Forgotten mathematician Henry Löwig (1904–1995)

Jindřich Bečvář

Löwig’s works in functional analysis


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The basic structures investigated in functional analysis are infinite-dimensional real and complex linear (vector) spaces endowed with a topology that is compatible with the algebraic structure. The topology can be induced by a metric, the metric can be derived from a norm, and the norm from an inner product. Consequently, functional analysis studies topological linear spaces, metric linear spaces, normed linear spaces, and inner product spaces. Every inner product space is also a normed linear space, every normed linear space is also a metric linear space, and every metric linear space is a topological linear space.

A topology compatible with the algebraic structure is sufficient to define concepts such as the limit of a sequence, the sum of an infinite series, continuity of a linear mapping (an operator or a functional). It allows us to speak about Cauchy (fundamental) sequences, complete spaces, to construct the completion of a given space, etc.

In an inner product space, it is also possible to introduce orthogonality (in a real linear space, one can define the angle between two nonzero vectors), orthogonal and orthonormal sets, etc.

∗ ∗ ∗

The notion of a linear space was already investigated by Hermann Günther Grassmann (1809–1877) in his obscure book *Die lineale Ausdehnungslehre* from 1844. Even the somewhat more intelligible version, which appeared in 1862 under the title *Die Ausdehnungslehre. Vollständig und in strenger Form bearbeitet*, met with a mild reception. Giuseppe Peano (1858–1932) decided to follow Grassmann’s ideas in his book *Calcolo geometrico secondo l’Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*; among other things, he gave the axiomatic definition of a linear space (*sistema lineare*), introduced linear mappings (*operazione distributiva, trasformazione lineare*), and clearly formulated certain basic facts which are now classified as belonging to linear algebra. Again, his book received almost no response. Peano’s axiomatic definition of linear spaces was later taken over by Salvatore Pincherle (1853–1936). In his extensive monograph *Le operazioni distributive e le loro applicazioni all’analisi* published in 1901, he presented a systematic theory of linear spaces (*sistema lineare, spazio lineare*), especially function spaces, inner product spaces, etc. The book’s title was chosen to emphasise its main goal, namely the study of linear operators on function spaces. Pincherle’s book is sometimes considered to be the first monograph on functional analysis (see e.g. [Me2]).

In a famous book by Hermann Weyl (1885–1955) entitled *Raum. Zeit. Materie. Vorlesungen über allgemeine Relativitätstheorie* and published in 1918, the axioms of a linear space reappeared in the context of affine spaces and
their axiomatic definition. The book was very successful and had its fifth edition published in 1923, while the English and French translations appeared in 1922.

Nevertheless, the notion of a linear space was still not a part of common knowledge. In 1922, the Polish mathematician Stefan Banach (1892–1945) found it necessary to provide a detailed explanation of this concept in his work Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales [Ba1]. A completely modern definition of a linear space can be found in the chapter Espaces vectoriels généraux of his book Théorie des opérations linéaires, which was published ten years later (see [Ba2], p. 26).

The concept of a linear space became generally known only in the 1930s, partially in connection with the emergence of functional analysis. At the same time, linear algebra started to constitute itself as an independent discipline, and later became one of the cornerstones of higher mathematics (see e.g. [B2]).

A basis is nowadays defined as a linearly independent spanning set. A simple construction proves the existence of a basis for finitely generated spaces; in the general case, it is sufficient to apply Zorn’s lemma.

The well-known fact that all bases have the same number of elements follows either from the Steinitz exchange lemma (for finitely generated spaces), or from the basic properties of infinite cardinals (see e.g. [B1], pp. 86–87). The dimension of a linear space is then defined as the cardinality of an arbitrary basis.

The concepts of dimension and \( n \)-dimensional space were already known to Grassmann in 1862 (Stufenzahl, Gebiet \( n \)-ter Stufe). Moreover, he stated the theorem concerning the dimensions of the intersection and the sum of two subspaces. In 1888, Peano was also familiar with the notions of basis and dimension (entiti di riferimento, numero delle dimensioni). Both of them restricted themselves to finite-dimensional spaces, but Peano mentioned the space of polynomials as an example of an infinite-dimensional space. Similarly in 1901, Pincherle initially studied these concepts in finite-dimensional spaces (sistema fondamentale, insieme lineare ad \( n \) dimensioni); his ideas on infinite-dimensional spaces were not yet completely rigorous. This should come as no surprise when we consider that set theory (including the basic facts concerning cardinalities) was first developed by Georg Cantor (1845–1918) in the period from 1873 to 1884, and that his extensive unifying work entitled Beiträge zur Begründung der transfiniten Mengenlehre\(^1\) was published in the years 1895 and 1897. Only in the beginning of the 20th century did set theory begin to influence other branches of mathematics.\(^2\)

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2. Let us recall the first study texts, textbooks, and monographs: A. Schoenflies: Die Entwicklung der Lehre von den Punktmannigfaltigkeiten (1st volume: Jahresbericht
In 1905, the German mathematician Georg Hamel (1877–1954) published a short paper entitled Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: \( f(x+y) = f(x) + f(y) \) [Ha] in the journal Mathematische Annalen, where he proved the existence of a basis of the linear space of all reals numbers over the field of rational numbers.³ His original result reads as follows:

Es existiert eine Basis aller Zahlen, d. h. es gibt eine Menge von Zahlen \( a, b, c, \ldots \) derart, daß sich jede Zahl \( x \) in einer und auch nur einer Weise in der Form

\[
x = \alpha a + \beta b + \gamma c + \ldots
\]

darstellen läßt, wo die Zahlen \( \alpha, \beta, \gamma, \ldots \) rational sind, aber in jedem einzelnen Falle nur eine endliche Anzahl von ihnen von Null verschieden ist. ([Ha], p. 459)

His proof made use of the so-called Zermelo’s theorem, which states that every set can be well-ordered.⁴ Using the last result, it was easy for him to conclude that an arbitrary mapping of a given basis into the set of real numbers can be uniquely extended into a linear mapping \( f \) (in general, the mapping is not continuous). In this way, he obtained a solution of the functional equation \( f(x+y) = f(x) + f(y) \).

We remark that the famous Zorn’s lemma, which plays a crucial role in the proof of the existence of a basis of a linear space, was published by Max Zorn (1906–1993) in his work A remark on method in transfinite algebra [Zo] from 1935 in the following form:

**DEFINITION 2.** A set \( \mathcal{A} \) of sets \( \mathcal{A} \) is said to be closed (right-closed), if it contains the union \( \Sigma_{\mathcal{B} \supseteq \mathcal{A}} B \) of every chain \( \mathcal{B} \) contained in \( \mathcal{A} \).

Then our maximum principle is expressible in the following form.

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³ It is well known today that this space is of uncountable dimension.

⁴ This result of Ernst Zermelo (1871–1953) was published in the paper Beweis, daß jede Menge wohlgeordnet werden kann [Z1] in 1904. It included a part of Zermelo’s letter to David Hilbert dating from 24th September 1904. Zermelo proved the well-ordering theorem using a new axiom, which became known as the axiom of choice. See e.g. [M] and [E]. This topic is also discussed in Zermelo’s later works [Z2], [Z3] from 1908.
(MP). In a closed set $\mathfrak{A}$ of sets $A$ there exists at least one, $A^*$, not contained as a proper subset in any other $A \in \mathfrak{A}$. ([Zo], p. 667)\footnote{5}

Both Zermelo’s theorem and Zorn’s lemma (as well as the Hausdorff maximal principle) are equivalent to the axiom of choice.\footnote{6}

* * *

The birth of functional analysis is usually dated to the 1920s and 1930s. However, the basic ideas have their origin in the works of several important mathematicians from the turn of the century, namely Salvatore Pincherle, Vito Volterra (1860–1940), David Hilbert (1862–1943), Jacques Hadamard (1865–1963), and Maurice Fréchet (1878–1973). In their later works, these mathematicians were analysing the ideas which led to the birth and development of functional analysis (see e.g. [Me1]), such as the study of differential and integral equations (often motivated by real-world phenomena), the investigation of function spaces, particular operators, functionals, etc. Their ideas were later picked up by other mathematicians – Ivar Fredholm (1866–1927), Felix Hausdorff (1868–1942), Ernst Fischer (1875–1954), Hans Hahn (1879–1934), Frigyes (Frédéric, Friedrich) Riesz (1880–1956), Paul Lévy (1886–1971), Stefan Banach, John von Neumann (1903–1957), and others. Nowadays, functional analysis is understood as the axiomatic theory of topological linear spaces and linear as well as nonlinear operators.

