, $A_1 = a_0 + a_1$, ..., $A_n = a_0 + a_1 + \ldots + a_n$.

For the sequence of partial sums, we form the consequent arithmetic averages:

$$\alpha_0 = A_0,\ \alpha_1 = \frac{A_0 + A_1}{2},\ \ldots,\ \alpha_n = \frac{A_0 + A_1 + \ldots + A_n}{n},\ \ldots;$$

If the obtained sequence has the limit A provided n→∞, then A is called the *generalized sum* (in the sense of Cesàro) of the given series [See: Fichtenholtz, 1974].

Power series are the series of the form:

$$a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n \ .$$

Harmonic analyser of Henrici. In 1894, Olaus Henrici (1840–1918) invented a tool to determine the fundamental and harmonic components for a complex sound wave. It is worth mentioning that nowadays this topic is also investigated [See: Leszczyński 2005].

Tschirnhausen's metod.

Tschirnhausen's transformation takes a polynomial $P(x)$ with roots x_1 ,..., x_n into the polynomial $Q(x)$ with roots $\varphi(x_1)$, ..., $\varphi(x_n)$, where $\varphi(x)$ is also a polynomial. The coefficients of Q can be expressed in terms of the coefficients of P and $\varphi(x)$, which can be used to solve the equations of the third and fourth degree and simplify the general form of equations of higher degree.

Examples of functions that have equal derivatives and whose difference is a constant; Most probably, Ruziewicz means here the example from his article *Sur les fonctions qui ont la meme dérivée et dont la difference n'est pas constante* published in the first volume of Fundamenta Mathematicae.

Sur les fonctions qui ont la même dérivée et dont la différence n'est pas constante.

Par

Stanisław Ruziewicz (Lwów).

M. Hahn a donné un exemple d'une infinité de fonctions qui ont la même dérivée (non partout finie) en tous les points d'un intervalle et dont la différence n'est pas cependant constante dans cet intervalle [1]). Le but de cette Note est de donner un exemple très simple de même nature.

Actually, in this paper Ruziewicz speaks about a family of functions of cardinality continuum such that these functions have the same derivative and the difference of two distinct elements of this family is not a constant.

Harmonic functions are the functions $f: R^n \to R$ satisfying the Laplace equation $\Delta f \equiv 0$. For a function defined in a domain of the Euclidean space, the Laplace operator in the cartesian coordinates has the form:

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

Continuum hypothesis. In 1884, G. Cantor formulated the following continuum hypothesis (conjecture):
There is no cardinal number α such that $\aleph_0 < \alpha < c$, where $\aleph_0$ denotes the countable cardinality and c denotes the cardinality of continuum (of the segment [0,1] or the real line). In 1940, K. Gödel published a paper, where he proved that the continuum hypothesis is consistent with the axioms of the set theory, and in 1963 P.J. Cohen proved that the continuum hypothesis is independent of the axioms of the set theory.

The **three-body problem** is the problem of taking an initial set of data that specifies the positions, masses, and velocities of three bodies in space, for some particular point in time, and then using that set of data to determine the motions of the three bodies, and to find their positions at other times, in accordance with Newton's laws of universal gravitation. Karl Frithiof Sundman (1873–1949) was a Finnish mathematician who used analytic methods to prove the existence of a convergent infinite series solution to the three-body problem in 1906 and 1909.

Recall that a function (a map) between measurable spaces is said to be **measurable** if the reimage of each measurable set is measurable. One can conjecture that the mentioned above talk by Sierpiński contained the result which was later published in his paper in the first issue of Fundamenta Mathematicae

Sur les fonctions convexes mesurables.

Par

Wacław Sierpiński (Warszawa).

Une fonction $f(x)$ de variable réelle est dite *convexe* dans un intervalle $\langle a, b \rangle$, lorsqu'elle satisfait à l'inégalité

$$2f\left(\frac{x_1+x_2}{2}\right) \leqslant f(x_1)+f(x_2) \quad \text{pour } a \leqslant x_1 \leqslant b \text{ et } a \leqslant x_2 \leqslant b.$$

The main result of the paper claims that any convex measurable function on an interval is continuous.

Two talks are devoted to algebra. In particular, to the group theory and the ring theory. Recall that a (right) **ideal** of a ring *R* is a subset *I* of *R* such that *I* is an abelian subgroup of *R* and *rI* ={*ri* | *i is an element of I*} is a subset of *I*.