A decisive step was the birth of the theory of normed linear spaces in the 1920s. Particular examples of various different spaces were already known at the time, and their common properties created an impetus for the development of a new theory. Mathematicians had new tools at their disposal, such as the axiom of choice, Zermelo’s theorem, or transfinite induction. Axiomatisation and abstraction witnessed a significant progress – it was no longer a problem to discard the particular nature of elements of specific spaces (functions, sequences, etc.) and consider general spaces, whose elements, instead of being given explicitly, were now characterised by a certain collection of axioms.

The notion of a Banach space, a complete normed linear space, appeared almost simultaneously in several places: in the doctoral dissertation of Banach from 1920, published in 1922 as a journal article under the title *Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales* [Ba1], in the work *Über Folgen linearer Operationen*\footnote{7} written by Hans Hahn, in the paper *Limit in terms of continuous transformations*,\footnote{8} whose author was Norbert Wiener (1894–1964), and also in the work *Über lineare

\footnote{6} See e.g. [Ku], pp. 23–25, and [Ru1].  
\footnote{7} Monatshefte für Mathematik und Physik 32 (1922), 1–88.  
\footnote{8} Bulletin de la Société Mathématique de France 50 (1922), 119–134.

The theory of normed linear spaces evolved successfully in the 1920s and in the first half of the 1930s. The main results were summarised in Banach’s monograph Théorie des opérations linéaires [Ba2] from 1932. The recent developments in functional analysis were also discussed at the International Congress of Mathematicians held in Bologna in 1928: Hadamard delivered the lecture Le développement et le rôle scientifique du calcul fonctionnel, and Fréchet chose the topic L’analyse générale et les espaces abstraits.\footnote{Atti del Congresso internazionale dei matematici, Bologna 3–10 settembre 1928, T. I, Bologna, 1929, 143–161, 267–274.} There was a clear distinction between functional analysis and general analysis at the time.\footnote{General analysis grew out of the ideas of Eliakim Hastings Moore (1862–1932), namely from his works On a form of general analysis with application to linear differential and integral equations (Atti del IV Congresso Internazionale dei matematici, Roma 6–11 Aprile 1908, Vol. II, Roma, 1909, pp. 98–114), Introduction to a form of general analysis (Yale University Press, 1910, 150 pp.), and On the foundation of the theory of linear integral equations (Bulletin of the American Mathematical Society 18 (1912), pp. 334–362). See e.g. [Bro].}

According to the present definition, a Hilbert space is a real or complex inner product space that is complete.\footnote{Halmos gave a clear and succinct definition in his book Introduction to Hilbert space from 1951: A Hilbert space is an inner product space which, as a metric space, is complete. His definition of a Banach space is similarly brief: \ldots a Banach space is a normed vector space which, as a metric space, is complete. ([H], p. 17)} The original definition of a Hilbert space included the requirement of separability (i.e. the existence of a countable dense subset). The abstract definition of a Hilbert space was formulated in the late 1920s and early 1930s.\footnote{The term Hilbert space was already coined by Schoenflies in the second part of his 1908 book Die Entwicklung der Lehre von den Punktmannigfaltigkeiten, see p. 266, 298.} For example, in 1930, von Neumann gave a thorough definition of a Hilbert space in his work Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren [N1]\footnote{See also Neumann’s work [N0] from 1927, pp. 15–17.}:

\begin{enumerate}
\item $\mathcal{H}$ ist ein linearer Raum. \ldots
\item Es gibt in $\mathcal{H}$ ein, zu dem der Vektorrechnung analoges, inneres Produkt, das eine Metrik erzeugt. \ldots
\item In der Metrik $|f - g|$ ist $\mathcal{H}$ separabel. D. h.: eine gewisse abzählbare Menge ist in $\mathcal{H}$ überall dicht.
\item $\mathcal{H}$ besitzt beliebig (endlich!) viele lin. unabh. Elemente.
\item $\mathcal{H}$ ist vollständig. D. h.: wenn eine Folge $f_1, f_2, \ldots$ in $\mathcal{H}$ der Cauchyschen
\end{enumerate}
Konvergenzbedingung genügt (zu jedem \(\varepsilon > 0\) gibt es ein \(N = N(\varepsilon)\), so daß aus \(m, n \geq N\) \(|f_m - f_n| \leq \varepsilon\) folgt), so ist sie konvergent (es existiert ein \(f\) aus \(\mathcal{F}\), so daß es zu jedem \(\varepsilon > 0\) ein \(N = N(\varepsilon)\) gibt, so daß aus \(n \geq N\) \(|f_n - f| \leq \varepsilon\) folgt). ([N1], pp. 63-66)

In a similar way, Riesz introduced the notion of a Hilbert space in his work "Über die linearen Transformationen des komplexen Hilbertschen Raumes" [R2] from 1930; its first paragraph, entitled "Der komplexe Hilbertsche Raum", reads as follows:

Erstens ist \(\mathcal{F}\) linear, d. h. …

Zweitens ist jedem Paare \(f, g\) eine bestimmte reelle oder komplexe Zahl, das innere Produkt \((f, g)\), zugeordnet …

Drittens ist \(\mathcal{F}\) vollständig, d. h. …

Viertens ist \(\mathcal{F}\) separabel, d. h. er enthält eine abzählbare, überall dichte Teilmenge. …

Fünftes enthält \(\mathcal{F}\) beliebig viele linear unabhängige Elemente, d. h. … ([R2], pp. 26–27)

This type of space was intensively studied in the 1930s, and became gradually known as the Hilbert space.\(^{15}\) This is evidenced e.g. by the works "Linear Transformations in Hilbert Space" [St1] and "Linear Transformations in Hilbert Space and their applications to analysis" [St2] written by Marshall Harvey Stone (1903–1989). It was in this period when it became clear that the assumption of separability is not always necessary (Löwig [L6] and Rellich [Re]) – and that’s why this requirement was dropped from the definition of a Hilbert space.

For a more detailed look at the history of functional analysis, see e.g. the following books:

A. F. Monna: *Functional Analysis in Historical Perspective* [Mo],

J. Dieudonné: *History of Functional Analysis* [Di],

A. Pietsch: *History of Banach Spaces and Linear Operators* [P].

A wealth of information is contained in a number of journal articles, e.g. [BK], [NB1], [NB2], [Bor], [Br1], [Br2], [Bo2], [Bro], [Du], [G], [Hor] [Hu1], [Hu2], [Ka], [K2], [K3], [K4], [Me1], [Me2], [Si1], [Si2], [Sm], and in the monographs [DS], [Kli]. Very interesting and inspiring is the publication

J.-L. Dorier (ed.): *On the Teaching of Linear Algebra* [Do].

Another book which is worth recommending is K. Saxe: *Beginning Functional Analysis* [Sa], which contains many historical notes, an extensive bibliography, and biographical sketches (Banach, Enflo, Fréchet, Fourier, Hilbert, Lebesgue, von Neumann, Riesz, Stone). The textbook

H. Schröder: *Funktionalanalyse* [Sch],

also provides numerous historical and bibliographic comments. Each chapter in the monograph

F. Deutsch: *Best Approximation in Inner Product Spaces* [De]

is also followed by historical notes. For the relationship between functional analysis and its applications, see the monograph

A. W. Naylor, G. R. Sell: *Linear Operator Theory in Engineering and Science* [NS],

or the books

E. Zeidler: *Applied Functional Analysis. Applications to Mathematical Physics* [Ze1],

E. Zeidler: *Applied Functional Analysis. Main Principles and their Applications* [Ze2],

H. Heuser: *Funktionalanalysis. Theorie und Anwendung* [Hu2].

Felix Hausdorff started his 1932 paper *Zur Theorie der linearen metrischen Räume* [H1] by providing definitions of certain basic linear algebraic notions: linear space (*linearer Raum*), linearly independent set (*linear unabhängige Menge*), linear subspace (*lineare Menge, lineare Teilmenge*), etc.

He denoted the linear span (*lineare Hülle*) of a set $A$ (the intersection of all subspaces containing the set $A$, or equivalently, the set of all linear combinations of the elements of $A$) by the symbol $A_\lambda$; he referred to a linearly independent set $A$ as the basis (*Basis*) of the subspace $A_\lambda$. He mentioned Zermelo’s theorem, and remarked that by discarding unnecessary elements, a set $A$ can always be reduced to a basis of $A_\lambda$. After a few preparatory paragraphs, he stated the following definition:

*Ein linearer metrischer Raum $E$ entsteht aus einem linearen Raum, wenn jedem Punkt $x$ eine reelle Zahl $|x|$, der Betrag von $x$, gemäß den Vorschriften*

\[
|0| = 0, \quad |x| > 0 \quad \text{für} \quad x \neq 0, \\
|\alpha x| = |\alpha||x|, \quad |x + y| \leq |x| + |y|
\]

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16 The monograph of Harro Heuser (1927–2011) includes an extensive historical treatise entitled *Ein Blick auf die werdende Funktionalanalyse* (pp. 599–663).

17 See also the commentary [Ch] written by Srishti D. Chatterji (born 1935).
Thus Hausdorff introduced a normed linear space and its corresponding metric linear space. This definition allowed him to discuss the convergence of sequences, sums of infinite series, and the closure of a set $A$ (Konvergenz einer Punktfolge, Summe einer unendlichen Reihe, abgeschlossene Hülle $\overline{A}$ von $A$). He stated that the closure of a subspace is a subspace again, and emphasized that the operations of closure and linear span are not interchangeable; in general, the closure $\overline{A}_\lambda$ of the linear span of a set $A$ and the linear span $(\overline{A})_\lambda$ of the closure of $A$ satisfy

$$(\overline{A})_\lambda \subseteq \overline{A}_\lambda.$$ 

He designated a linearly independent set $A$ as the fundamental set ($\text{Grundmenge}$) of the closed linear span $\overline{A}_\lambda$. We quote the corresponding excerpt:

* * *

According to Hausdorff, a space is called separable if it has a countable dense subset. He emphasised that a separable space has a finite or countable fundamental set, but its basis can be finite, countable as well as uncountable.

* * *

The following text is devoted to Löwig’s works [L6], [L7], and [L8]. Their character is illustrated by several excerpts. However, we remark that the terminology has been subject to many changes over the years. This is evident when we compare works from the first decades of the 20th century with the latest monographs and textbooks (e.g. [Sa], [De], [Sch]), or even with books which already belong to the classics (e.g. [Ru2]).

* * *
In 1934, the journal Acta Litterarum ac Scientiarum (Szeged) published Löwig’s work *Komplexe euklidische Räume von beliebiger endlicher oder transfiniter Dimensionszahl* [L6] consisting of thirty-three pages.\footnote{It was received by the journal on 24th February 1934.} In February 1934, he submitted the same work as a habilitation thesis to the Faculty of Science of the German University in Prague, and successfully became a Privatdozent in 1935.

As he explained in the introduction, his motivation was to show that many statements concerning Hilbert spaces (as they were understood at the time – see the above mentioned definitions by von Neumann and Riesz) are still valid for arbitrary complete inner product spaces. In other words, the assumption of separability, which was then a usual part of the definition of a Hilbert space, is not really substantial:

*In dieser Abhandlung soll gezeigt werden, daß viele Sätze, welche man bisher nur für den Hilbertschen Raum bewiesen hat, auch für beliebige euklidische Räume, d. h. lineare metrische Räume, in denen ein inneres Produkt definiert ist, gelten, oder – mit andern Worten – daß die Voraussetzung der Separabilität des Raumes, die man beim Beweise dieser Sätze bisher zu machen pflegte, unwesentlich ist.* ([L6], p. 1)

The first paragraph entitled *Vorbemerkungen über allgemeine komplexe lineare metrische Räume* ([L6], pp. 1–9) served to introduce the reader to the topic. Löwig first recalled the notions of a complex linear space\footnote{He made a reference to the laws of affine vector algebra.} (komplexer linearer Raum $\mathcal{R}$) and norm (absoluter Betrag des Elements $r$), which enabled him to define a complex normed linear space (komplexer linearer metrischer Raum $\mathcal{R}$). As a next step, he proposed to call two subsets $\mathcal{M}, \mathcal{N}$ of the spaces $\mathcal{R}, \mathcal{S}$ isomorphic if there is a bijection between $\mathcal{M}$ and $\mathcal{N}$, and for the corresponding elements $r_k, n_k$ and arbitrary complex numbers $a_k$, we have

$$\left| \sum_{k=1}^{n} a_k r_k \right| = \left| \sum_{k=1}^{n} a_k n_k \right|.$$  

With the norm at his disposal, he was able to define the neighbourhood of an element $r_0$ (starke Umgebung der Stelle $r_0$), the concepts of a limit point (starke Häufungsstelle einer Teilmenge), closed subset (starkabgeschlossene Menge), and convergent sequence (stark konvergente Folge). The closed linear span of a set $\mathcal{M}$ is the set of all limit points of the span of $\mathcal{M}$.

In the following pair of definitions, he introduced fundamental sequences (starke Fundamentalfolge) and complete sets (starkvollständige Teilmenge). He noted that complete sets are necessarily closed.

In the case when the closed linear span of a set $\mathcal{M}$ is complete, he referred to it as the complete linear span of the set $\mathcal{M}$. He proved that if two subsets of normed linear spaces are isomorphic, then their complete linear spans are also...
isomorphic (provided they exist). In the usual way, he showed the construction of the completion of a linear metric space:

Jeden komplexen linearen metrischen Raum $\mathbb{R}$, welcher nicht starkvollständig ist, kann man zu einem starkvollständigen komplexen linearen metrischen Raum erweitern ... Man betrachte als die Elemente von $\mathbb{R}^*$ die Gesamtheiten äquivalenter starker Fundamentalfolgen von $\mathbb{R}$. ... wir wollen diese Erweiterung die kleinste starkvollständige Erweiterung von $\mathbb{R}$ nennen. ([L6], p. 4)

Löwig also recalled the concepts of a linear mapping between linear spaces (lineare Abbildung), bounded linear mapping between normed linear spaces (beschränkte lineare Abbildung), he mentioned that the set of all bounded linear operators between two spaces $\mathbb{R}$ and $\mathcal{S}$ forms a normed linear space, and defined the notions of a linear operator (linearer Operator) and linear functional (lineares Funktional), which led him to the concept of a weak neighbourhood.

Es seien $L_k$ ($k = 1, 2, \ldots, n$) endlich viele beschränkte lineare Funktionale in $\mathbb{R}$ und $\varepsilon$ eine beliebige positive reelle Zahl. Ferner sei $r_0$ ein beliebiges Element von $\mathbb{R}$. Dann heiße die Gesamtheit der Stellen $r$ von $\mathbb{R}$, welche den Ungleichungen

$$|L_k (r - r_0)| < \varepsilon \quad (k = 1, 2, \ldots, n)$$

genügen, eine schwache Umgebung der Stelle $r_0$. ([L6], p. 5)

He noted that this definition generalises the definition proposed by von Neumann and that the four axioms of neighbourhood formulated by Hausdorff in his book *Grundzüge der Mengenlehre* are satisfied.