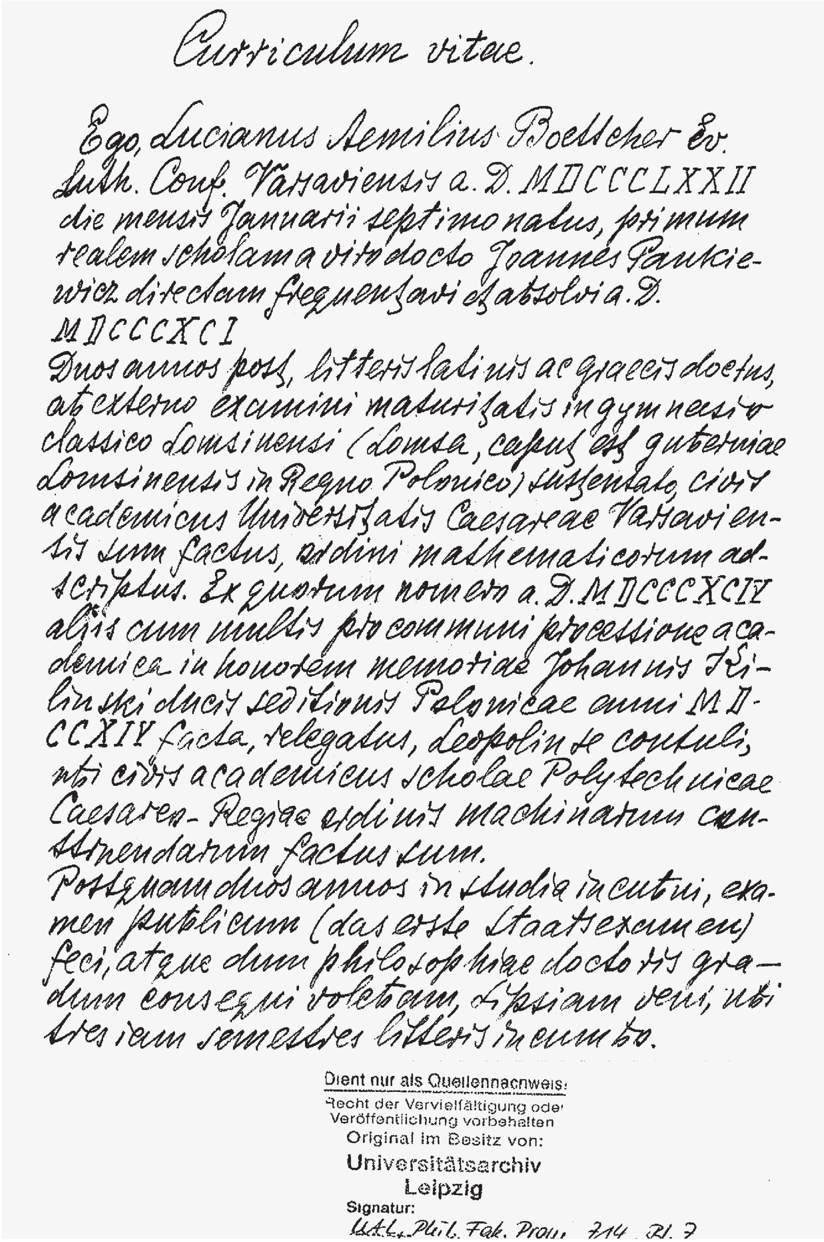
Unfortunately, we have no data about the contents of Steinhaus's report at a meeting of the Society. His articles on the subject were published much later. We believe that if the text of the report was available, it would have added more information about the history of functional analysis, especially of its initial stages.

Analyzing the subject of reports in its generality we come to the conclusion that in general it reflected the development of a large part of the contemporary mathematics. The scientists of the city followed the mathematical innovations and this had an inevitable influence on the subjects of research of the Lvov mathematical school.

5.5. A forgotten mathematician

Lucyan Böttcher (1872–1937)

Böttcher is forgotten as a mathematician, which is a great pity for the history of mathematics in Poland. Böttcher was an interesting personality, related to Lvov, but not to the Lvov Mathematical School. Most probably, this circumstance explains why is he considerably less known than the other mathematicians from Lvov.



Curriculum vitae.

Ego, Lucianus Aemilius Boettcher Ev. Luth. Conf. Varsaviensis a. D. MDCCCLXXII die mensis Januarii septimo natus, primum realem scholam a viro docto Joannes Pankiewicz directam frequentavi et absolvi a. D. MDCCCXCI

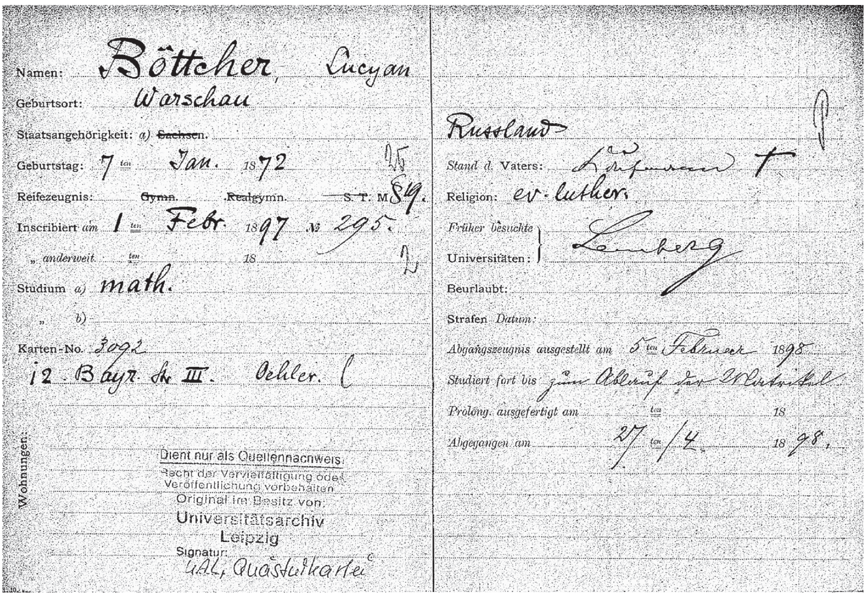
Duos annos post, litteris latinis ac graecis doctus, at externo examini maturitatis in gymnasio classico Lomsinensi (Lomsa, caput est guberniae Lomsinensis in Regno Polonico) sustentato, civis academicus Universitatis Caesareae Varsaviensis sum factus, ordini mathematicorum adscriptus. Ex quorum numero a. D. MDCCCXCIV aliis cum multis pro communi processione academica in honorem memoriae Johannis Kilinski ducis seditionis Polonicae anni MDCCXIV facta, relegatus, Leopolin se contuli, ubi civis academicus scholae Polytechnicae Caesareo-Regiae ordinis machinarum construendarum factus sum.

Postquam duos annos in studia incubui, examen publicum (das erste Staatsexamen) feci, atque dum philosophiae doctoris gradum consequi volebam, Lipsiam veni, ubi tres iam semestres litteris incumbo.