The definition of a weak neighbourhood was followed by a number of related concepts: weak limit point of a set $M$ (schwache Häufungsstelle einer Menge $M$), weakly closed set (schwachabgeschlossene Menge), weakly convergent sequence (schwach konvergente Folge); he introduced the notation

$$\lim_{n \to \infty} r_n = r \quad \text{and} \quad \lim_{n \to \infty} \sup r_n = r$$

for the strong and weak convergence, respectively.

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20 A linear mapping $A$ is bounded if the set of all numbers $|Ar|$, where $|r| \leq 1$, is bounded. The supremum of this set is then referred to as the norm of $A$.

21 Schwache Topologie in $\mathcal{H}$. Sei $U_2(f_0; \varphi_1, \ldots, \varphi_s, \varepsilon)$ die Menge aller $f$ mit $|(f - f_0, \varphi_1)| < \varepsilon$, $\ldots$, $|(f - f_0, \varphi_s)| < \varepsilon$; alle $U_2(f_0; \varphi_1, \ldots, \varphi_s, \varepsilon)$ mit beliebigen $\varphi_1, \ldots, \varphi_s$ (aus $\mathcal{H}$, $s$ beliebig) und $\varepsilon > 0$ sind die Umgebungen von $f_0$. ([N2], p. 379)

22 Umgebungsaxiome:

(A) Jedem Punkt $x$ entspricht mindestens eine Umgebung $U_x$; jede Umgebung $U_x$ enthält den Punkt $x$.

(B) Sind $U_x$, $V_x$ zwei Umgebungen desselben Punktes $x$, so gibt es eine Umgebung $W_x$, die Teilmenge von beiden ist ($W_x \subseteq \mathcal{D}(U_x, V_x)$).

(C) Liegt der Punkt $y$ in $U_x$, so gibt es eine Umgebung $U_y$, die Teilmenge von $U_x$ ist ($U_y \subseteq U_x$).

(D) Für zwei verschiedene Punkte $x$, $y$ gibt es zwei Umgebungen $U_x$, $U_y$ ohne gemeinsamen Punkt ($\mathcal{D}(U_x, U_y) = 0$). ([H2], p. 213)
He mentioned the fact that a sequence $r_n$ is weakly convergent to an element $r$ if and only if the sequence of numbers $Lr_n$ converges to $Lr$ for every bounded linear functional $L$ (Satz 2). He pointed out that a weak limit point of a set $\mathcal{M}$ need not be the weak limit of any sequence of elements from $\mathcal{M}$ (and gave a reference to the work of von Neumann [N2], pp. 380–381).

Then he recalled the following result (Satz 3): If $\mathcal{M}$ is a subset of a normed complex linear space and the element $r_0$ lies outside the closed linear span of $\mathcal{M}$, then there exists a bounded linear functional $L$ which is nonzero at $r_0$ and vanishes on $\mathcal{M}$.

As he remarked, this theorem is usually stated and proved for real linear spaces; however, he showed that a real-linear functional can be simply extended to a complex-linear functional by setting

$$Lr = Rr - iR(ir),$$

where $R$ is the real-linear functional. The previous result leads to the following theorem:

Satz 4. Jede schwache Häufungsstelle einer Menge $\mathcal{M}$ gehört der starkabgeschlossenen linearen Hülle von $\mathcal{M}$ an. ([L6], p. 6)

As a consequence, he obtained the following pair of corollaries:

Satz 5. Ist

$$\lim_{n \to \infty} r_n = r,$$

dann gehört $r$ der starkabgeschlossenen linearen Hülle der Menge der Elemente $r_n$ ($n = 1, 2, 3, \ldots$) an.

Satz 6. Jede starkabgeschlossene lineare Mannigfaltigkeit ist auch schwachabgeschlossen. ([L6], p. 7)

Thus, given a linear subspace (or a linear span), it is enough to consider its closedness. Löwig then stated the following definition:

Eine Teilmenge $\mathcal{M}$ von $\mathcal{R}$ heiße schwachvollständig, wenn sie in der kleinsten starkvollständigen Erweiterung von $\mathcal{R}$ schwachabgeschlossen ist. ([L6], p. 7)

Every weakly complete set is also weakly closed, but the converse is not necessarily true.

Eine Folge $r_n$ ($n = 1, 2, 3, \ldots$) von Elementen von $\mathcal{R}$ heiße eine schwache Fundamentalfolge, wenn für jedes beschränkte lineare Funktional $L$ in $\mathcal{R}$

$$\lim_{n \to \infty} Lr_n$$

existiert. ([L6], p. 8)

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23 Löwig cited Hausdorff’s work [H1], p. 306.
24 This theorem was also formulated by Banach in [Ba1], p. 134.
25 The theorem generalises a result due to Schmidt [S]. See also von Neumann [N2], p. 396.
26 The reader should be aware that in the existing literature, the notions of weak closedness and weak completeness usually have a different meaning.
In the first paragraph, Löwig cited Hausdorff’s work [H1], Banach’s monograph [Ba2], von Neumann’s work [N2], Schmidt’s article [S], and Mazur’s paper [Ma]; these works were published (except the article [S]) in the period from 1930 to 1932.

Löwig divided the core of his work in two parts, depending on whether or not he made use of Zermelo’s theorem.

In the second paragraph entitled Sätze über komplexe euklidische Räume, die ohne Benützung des Wohlordnungssatzes bewiesen werden können (pp. 9–25), he first gave the definition of a general inner product (hermiteische bilineare Funktion). In the next theorem, he showed that if the inner product is also positive definite, then it induces a norm, which satisfies the Cauchy-Schwarz inequality and the triangle inequality. Thus a linear space with a positive definite inner product (innere Produkt) is also a normed linear space; the \( n \)-dimensional complex Euclidean space (komplexer euklidischer Raum) and complex Hilbert space (komplexe Hilbertsche Raum) are included as special cases.

The following series of theorems is concerned with various properties of Hilbert spaces. We quote some of them in Löwig’s original wording.

\[ \text{Satz 11. Zu jedem beschränkten linearen Funktional } L \text{ eines vollstän digen komple xen euklidischen Raumes } \mathcal{R} \text{ gibt es ein (und daher auch nur ein) erzeugendes Element, d. h. ein Element } u \text{ von } \mathcal{R} \text{ von der Beschaffenheit, daß für jedes Element } r \text{ von } \mathcal{R} \]

\[ Lr = (r, u) \]

ist. ([L6], p. 11)

Löwig noted that Theorem 11 is well known to be true for complex Euclidean finite-dimensional spaces as well as for complex Hilbert spaces (he cited Riesz’s work [R2]), and that Theorem 12 is a corollary of these statements. He then proved Theorem 13, generalised the statement of Theorem 12 to arbitrary complex spaces, and finally proved Theorem 11.

\[ \text{Satz 12. Ist } L \text{ ein beschränktes lineares Funktional in einem endlichdimensio nalen komplexen euklidischen Raum oder im komplexen Hilbertschen Raum, dann gibt es stets ein Element } u \text{ des betreffenden Raumes mit } |u| = |L| \text{ und } Lu = |u|^2. \]  

([L6], p. 11)

\[ \text{Satz 13. Gibt es zu einem beschränkten linearen Funktional } L \text{ in einem komple xen euklidischen Raum } \mathcal{R} \text{ ein Element } u \text{ von } \mathcal{R} \text{ mit } |u| = |L| \text{ und } Lu = |u|^2, \text{ dann ist identisch } \]

\[ Lr = (r, u). \]

Löwig introduced the notion of a “regular subspace“ – a subspace whose orthogonal complement is also its algebraic complement – and in the following pair of theorems, he explained the relation between regularity, completeness and closedness.
Definition 10. Eine lineare Mannigfaltigkeit $M$ eines komplexen euklidischen Raumes heiße regulär, wenn man jedes Element $r$ von $R$ in Bezug auf $M$ in eine Tangentialkomponente $t$ und eine Normalkomponente $n$ zerlegen kann, d. h. wenn man zu jedem Element $r$ von $R$ zwei ebensolche Elemente $t$ und $n$ angeben kann, so daß

$$r = t + n$$

ist, $t M$ angehört und $n$ zu allen Elementen von $M$ orthogonal ist. ([L6], p. 12)


Satz 15. Für die Regularität einer linearen Mannigfaltigkeit $M$ von $R$ ist hinreichend, aber nicht notwendig, daß $M$ vollständig sei. ([L6], pp. 11–14)

He summarised these results in the following lucid form:

Vollständigkeit $\rightarrow$ Regularität $\rightarrow$ Abgeschlossenheit.


Löwig also studied adjoint and self-adjoint operators and derived some of their basic properties.27

The presentation of his definitions and theorems is completely modern.

Definition 11. Ein linearer Operator $B$ in einem komplexen euklidischen Raum $R$ heiße zu dem linearen Operator $A$ in $R$ adjungiert, wenn für beliebige Elemente $r$ und $n$ von $R$

$$(Ar, n) = (r, Bn)$$

ist.

Given a linear operator $A$, he noted there is at most one adjoint operator, which he denoted by $A^*$. 


Satz 23. Ist $R$ vollständig, dann existiert zu jedem beschränkten linearen Operator $A$ in $R$ der adjungierte lineare Operator. ([L6], p. 20)

He also studied operators $E$ satisfying $E^2 = E$ (so-called Einzeloperator).

In the second part of his treatise, Löwig cited the works of Hilbert [Hi], Mazur [Ma], von Neumann [N1], and Riesz [R1], [R2].

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27 This topic originated in the works of Schmidt. Self-adjoint operators (hypermaximalen Operatoren) were studied by von Neumann in his 1930 work [N1], the term (self-adjoint) was coined by Stone in 1932. von Neumann and Stone gave a fairly complete presentation of the theory of self-adjoint operators in their monographs [N3] and [St2] published in 1932.
The third paragraph is called *Beweis weiterer Sätze über komplexe euklidische Räume unter Anwendung des Wohlordnungssatzes* (pp. 25–32). Löwig starts by introducing the space $\mathfrak{R}$ of complex functions $\varphi$ defined on a certain set $\mathfrak{N}$, such that $\varphi$ vanishes with the exception of at most countably values (i.e., the function has a countable support), and such that the series

$$\sum_{n=1}^{\infty} |\varphi(u_n)|^2$$

is convergent for every sequence $u_n$ with the property that $\varphi(u) = 0$ for every $u \neq u_n$ (i.e., the sequence “covers” the support). Given a pair of functions $r = \varphi(u), n = \psi(u)$ from $\mathfrak{R}$, he defined their inner product by the formula

$$(r, n) = \sum_{n=1}^{\infty} \varphi(u_n) \psi(u_n),$$

where $\varphi(u) = 0$ and $\psi(u) = 0$ for every $u \neq u_n$. In this way, he obtained a complete complex Euclidean space (i.e., a Hilbert space). Moreover, the collection of all functions satisfying $\varphi(u_0) = 1$ for a single value $u_0$ and $\varphi(u) = 0$ for all remaining $u$ forms a maximal orthonormal set (see below). We remark that von Neumann [N3] (pp. 37–38) investigated the case when $\mathfrak{N}$ is the set of all real numbers.

For an arbitrary cardinal number $\aleph$, Löwig used the symbol $\mathfrak{R}_{\aleph}$ to denote the above mentioned space of complex functions defined on a set $\mathfrak{N}$ whose cardinality is $\aleph$. The space $\mathfrak{R}_{\aleph}$ has a finite dimension if $\aleph$ is finite, and $\mathfrak{R}_{\aleph}$ is a Hilbert space if $\aleph$ is countable.

Löwig defined a maximal orthonormal set (*vollständiges normiertes Orthogonalsystem*) in a complete complex Euclidean space, and proved its existence using Zermelo’s theorem:

*Satz 28.* In jedem vollständigen komplexen euklidischen Räume gibt es mindestens ein vollständiges normiertes Orthogonalsystem. ([L6], p. 27)\textsuperscript{28}

He developed the theory further and obtained the following results (theorems 29, 30, 31, 32 on pages 27–29):

- The complete linear span of every maximal orthonormal set in a complete complex Euclidean space coincides with the whole space.
- Two complete complex Euclidean spaces possessing maximal orthonormal sets of the same cardinality are necessarily isomorphic.\textsuperscript{29}

\textsuperscript{28} Although it is possible to define a maximal orthonormal set and prove its existence in a general inner product space (without the assumption of completeness), Löwig was interested only in the theory of complete spaces.

\textsuperscript{29} *Satz 30.* Besitzen zwei vollständige komplexe euklidische Räume vollständige normierte Orthogonalsysteme von gleicher Mächtigkeit, dann sind sie isomorph. ([L6], p. 27)
- A complete complex Euclidean space with a maximal orthonormal set of cardinality $\mathbb{N}$ is necessarily isomorphic to $\mathbb{R}_\mathbb{N}$.\(^{30}\)

- If $\mathfrak{M}$ is a maximal orthonormal set in a complete complex Euclidean space, then for every element $r$, only countably many of the numbers $(r, e), e \in \mathfrak{M}$, might be nonzero. If $e_n \in \mathfrak{M}$ and $(r, e) = 0$ for every $e \neq e_n$, then $r = \sum_{n=1}^{\infty} (r, e_n) e_n$.\(^{31}\)

Löwig then showed the important fact that all maximal orthonormal sets in a complete complex Euclidean space have the same cardinality:

**Satz 33.** Zwei verschiedene vollständige normierte Orthogonalsysteme eines vollständigen komplexen euklidischen Raumes $\mathfrak{R}$ haben stets die gleiche Mächtigkeit. ([L6], p. 31)

In the proof of this statement, he had to use the properties of infinite cardinal numbers; he gave a reference to both editions of Hausdorff's monograph on set theory.\(^{32}\) Theorem 33 then enabled him to give the following definition of a dimension:

**Definition 20.** Ist $\mathfrak{R}$ ein vollständiger komplexer euklidischer Raum und $\mathfrak{M}$ ein vollständiges normiertes Orthogonalsystem von $\mathfrak{R}$, dann heiße die Mächtigkeit von $\mathfrak{M}$ die Dimensionszahl von $\mathfrak{R}$. ([L6], p. 32)

As Löwig noted, it follows that the dimension of the complete complex Euclidean space $\mathfrak{R}_\mathbb{N}$ is $\mathbb{N}$. He then defined the dimension of an arbitrary complex Euclidean space as the dimension of its smallest completion. He concluded by saying that the method of proof of Theorem 33 can be used to prove the fact that all bases of a complex linear space have the same number of elements. He recalled that the definition was given by Hausdorff in [H1], p. 395.

In the third part of his work, Löwig cited Hausdorff's publications [H1], [H2], [H3], and von Neumann's work [N3].

\star \star \star

Let us remark that Löwig's work [L6] published in the journal Acta Litterarum ac Scientiarum (Szeged) was immediately followed by the five-page article *Zur Theorie des Hilbertschen Raumes* [R3]\(^{33}\) written by Riesz, who cited

\(^{30}\) *Satz 31. Hat ein vollständiges normiertes Orthogonalsystem eines vollständigen komplexen euklidischen Raumes die Mächtigkeit $\mathbb{N}$, dann ist dieser Raum $\mathfrak{R}_\mathbb{N}$ isomorph. ([L6], p. 27)\(^{31}\)

\(^{31}\) *Satz 32. Ist $\mathfrak{M}$ ein vollständiges normiertes Orthogonalsystem eines vollständigen komplexen euklidischen Raumes $\mathfrak{R}$ und $r$ ein beliebiges Element von $\mathfrak{R}$, dann sind von den Zahlen $(r, e)$ mit $e \in \mathfrak{M}$ höchstens abzählbar viele von Null verschieden. Ist weiter $e_n \in \mathfrak{M}$ ($n = 1, 2, 3, \ldots$) und $(r, e) = 0$ für $e \in \mathfrak{M}$, $e \neq e_n$ ($n = 1, 2, 3, \ldots$), dann ist $r = \sum_{n=1}^{\infty} (r, e_n) e_n$, wobei das Summenzeichen im Sinne der starken Konvergenz zu verstehen ist. ([L6], pp. 28–29)\(^{32}\)


\(^{33}\) It was received by the journal on 30th April 1934.
Löwig’s work [L6]. The paper represented a continuation of Riesz’s previous work Über die linearen Transformationen des komplexen Hilbertschen Raumes [R2] published in the same journal; it contained a very detailed definition of a Hilbert space.