L. Böttcher was born on January 21, 1872 in Warsaw. In his CV, which supplemented his PhD thesis in Leipzig, he indicated that he belonged to the Evangelical-Lutheran church. In Warsaw he finished a so-called real school and in 1893 he graduated from the gymnasium in Łomża. Then Böttcher started his studies at the university in Warsaw, which was then a Russian university called

Tsar's University. Because of his participation in demonstrations dedicated to J. Kiliński, Böttcher was forced to leave Warsaw and entered the Department of Machine Construction at the Polytechnical School in Lvov. Two years later, he interrupted his studies in Lwow and moved to the Leipzig University. In Leipzig, he also attended lectures of Professors: Lie [Theory of differential invariants, Theory of differential equations, Theory of continuous transformation groups; seminars Theory of integral invariants and Differential equations], Mayer [Higher analytical mechanics], Engel [Differential equations, Algebraic equations, Non-Euclidean geometry], F. Hausdorff [Similarity maps].

In 1898, Böttcher was awarded doctoral degree for his thesis *Beiträge zur Theorie der Iterationsrechnung.*



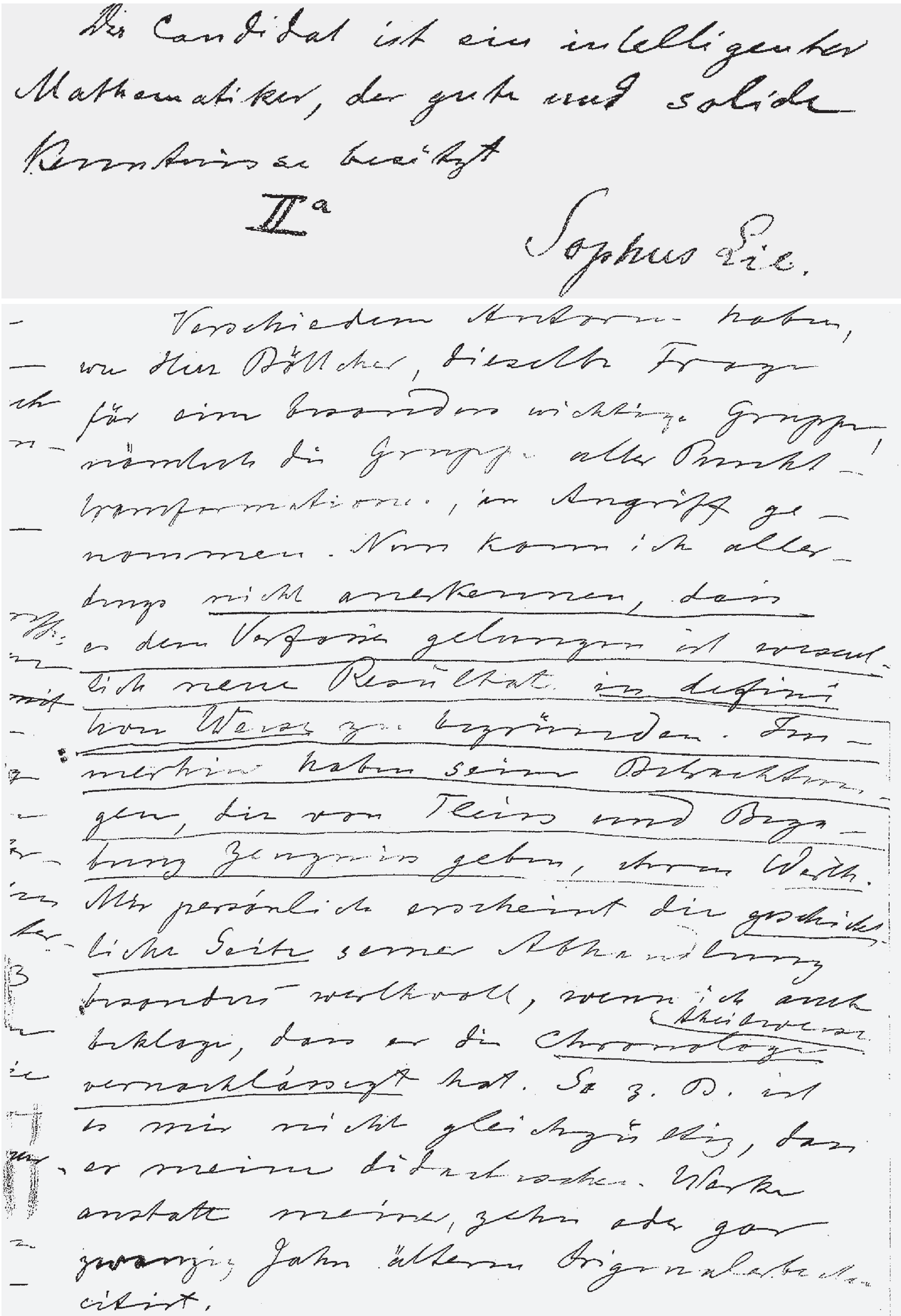
Namen: Böttcher, Lucyan
Geburtsort: Warschau
Staatsangehörigkeit: a) ~~Sachsen~~. Russland
Geburtstag: 7 ten Jan. 1872
Reifezeugnis: ~~Gymn.~~ ~~Realgymn.~~ ~~S. T.~~ M. 849.
Inscribiert am 1 ten Febr. 1897 № 295.
„ anderweit ten 18
Studium a) math.
„ b)
Karten-No. 3092
12 Bayr. Str. III. Oehler.
Wohnungen:
Dient nur als Quellennachweis. Recht der Vervielfältigung oder Veröffentlichung vorbehalten. Original im Besitz von: Universitätsarchiv Leipzig. Signatur: UAL, Quästurkartei

Stand d. Vaters: Kaufmann †
Religion: ev.-luther.
Früher besuchte Universitäten: Lemberg
Beurlaubt:
Strafen Datum:
Abgangszeugnis ausgestellt am 5 ten Februar 1898
Studiert fort bis zum Ablauf der Matrikel
Prolong. ausgefertigt am ten 18
Abgegangen am 27 ten / 4. 1898.