In [R3], Riesz proved the following two well-known theorems (he still found it necessary to recall the definitions of the basic notions) and referred the reader to [R2], which also included these two theorems (see [R2], p. 28):

Die folgenden anspruchslosen Bemerkungen betreffen die beiden wohlbe-kannten, für die Theorie des reellen oder komplexen Hilbertschen Raumes grundlegenden Sätze, die ich auch in einer früheren Arbeit der Behandlung an die Spitze stellte.

Satz A. Ist eine lineare Mannigfaltigkeit $L$ des Hilbertschen Raumes $H$ nicht überall dicht in $H$, so gibt es ein Element $g$ aus $H$ mit $|g| = 1$, das zu allen Elementen aus $L$ orthogonal ist.

Satz B. Für jede lineare Funktion $l(f)$ gibt es ein eindeutig bestimmtes „erzeugendes“ Element $g$, so daß

$$l(f) = (f, g).$$

Dabei ist unter linearen Mannigfaltigkeit von $H$ eine Teilmenge zu verstehen, die mit $f$ für jede komplexe Zahl $c$ auch $cf$ und mit $f$ und $g$ auch $f + g$ enthält. Eine in $H$ definierte (skalare) Funktion $l(f)$ heißt linear, wenn sie den Gleichungen $l(cf) = cl(f)$, $l(f + g) = l(f) + l(g)$ genügt und auf der Einheitskugel $|f| = 1$ beschränkt ist.

Gewöhnlich stützt man den Beweis dieser beiden Sätze auf die Separabilität ... ([R3], p. 34)

* * *

Löwig’s work [L6] was closely related to certain essential results of functional analysis, which were discovered in the previous three decades and had their origin in the works of Riesz, Fischer, and Fréchet from the beginning of the century. These results were concerned with the isometry between the spaces $L_2$ and $l_2$, its construction based on maximal orthonormal sets, completeness of these spaces, the general form of a continuous (bounded) linear functional on $L_2$ and $l_2$, etc.

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34. Die unmittelbare Anregung zu diesen Zeilen verdanke ich der voranstehenden Arbeit (i.e. the work [L6]), in welcher unter andern gezeigt wird, daß auch Satz B ohne Dimensions-abgrenzung nach oben richtig ist. ([R3], p. 35)
35. Cf. Löwig’s Satz 16.
36. Cf. Löwig’s Satz 11.
discovered in the late 1920s. Löwig’s main contribution was his consideration of complex linear spaces, the study of Hilbert spaces without the assumption of separability, the investigation of orthonormal sets, etc.

The work [L6] received a wide response. It was reviewed in the journal Zentralblatt für Mathematik by Stone, who also mentioned the paper [L6] in his subsequent review of Riesz’s work [R3].

Georg Aumann (1906–1980) wrote a report on Löwig’s work [L6] for the journal Jahrbuch über die Fortschritte der Mathematik. In the same journal, Riesz’s work [R3] was reviewed by Gustav Heinrich Adolf Doetsch (1892–1977), who also recalled the results obtained by Löwig in [L6].

On 12th September 1934, Franz Rellich (1906–1955) delivered the lecture Abstrakte Spektraltheorie und fastperiodische Funktionen at the meeting of the Deutsche Mathematiker-Vereinigung; Löwig’s work [L6] was also presented on that occasion.


The hyper-Hilbert spaces (without E) have been first discussed by H. Löwig, Acta Szeged, vol. 7 (1934), pp. 1–33. ([JN], p. 719)

In 1936, Francis Joseph Murray (1911–1996) and John von Neumann cited Löwig’s article [L6] in their extensive work On rings of operators [MN] consisting of more than 110 pages; the article [L6] is one of the 22 items in their list of references. In the same year, Oswald Teichmüller (1913–1943) cited the paper [L6] in his work Operatoren im Wachsschen Raum [Te].

Béla Adalbert Lengyel (1910–2002) and Marshall Harvey Stone cited the work [L6] in their 1936 paper Elementary Proof of the Spectral Theorem [LS]. They wrote:


39 Zbl 0009.25901, Zbl 0009.25902.

40 See JFM 60.0324.01 and JFM 60.0325.01.

Since the dimensionality postulates occurring in the definition of Hilbert space are actually so little used in the theory, it is convenient to suppress them. Thus we shall consider those spaces which are called following Löwig [12], complex Euclidean spaces. ([LS], pp. 853–854)

Their article was reviewed by Edgar Raymond Lorch (1907–1990), who also mentioned Löwig’s work [L6]:

Der Beweis wird für den allgemeineren Fall eines komplexen euklidischen Raumes \( \Sigma \) gegeben (vgl. Löwig ...), was bekanntlich keine wesentlichen neuen Schwierigkeiten macht.


In the same year, the paper [L6] was cited by Tosio Kitagawa in the work The Parseval theorem of the Cauchy series and the inner products of certain Hilbert spaces [Ki]. He appreciated Löwig’s precise formulation of the basic concepts.

Also in the same year, Franz Wecken (1912 – before 1945) cited Löwig’s work [L6] in the article Unitärinvarianten selbstadjungierter Operatoren [W]:

Während nun der Spektralsatz für selbstadjungierte Operatoren im Hilbertschen Raum nahezu unverändert auch im nichtseparablen Raum gilt (Rellich [11], Löwig [6]), treten für die Theorie der unitären Invarianten beim Übergang von abzählbarer zu nichtabzählbarer Dimension die im folgenden angedeuteten neuen Verhältnisse ein. ([W], p. 422)


A Banach space is a linear, normed, complete space [3, chap. 5]. A euclidean space of dimension \( \alpha \) where \( \alpha \) is any cardinal number, is defined to be the Banach space of sequences \( x_\nu \) of real numbers where \( \nu \) ranges over a class of cardinal number \( \alpha \), and \( \sum x_\nu^2 \) is finite and equal to the square of the norm [4]. ([Ph], p. 930)

In 1941, the name of Löwig appeared in the cyclostyled lectures Invariant measures [N5] by John von Neumann with comments due to Paul Richard Halmos (1916–2006).

In the same year, the article [L6] was cited by J. W. Calkin in his extensive work Two-sides ideals and congruences in the ring of bounded operators in Hilbert space [Ca] consisting of 35 pages.

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42 See JFM 62.0450.02.
44 Page 124 or page 74 in the printed version from 1999, respectively.
In 1941, Abram Iezekiilovič Plesner (1900–1961) cited the work \([L6]\) in the extensive article *Spektralnaja teorija linejnych operatorov* [Pl].

In 1947, Gottfried Maria Hugo Köthe (1905–1989) made use of Löwig’s results from \([L6]\) in his paper *Eine axiomatiche Kennzeichnung der linearen Räume* [KÖ1].

In the same year, Robert C. James (1918–2003) cited Löwig’s Theorem 32 from \([L6]\) in his article *Inner products in normed linear spaces* [J1]:

Any complete normed linear space \(T\) which has an inner product is characterized by its (finite or transfinite) cardinal “dimension-number” \(n\). It is equivalent to the space of all sets \(x = (x_1, x_2, \ldots)\) of \(n\) real numbers satisfying \(\sum_i x_i^2 < +\infty\), where \(||x|| = \left(\sum_i x_i^2\right)^{1/2}\) \([7, \text{Theorem 32}]\). ([J1], p. 559)

Also in the same year, he cited the work \([L6]\) in his paper *Orthogonality and linear functionals in normed linear spaces* [J2]; he made a reference to Löwig’s Theorems 11 and 16 (see the above mentioned *Satz 11* and *Satz 16*).

In 1948, Helmut Schiek (1915–1981) cited the work \([L6]\) in the paper *Mengen mit affiner Anordnung* [Sc].

Naum Il’jič Achiezer (also Achieser, 1901–1980) and Izrail’ Markovič Glazman (also Glasmann, 1916–1968) included Löwig’s work \([L6]\) among the references in their monograph *Teorija linejnych operatorov v Gil’bertovom pros-transtve* [AG] published in 1950; the second edition of the book appeared in 1966. The monograph was also translated to German and English.

Frédéric Riesz and Béla Szökefalvi-Nagy (1913–1998) cited Löwig’s work \([L6]\) in their monograph *Leçons d’analyse fonctionnelle* [RS] from 1952. Another French edition of this book appeared in the next year, followed by several English, German, and Russian editions. The authors also mentioned Riesz’s work *Zur Theorie des Hilbertschen Raumes* [R3], the work of Rellich entitled *Spektraltheorie in nichtseparabeln Räumen* [Re], and von Neumann’s work *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren* [N1]. The chapter *Espace de Hilbert abstrait* contains the following remark related to the definition of a Hilbert space:

*Cf. J. v. Neumann [1]; cet auteur énonce encore deux axiomes exigeant que l’espace soit séparable et de dimension infinie (donc de dimension dénombrablement infinie). Nous préférons de ne pas exclure dès commencement les espaces de dimension finie et les espaces non séparables. Voir, pour les espaces non séparables, Löwig [1], Riesz [15], Rellich [1]. ([RS], 1952, p. 195)


It was the work \([L6]\) which Bourbaki had in mind when referring to Löwig...
in the treatise on the history of topological vector spaces:

_Dès ce moment les points essentiels de la théorie des espaces hilbertiens peuvent être considérés comme acquis; parmi les progrès plus récents, il faut notamment mentionner la présentation axiomatique de la théorie donnée vers 1930 par M. H. Stone et J. von Neumann, ainsi que l’abandon des restrictions de «séparabilité», qui s’effectue aux environs de 1934, dans les travaux de Rellich, Löwig, et F. Riesz (IXe). ([Bo1], 1955, p. 168)\textsuperscript{45}

In 1958, Nelson Dunford (1906–1986) and Jacob T. Schwartz (1930–2009) cited the work [L6] in the extensive monograph Linear Operators, which was later translated into Russian. They mentioned Löwig’s proof of the facts that all orthonormal bases of a Hilbert space have the same cardinality (see [L6], Satz 33, p. 31 – see above; see also Rellich’s work [Re], p. 355), and that two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension (see [L6], Satz 30, Satz 31, p. 27 – see above). They also acknowledged Löwig’s contribution in suppressing the requirement of separability in the definition of a Hilbert space.\textsuperscript{46}


Pietsch mentioned Löwig’s work [L6] in the epilogue to his 1989 publication [HS]\textsuperscript{47}, as well as in his extensive monograph History of Banach spaces and linear operators [P] published in 2007.\textsuperscript{48}

The representation theorem for abstract non-separable Hilbert spaces had to wait for Löwig [1934a, p. 11] and Riesz [1934, p. 34]:

Für jede lineare Funktion \( l(f) \) gibt es ein eindeutig bestimmtes erzeugendes Element \( g \), so dass \( l(f) = (f, g) \). ([P], p. 31)

... formula (2.3.6.a) was already discovered by Löwig [1934a, p. 6]. ([P], p. 39)\textsuperscript{49}

In 2001, Löwig’s work [L6] was cited in the book Best Approximation in Inner Product Spaces [De] by Frank Deutsch (born 1936); the information on Löwig’s results is taken over from the monograph [DS].


\textsuperscript{45} The reference (IXe) corresponds to Riesz’s work [R3]. For the above quotation, see [Bo2]: p. 213 in the English version from 1994, p. 248 in the German version from 1971, or p. 224 in the Russian version from 1963, respectively. The reference to Rellich stands for his work [Re].

\textsuperscript{46} See [DS], p. 372 and p. 373, or p. 407 and p. 414 in the Russian edition, respectively.

\textsuperscript{47} Nicht-separable Hilberträume sind erstmalig von Löwig (1934) und Rellich (1934) betrachtet worden. ([HS], p. 287)

\textsuperscript{48} See [P], p. 13, p. 31, and p. 39.

\textsuperscript{49} The number (2.3.6.a) refers to Löwig’s formula \( Lr = Rr - iR(ir) \).
In 2005, Femi O. Oyadare cited Löwig’s work [L6] in his paper *Construction of higher orthogonal polynomials through a new inner product, \( \langle \cdot, \cdot \rangle_p \) in a countable real \( L^p \)-space* [Oy].

Let us return to the quotation from Pietsch’s monograph concerning Löwig’s formula

\[ Lx = Rx - iR(ix), \]

which describes the structure of a functional on a complex space.

In 1938, Henri F. Bohnenblust (1906–2000) and Andrew Sobczyk (1915–1981) came up with a similar idea; in their short paper *Extensions of functionals on complex linear spaces* [BS], they stated the following analogue of the Hahn-Banach theorem:

**Theorem 1.** Let \( l \) be any complex linear subspace of a normed complex linear space \( L \). Let \( f(x) \) be any complex linear functional defined on \( l \), having a norm \( M \). Then there always exists a complex linear functional \( F(x) \) defined on \( L \), which coincides with \( f(x) \) in \( l \), and which has the same norm \( M \) on \( L \). ([BS], p. 91)

They expressed the complex functional \( f \) in the form

\[ f(x) = f_1(x) + i f_2(x) \]

and noted that \( f_2(x) = -f_1(ix) \). The extended functional \( F \) is then obtained by letting

\[ F(x) = F_1(x) - i F_1(ix), \]

where \( F_1 \) is a real functional from the classical version of the Hahn-Banach theorem.

Moreover, they mentioned the existence of a similar proof given by Murray in the work *Linear transformations in \( L^p \), \( p > 1 \)* [Mu] in 1936. However, Murray was aware of Löwig’s work [L6] at the time (or a few weeks later), because he cited it (together with von Neumann) in the article [MN].

Soon after, the above mentioned result of Bohnenblust and Sobczyk became generally known; for example, it was cited by Salomon Bochner (1899–1982) and Angus Ellis Taylor (1911–1999) in their paper *Linear functionals on certain spaces of abstractly-valued functions* [BT] from 1938.

Later, it became apparent that a similar result appeared in Suchomlinov’s paper *O prodolženii linejnych funkcionalov v komplexnom i kvaternionnom linejnom prostranstve* [Su] from 1938. Löwig’s result dating back to 1934 was thus completely overlooked.

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50 The work [Mu] was presented on 23rd February 1935, the work [MN] was received by the journal on 3rd April 1935.
As late as 1975, John A. R. Holbrook wrote the following words in his paper *Concerning the Hahn-Banach theorem* [Ho]:

... the Hahn-Banach extension theorem for real spaces dates from papers by H. Hahn [3] in 1927 and S. Banach [1] in 1929, while the familiar trick deriving the theorem for other scalars by reduction to the real case was not forthcoming until 1938: H. F. Bohnenblust and A. Sobczyk [2] (complex scalars); G. A. Soukhomlinoff [9] (complex or quaternionic scalars). ([Ho], p. 322)

In 2008, Lawrence Narici and Edward Beckenstein acknowledged the contribution of Löwig’s work [L6] in their article *The Hahn-Banach theorem and the sad life of E. Helly* [NB2]. They remarked that the important moment was the discovery of the relation between the real and complex component of a linear functional $f$ on a complex space, i.e. the relation

$$\Re f(ix) = \Im f(x).$$

Although usually credited to F. Murray [1936], H. Löwig discovered this in 1934. Murray reduced the complex case to the real case, then used the real Hahn-Banach theorem to prove the complex form for subspaces of $L_p[a,b]$ for $p > 1$. Murray’s perfectly general method was used and acknowledged by Bohnenblust and Sobczyk [1938] who proved it for arbitrary complex normed spaces. They, incidentally, were the first to call it the Hahn-Banach theorem. Also by reduction to the real case, Soukhomlinov [1938] and Ono [1953] obtained the theorem for normed spaces over the complex numbers and the quaternions. ([NB2], p. 105)

In 2009, Barbara D. MacCluer mentioned Löwig’s contribution concerning complex spaces in her book *Elementary Functional Analysis*:

The extension of the proof of the Hahn-Banach theorem from the real case to the complex case is outlined in Exercises 3.2 and 3.3. While it is not hard, historically there was a span of nearly ten years between the work on the real case by Banach and the extension to the complex case by H. Bohnenblust and A. Sobczyk in 1938. Perhaps not coincidently, Banach’s esteemed 1932 treatise *Opérations Linéaires* deals only with real Banach spaces. In the particular setting of $X = L^p$ the complex case appeared in 1936 in the work of F. Murray; see also the comment in Exercise 3.3 on an earlier contribution by H. Löwig.