L. Böttcher complied with the procedure of obtaining the doctoral degree in Leipzig in 1897–1898.

His results of this period were connected with the theory of transformation groups. The following is the opinion of one of the most outstanding mathematicians in this area, S. Lie, written in 1898. It clearly shows that in some way Böttcher participated in the creation of the theory of Lie groups:

I consider it to be understood that Mr. Scheibner, as the situation indicates, after much speculation comes to the conclusion that he does not want to give any official paper. But I think that his not clear enough attention to prof. Engel is more striking. The thing!



Der Candidat ist ein intelligenter
Mathematiker, der gute und solide
Kenntnisse besitzt

II a

Sophus Lie.

Verschiedene Autoren haben,
wie Herr Böttcher, dieselbe Frage
für eine besonders wichtige Gruppe,
nämlich die Gruppe aller Punkt-
transformationen, in Angriff ge-
nommen. Nun kann ich aller-
dings nicht anerkennen, dass
es dem Verfasser gelungen ist wesent-
lich neue Resultate in defini-
tiver Weise zu begründen. Im-
merhin haben seine Betrachtun-
gen, die von Fleiss und Bega-
bung Zeugniss geben, ihren Werth.
Mir persönlich erscheint die geschicht-
liche Seite seiner Abhandlung
besonders werthvoll, wenn ich auch
beklage, dass er die Chronologie (theilweise)
vernachlässigt hat. So z. B. ist
es mir nicht gleichgültig, dass
er meine diesbezüglichen Werke
anstatt meiner, zehn oder gar
zwanzig Jahre älteren Originalarbeiten
citirt.

Fragment of S. Lie’s opinion regarding Böttcher’s dissertation
(University Archives in Leipzig)

If both the author and Mr. Scheibner indicate the relationship of the submitted work to my concept of groups of unitary transformations, I agree in part with these comments. The relationship, however, lies a little deeper. In 1874, I thought that every finite transformation of a finite continuous group is contained in a unitary subgroup. In 1883, I formulated the question whether this fundamental theorem also applies to the infinite continuous groups. However, since this issue does exceed not only my strength, but also the strength of the current analysis, I restricted myself mainly to show, only for specific examples, that this question can be answered in the affirmative.

Various authors, as well as Mr. Böttcher, considered the same issue for a particularly important group, namely the group of all point transformations. ***However, I cannot recognize that the author has managed to definitively substantiate significant new results. Despite all of this, his considerations, which testify to the carefulness and talent, have their value.*** *The historical side of his hearing seems to be particularly valuable personally to me, although I regret that he disregarded the chronology. So it is not indifferent to me, for example, that he cites my work of teaching, instead of my ten or even twenty years older works of authorship.*

In any case, I (as well as Mr. Scheibner) agree that this attempt be accepted as a thesis and we also agree regarding the evaluation II. I choose such good score, because Mr Böttcher himself chose his topic and developed it independently. Further, I formulate the following proposals:

1) Mr. Vice-Dean will ask Mr. Mayer to undertake the second communication. I hope he will agree with my and Mr. Scheibner's opinion.

2) Mr. Vice-Dean will tell the candidate that he must take care that his work be printed in correct German. Otherwise, he will not receive a diploma. I hope Mr. Meyer, together with me, will be in control, as necessary, of this and other development, and that he will then communicate with Mr. Vice-Dean.

3) Before printing the dissertation, the candidate must report to me (or possibly to Mr. Mayer) to obtain information about various changes in content. Finally I mention that I saw the first draft in January. Under the conditions mentioned above, I support the acceptance of the dissertation with evaluation II, and admission to the oral exam.

S. Lie

In the same year Böttcher returned to Lvov and started his activity at the Polytechnical School, initially as an Assistant Professor of the Chair of Mechanical Technology and, in the next year, of the renewed Chair of Mathematics. In 1910, he occupied an Associate Professor position and in 1912 a Privatdozent position at the Polytechnical School.

Curriculum vitae

Niżej podpisany urodził się w Królestwie Polskiem dnia 7 Stycznia 1872 r. w Warszawie i jest wyznania ewangielicko-luterskiego. Uczęszczał do szkół w Warszawie, a mianowicie w latach 1881–1885 uczęszczał do czteroklassowej, prywatnej szkoły realnej p. Hermana Benniego, a następnie w latach 1886–1891 uczęszczał do sześcioklassowej prywatnej szkoły realnej p. Józefa Pankiewicza, którą ukończył, zdawszy egzamin z sześciu klas szkoły realnej w państwowej szkole realnej w Warszawie.

Niżej podpisany, powziąwszy zamiar poświęcić się studyom matematycznym uniwersyteckim zdał egzamin maturalny 1893 roku w gimnazyum klassycznem w Łomży (Królestwo Polskie) poczem wstąpił do Cesarskiego Warszawskiego Uniwersytetu na Wydział matematyczno fizyczny, na który uczęszczał w roku naukowym 1893/4 i słuchał następujących wykładów:

a) matematycznych: prof. Sonina (Analiza matematyczna); prof. Anisimowa (Analityczna geometrya) i p. Ehrenfeuchta (Astronomia opisowa)

b) przyrodniczych: prof. Potylicyna (Chemia ogólna) i prof. Ziłowa (Fizyka).

Niżej podpisany musiał w 1894 roku Uniwersytet Warszawski opuścić, obwiniony bowiem o udział w demonstracyi politycznej, którą studenci Uniwersytetu Warszawskiego na cześć pułk. Kilińskiego urządzili, wykluczony został z tegoż Uniwersytetu.

Opuściwszy Warszawski Uniwersytet udał się do Lwowa i zapisał w poczet słuchaczów zwyczajnych c.k. Szkoły Politechnicznej Lwowskiej i uczęszczał w przeciągu czterech przeszło półroczy na Wydział Budowy Maszyn.

Zdawał egzamin pierwszy państwowy w 1896 roku, na podstawie którego otrzymał Certyfikat z kwalifikacyą znamienicie uzdolnionego w [redacted]-tach objętych programem pierwszego państwo-

wego egzaminu Wydziału Budowy Maszyn.
Chcąc ukończyć studya uniwersyteckie udał się on
do Lipska, gdzie w przeciągu trzech semestrów w cha-
rakterze zwyczajnego słuchacza wydziału filozo-
ficznego uczęszczał na następujące wykłady:
a) matematyczne: prof. Lie (Teorya niezmien-
ników różniczkowych) Teorya równań różniczko-
wych o wiadomych przekształceniach infinitezy-
malnych. Teorya ciągłych grup przekształcenia.
Seminaryum: Teorya niezmienników całkowych.
Równania różniczkowe.
prof. Mayer (Wyższa mechanika analityczna)
prof. Engel (Równania różniczkowe. Równania
algebraiczne. Geometrya nieeuklidesowa)
doc. Hausdorff (Odwzorowanie podobne)
b) fizykalne: prof. Wiedemann (Ćwiczenia fi-
zykalne) prof. Drude (Elektryczność i magnetyzm)
c) filozoficzne: prof. Wundt (Psychologia).
Studya uniwersyteckie zamknąłem egzaminem
na stopień doktora filozofii, który uzyskałem na pod-
stawie rozprawy p.t. „Beiträge zu der Theorie der
Iterationsrechnung. Leipzig 1898. VII. 78) tudzież
egzaminów z matematyki, geometryi i fizyki.
Po ukończeniu studyów, otrzymałem posadę asysten-
ta przy c.k. Szkole Politechnicznej Lwowskiej, po-
czątkowo przy Katedrze Technologii Mechanicznej,
następnie przy Katedrze Matematyki. Posadę tę
dotąd zajmuję.
Działalność naukowa moja przedstawia się w na-
stępujący sposób: 1895 roku litografowałem „Repe-
tytoryum Wyższej Matematyki." (Rachunek
różniczkowy str. 55. Rachunek całkowy str. 47).
1897 roku ogłosiłem w „Pamiętniku Towarzystwa Po-
litechnicznego" artykuł „Zasadnicze podstawy
Teoryi Iteracyi"; 1898 roku drukowałem rozprawę
doktorską „Beiträge zu der Theorie der Iterations-
rechnung." 1899 roku ogłosiłem w „Czasopiśmie
Technicznem" artykuł „Kilka słów z dziedziny
rachunku iteracyjnego" a w „Pracach matema-
tyczno fizycznych. Tom X. ogłosiłem przerobiony
przekład pierwszych dwóch rozdziałów mojej
rozprawy doktorskiej. Obecnie opracowuję sy-
stematyczny wykład Teoryi równań funk-
cyjnych.

Lucyan Emil Böttcher

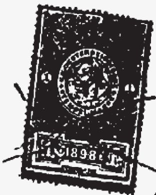
That is what a short biography of Böttcher could look like. However, some new materials from archives in Lwow and Leipzig open up for us other possibilities of critical analysis of his activity. We know from Böttcher's CV that he attended lectures of famous Russian professors Sonin and Anisimov.

Summing up, we see that Böttcher worked for about 37 years (until his retirement in 1935) for the Polytechnical School. He taught applied mathematics (solving and discussing mathematical problems important for technical applications), vector theory (the course contained fundamental facts on vectors and operations on them as well as applications of vectors in geometry and mechanics), differential equations, concepts and methods of elementary mathematics.

About 1912, Böttcher got interested in spiritism and metapsychology (whatever this term means), as well as occultism. One may speculate whether this direction of Böttcher´s interests caused his lack of contact with other Lvov mathematicians working at the beginning of the 20th century and later with the members of the famus Lvov mathematical school.

Evidently, these interests also affected the style of Böttcher's mathematical life: unlike his active participation in different mathematical events (e.g. conventions of Polish Physicians and Naturalists: Cracow, July, 1899, the title of his talk was *Substitutional functional equations*, Lvov, July, 1907, talk *From the theory of functional equations*; Cracow, 1900, session of Academy of Skills talk *Fundamental properties of grevians*), before this period, no similar activity can be noticed in the following years. In addition to presenting mathematical talks, Böttcher paid a lot of attention to the didactics of mathematics; this is witnessed by his numerous articles in Polish periodicals devoted to teaching mathematics.

In 1901, he first applied for admission to habilitation at the Lvov University. Note also that had to make a nostrification (official recognition) of his doctorate diploma from Leipzig.



3 / 11

Do Świetnego
Kollegium Profesorów
Wydziału Filozoficznego
c.k. Uniwersytetu we Lwowie.

Niżej podpisany prosi Świetne Kollegium Profesorów Wydziału Filozoficznego c.k. Uniwersytetu we Lwowie o dopuszczenie go do habilitacyi na docenta w zakresie nauk matematycznych.

Na poparcie swej prośby załącza niżej podpisany następujące dokumenty:

1° Dwie rozprawy: Zasady rachunku iteracyjnego. Część pierw. Odbitka z Prac Mat. Fiz.
O własnościach pewnych wyznaczników funkcyjnych. Odbitka z Rozpraw Wydz. mat. przyr. Akad. Um. w Krakowie.

2° Nostryfikowany na Lwowskiej Wszechnicy dyplom doktorski, uzyskany na Wszechnicy w Lipsku.

3° Curriculum vitae.

4° Program wykładów na 4 półrocza.

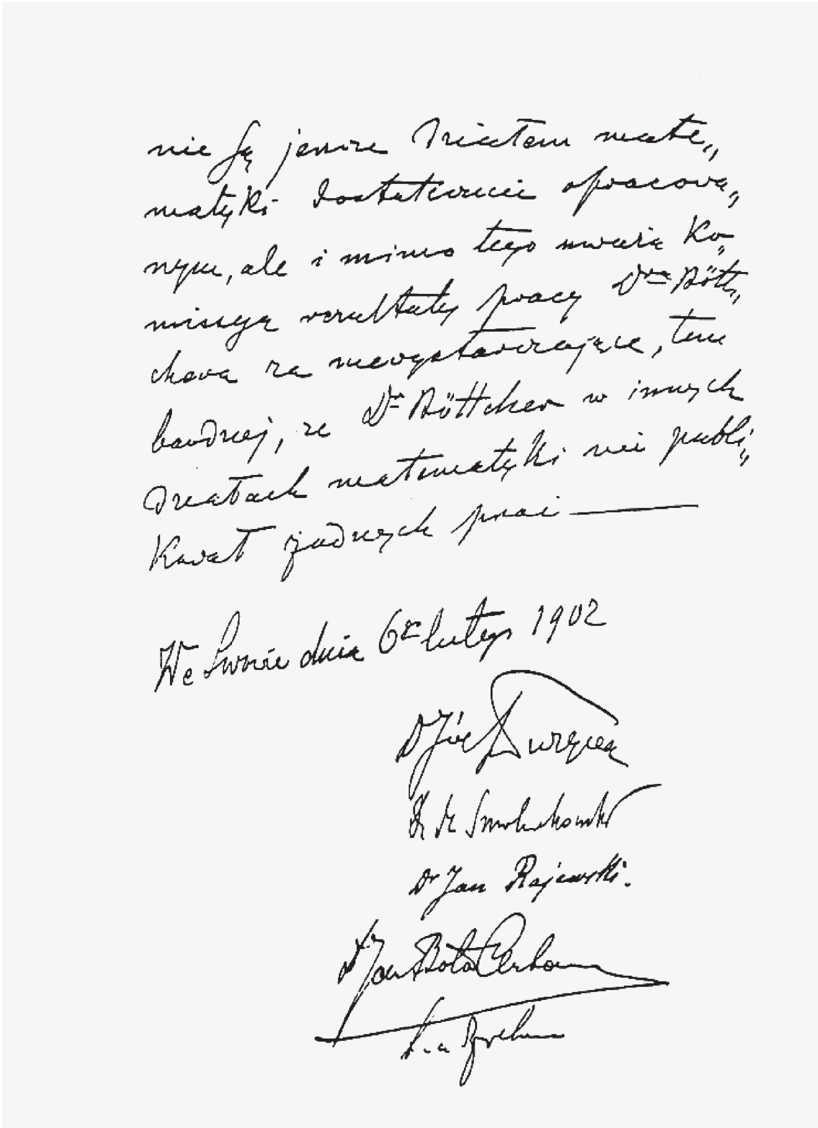
Lwów dnia 17/X 1901. Dr Lucyan Emil Böttcher
Asystent przy Katedrze Matematyki
c.k. Szkoły Politechnicznej
we Lwowie

Böttcher's application was considered by a commission (Profs L. Finkel, the Dean, J. Puzyna, J. Rajewski, M. Smoluchowski). The decision was to postpone the habilitation to the time of publication of the Thesis *Fundamentals of iteration calculus. Part III.* The meeting of the commission was held on January 13, 1902, on February 6, 1902, the commission presented its unanimous decision concerning Böttcher‘s habilitation. The commission recommended to the Council of the Faculty not to admit dr Böttcher‘s habilitation. The reasons for such a decision were the following:

- Böttcher's thesis was only a small contribution to studying properties of a Wrońskian (however, the commission considers the results as correct ones);
- As for the second thesis, *An application of the convergence and iteration to solving elementary equations*, the commission‘s opinion was that the author restricts himself with only one particular case.
- In addition, it was asserted that Böttcher usually exposes investigations of other authors (among them there are Greve, Köning e.a.) and pays less attention to his own results.

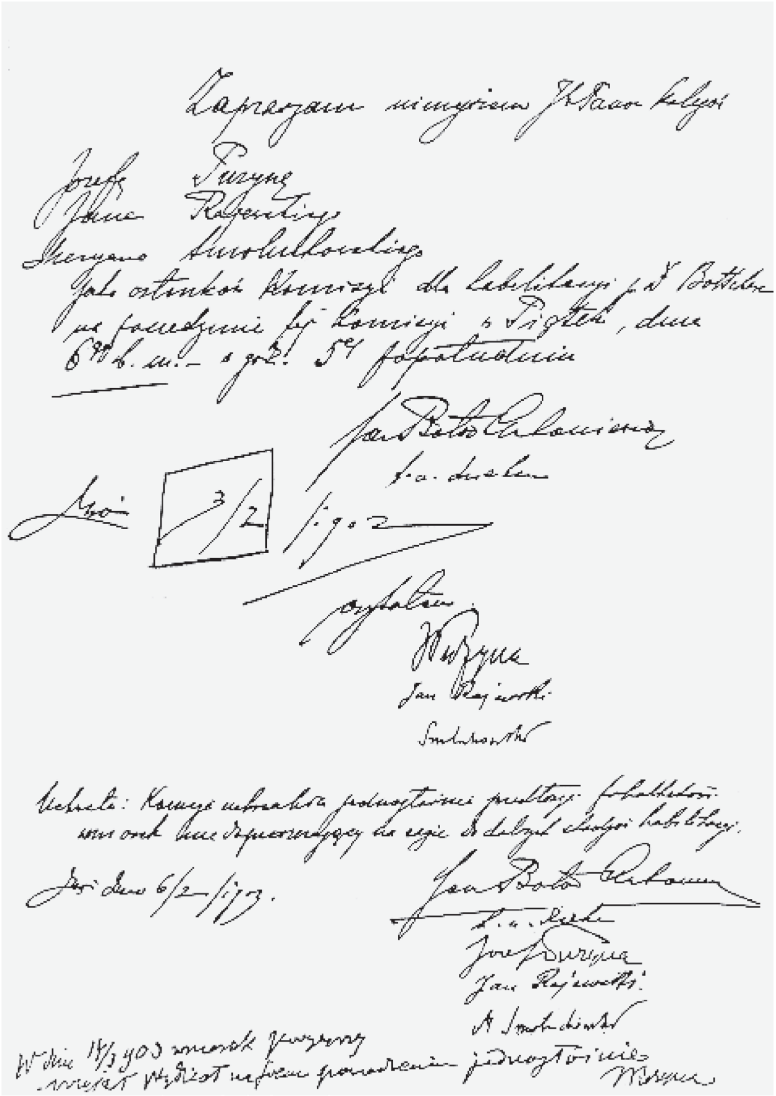
Also, the commission concluded that Böttcher‘s investigations were only a formal contribution to the problem of solving of functional equations. Finally, the iteration theory was hardly a very developed part of mathematics. However, the results of Böttcher were not sufficient for habilitation. The commision also remarks that Böttcher did not publish his results in other fields of mathematics.

The decision was signed, in particular, by Puzyna and Smoluchowski.


nie są jeszcze ścisłem mate-
matyki dostatecznie opracowa-
nym, ale i mimo tego uważa Ko-
misya rezultaty pracy Dra Bött-
chera za niewystarczające, tem
bardziej, że Dr Böttcher w innych
działach matematyki nie publi-
kował żadnych prac —

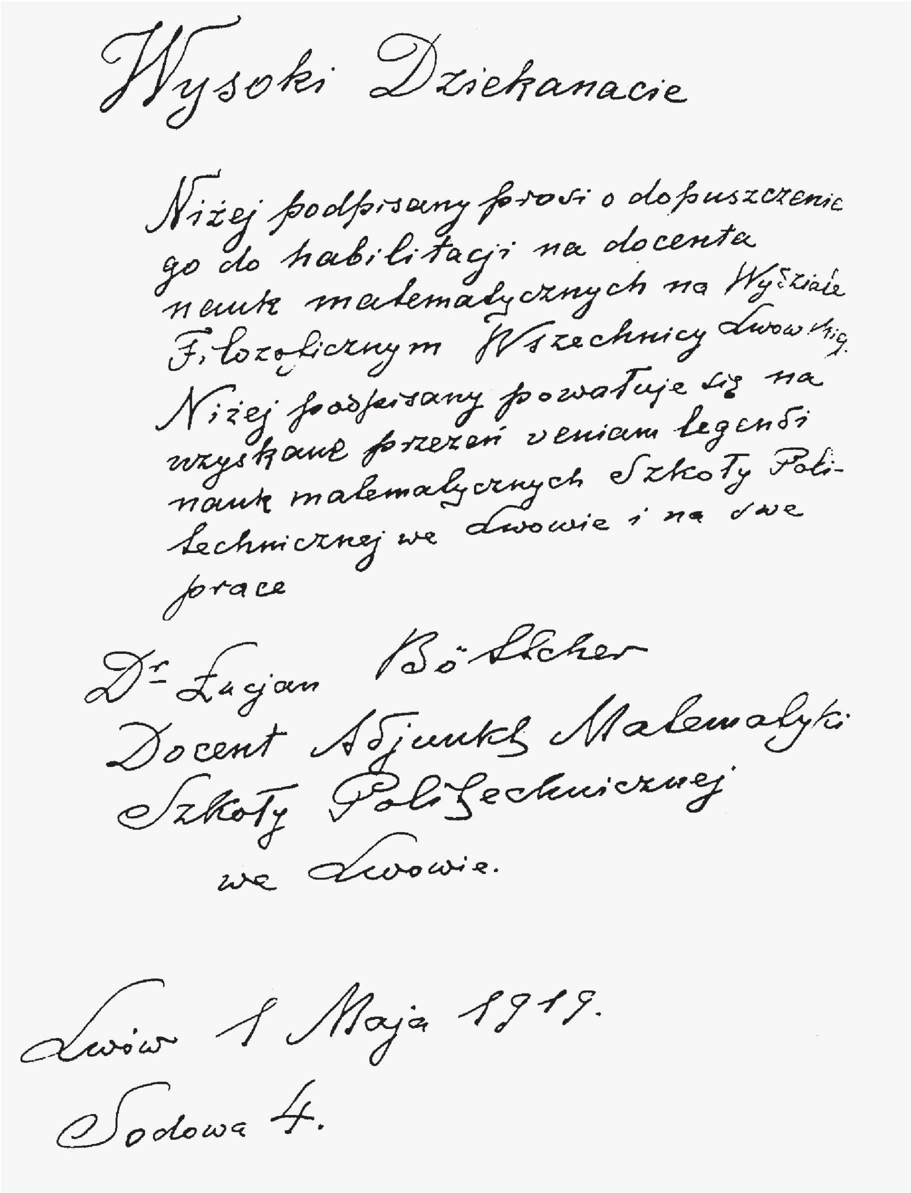
We Lwowie dnia 6go lutego 1902

Dr Józef Puzyna
Dr M. Smoluchowski
Dr Jan Rajewski.
[illegible]
[illegible]



It looks that already in 1903 Böttcher asked for admitting to habilitation at the University of Lvov. The commission confirmed the decision of the previous commission. In the decision of the commission (in 1918) one can find strong enough negative formulations that concerned Böttcher's results and publications: "the investigations do not bring anything new to the development of mathematics", "there are particular mistakes in the papers" etc. The commission analysed Böttcher's results in details and, as we already told, the decision was again unanimous: do not admit the applicant to the next stages of habilitation.

However, Böttcher did not agree. The following letter is dated on May 1, 1919.



Wysoki Dziekanacie

Niżej podpisany prosi o dopuszczenie
go do habilitacji na docenta
nauk matematycznych na Wydziale
Filozoficznym Wszechnicy Lwowskiej.
Niżej podpisany powołuje się na
uzyskaną przezeń veniam legendi
nauk matematycznych Szkoły Poli-
technicznej we Lwowie i na dwie
prace

Dr. Lucjan Böttcher
Docent Adjunkt Matematyki
Szkoły Politechnicznej
we Lwowie.

Lwów 1 Maja 1919.
Sodowa 4.

In it, Böttcher asked again to admit him to habilitation at the University. Note that, at that time, he was already a docent (associate professor, since 1912) at the Polytechnical School in Lvov.

Böttcher's results are widely cited in contemporary publications[147]. Böttcher's name is today mainly related to the so called Böttcher functional equation which plays an important role in iteration theory of polynomials and rational functions in the complex domain. This equation already appears in Böttcher's doctoral thesis in Leipzig , and it expresses the fact that a local analytic function *f*,

$$f(z) = a_d z^d + ..., d \geq 2, a_d \neq 0, |z| < \rho\ ,$$

is locally and analytically conjugate to the polynomial $p(z)=a_d z^d$. Since this part of iteration theory started around 1920 with the work of Fatou and Julia and was systematically developed only much later (1970–1980), it seems that Böttcher's ideas were neglected for a rather long time. Also his other contributions to iteration theory, like his papers[148] were not taken up immediately by his contemporarians, but deserve a systematic reinvestigation, although Böttcher could not give to his ideas the full precision and complete treatment which is possible and necessary today. The list of publications and talks shows his interest in various classes of functional equations, long before a systematic study and presentation of this topic was undertaken by János Aczél and Marek Kuczma. In particular, Böttcher showed an interest in what he calls "substitutional functional equations", nowadays called functional equations of iterative type. These are the most difficult functional equations problems, and are studied now in detail by many authors. It is interesting to see that the famous Böttcher equation, mentioned above, and certain generalizations[149] are today an important tool in these more recent efforts, e.g. in the theory of generalized Dhombres equations (Reich, 2004) and in the (purely algebraic) theory of the substitutional decomposition of polynomials [Dorfer et al.]. The remarkable is Böttcher's interest in Gréve determinants, today after called Casarati determinants, which are a kind of substitute of Wrońskians when dealing with linear dependence of functions without differentiability, and are frequently used today.

Summarizing, it seems that the very interesting mathematician L.E. Böttcher worked rather independently and lost connection with scientists in his neighbor-

---

[147] F. Balibrea, L. Reich, J. Smital, *Iteration theory: dynamical systems and functional equations*, International Journal of Bifurcation and Chaos 13(2003), no 7, pp. 1627–1647.

W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. 29(1993), no 2, pp. 151–188.

O. Jones, Multivariate *Böttcher equations for polynomials with nonnegative coefficients*, Math. Research Report No. MRR 99.014

P. Poggi-Corradini, *Canonical conjugations at fixed points other than the Denjoy-Wolff point*, Annales Academiae Scientiarum Fennicae Mathematica 25(2000), pp. 487–499.

L. Reich, *Generalized Böttcher equations in the complex domain, Symp. On Complex Differential and Functional Equations*, University Joensuu Dept. Math. Rep. 6(2004), pp. 135–137.

[148] *The principal laws of convergence of iterates and their application to analysis* [in Russian], Bulletin Kasan Mathematical Society 14,1904, pp. 155–234.

[149] L. Reich, Generalized *Böttcher equations in the complex domain*,

hood, but was able to create ideas which became influential in later research, up to the present day. As D. Gronau showed in *Gottlob Frege, A Pioneer in Iteration Theory* [150]. Frege has to be considered as an early pioneer of iteration theory, but the same is true for L.E. Böttcher.

In the book *Dynamics of one complex variable* by John Milnor, one can find the name of Böttcher in the list of founders of complex dynamics.

Following is a list of some of the founders of the field of complex dynamics.

Ernst Schröder	1841–1902
Hermann Amandus Schwarz	1843–1921
Henri Poincaré	1854–1912
Gabriel Kœnigs	1858–1931
Léopold Leau	1868–1940(?)
Lucjan Emil Böttcher	1872– ?
Samuel Lattès	1873–1918
Constantin Carathéodory	1873–1950
Paul Montel	1876–1975
Pierre Fatou	1878–1929
Paul Koebe	1882–1945
Arnaud Denjoy	1884–1974
Gaston Julia	1893–1978

One section of the Milnor's book is devoted to Böttcher's theorem. The following terms used in the book witness for the importance of Böttcher's contribution in the theory: Böttcher coordinate, Böttcher domain, Böttcher isomorphism, Böttcher map. These notions are also used in another publications in this direction[151] .

Böttcher died on May 29, 1937 in Lvov.

[150] D. Gronau, *Gottlob Frege, A Pioneer in Iteration Theory*, in: *Iteration Theory* (ECIT 94), Proceedings of the European Conference on Iteration Theory, Opava Grazer mathematische Berichte 334(1997), pp. 105–119, see: http://www.math.slu.cz/Konference/ECIT/PDF/105-119.pdf.

[151] E.g., **Dynamics on the Riemann sphere: a Bodil Branner festschrift by** Bodil Branner, Poul Hjorth,Carsten Lun de Petersen; in this book one can find also the term Böttcher potential)