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52 The references correspond to [Mu], [BS], [Su], [O]. It is interesting that Narici and Beckenstein did not mention Löwig’s work [L6] in their earlier paper *The Hahn-Banach theorem: the life and times* [NB1] from 1997 – they cited only [Mu], [BS], [Su] and [O], where the work [O] is cited incorrectly under the number [60]; on the other hand, they also recalled the work [Ho] from 1975. It is most probable that they learned about Löwig’s result in the above mentioned monograph [P] by Pietsch.
The work of Bohnenblust and Sobczyk may be the first place that the result is referred to as the “Hahn-Banach Theorem.” ([Mc], p. 54)

Löwig’s article Über die Dimension linearer Räume [L7] consisting of six pages was published in the journal Studia mathematica in 1934; it was received by the journal on 17th March 1934.53

Löwig’s point of departure were two important works published in 1932, namely the long paper Zur Theorie der linearen metrischen Räume [H1] by Felix Hausdorff, and the monograph Théorie des opérations linéaires [Ba2] by Stefan Banach. He also cited his previous paper Komplexe euklidische Räume von beliebiger endlicher oder transfiniter Dimensionszahl [L6].

He considered both real and complex normed linear spaces, and referred the reader to the concepts introduced by F. Hausdorff in his work [H1], e.g. linearer metrischer Raum, lineare Hülle, and abgeschlossene lineare Hülle. The article [L7] contains the following two definitions of a dimension:

- The affine dimension of a linear space $\mathcal{R}$, which coincides with the classical dimension of a linear space $\mathcal{R}$.

It is defined as the smallest cardinal number among the cardinalities of all sets whose linear span is the whole space $\mathcal{R}$.

- The metric dimension of a normed linear space $\mathcal{R}$.

It is defined as the smallest cardinal number among the cardinalities of all sets whose closed linear span is the whole space $\mathcal{R}$. We feel Zermelo’s well-ordering theorem behind both definitions.

Definition 1. Unter der affinen Dimensionszahl eines linearen Raumes $\mathcal{R}$ verstehe man die kleinste Kardinalzahl $\kappa$ von der Eigenschaft, daß es mindestens eine Teilmenge von $\mathcal{R}$ von der Mächtigkeit $\kappa$ gibt, deren lineare Hülle mit $\mathcal{R}$ zusammenfällt. ([L7], p. 18)

Definition 2. Unter der metrischen Dimensionszahl eines linearen metrischen Raumes $\mathcal{R}$ verstehe man die kleinste Kardinalzahl $\kappa$ von der Eigenschaft, daß es mindestens eine Teilmenge von $\mathcal{R}$ von der Mächtigkeit $\kappa$ gibt, deren abgeschlossene lineare Hülle mit $\mathcal{R}$ zusammenfällt. ([L7], p. 18)

Löwig noted the following relation between the two concepts: the affine dimension is never smaller than the metric dimension.

Es ist klar, daß die affine Dimensionszahl eines linearen metrischen Raumes niemals kleiner sein kann als seine metrische Dimensionszahl. ([L7], p. 19)

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53 The journal Studia mathematica was founded by Stefan Banach in 1929 to deal mainly with the questions of functional analysis.
He introduced the notion of a *basis* of a linear space (a linearly independent spanning set). His wording is cumbersome by modern standards; he was trying to avoid saying that an infinite set is linearly independent.

**Definition 3.** Eine Teilmenge $M$ eines linearen Raumes $R$ heiße eine Basis von $R$, wenn endlich viele Elemente von $M$ stets linear unabhängig sind und die lineare Hülle von $M$ mit $R$ zusammenfällt. ([L7], p. 19)

He remarked that the fact that every linear space has a basis is well known. In the subsequent theorem, he proved that all bases of a given linear space have the same cardinality, because the cardinality of every basis equals the affine dimension introduced earlier. He made use of some basic facts concerning infinite cardinal numbers. Nowadays, this topic is usually classified as belonging to linear algebra.

**Satz 1.** Jede Basis eines linearen Raumes $R$ hat eine Mächtigkeit, welche der affinen Dimensionszahl von $R$ gleich ist. ([L7], p. 19)

In the next theorem, he compared the cardinality of a linear space with its affine dimension. Again, he needed the basic properties of infinite cardinal numbers.

**Satz 2.** Ein linearer Raum $R$, der nicht nur aus einem Nullelement besteht, hat die Mächtigkeit des Kontinuums, wenn seine affine Dimensionszahl kleiner als die Mächtigkeit des Kontinuums ist, und sonst eine Mächtigkeit, welche gleich dieser affinen Dimensionszahl ist. ([L7], p. 20)

For normed linear spaces, Löwig introduced the notion of a *fundamental set*, showed that it corresponds to the metric dimension, and that a subspace and its closure have the same metric dimension.

**Definition 4.** Eine Teilmenge $M$ eines linearen metrischen Raumes $R$ heiße eine Grundmenge von $R$, wenn erstens kein Element von $M$ der abgeschlossenen linearen Hülle der übrigen Elemente von $M$ angehört und zweitens die abgeschlossene lineare Hülle von $M$ mit $R$ zusammenfällt. ([L7], p. 20)

Löwig compared his concept of a fundamental set with similar notions defined in the above mentioned Hausdorff’s work [H1] as well as in Banach’s monograph [Ba2]. He noted that in Hausdorff’s notion of a *Grundmenge*, the first condition is replaced by linear independence, while Banach introduced his *ensemble fondamental* using the second condition only. According to the

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54 Wenn $A$ linear unabhängig ist, so heiße sie eine Grundmenge ihrer abgeschlossenen linearen Hülle $\mathcal{L} \ldots$ ([H1], p. 295)

55 Un ensemble $G \subset E$ s’appelle fondamental, lorsque l’ensemble de toutes les combinaisons linéaires d’éléments de $G$ est dense dans $E$; il s’appelle total, lorsque toute fonctionnelle linéaire $f(x)$ qui s’annule pour chaque $x \in G$, s’annule aussi pour chaque $x \in E \ldots$. Théorème 7. Pour qu’un ensemble $G \subset E$ soit fondamental, il faut et il suffit qu’il soit total. ([Ba2], p. 58)
next theorem, the notion of a fundamental set is closely related to the notion of a metric dimension.

\begin{quote}
Satz 3. Jede Grundmenge eines linearen Raumes \( \mathcal{R} \) besitzt eine Mächtigkeit, welche gleich der metrischen Dimensionszahl von \( \mathcal{R} \) ist. ([L7], p. 21)
\end{quote}

\begin{quote}
Satz 4. Eine lineare Mannigfaltigkeit \( \mathcal{B} \) eines linearen metrischen Raumes und ihre abgeschlossene Hülle \( \mathcal{B} \) haben stets die gleiche metrische Dimensionszahl. ([L7], p. 22)
\end{quote}

Lōwig then remarked that in a complete Euclidean space, a maximal orthonormal set (\textit{vollständiges normiertes Orthogonalsystem}) is always a fundamental set, and its cardinality equals the metric dimension of the space.\footnote{The term Euclidean space refers to a real or complex linear inner product space, while a complete normed orthogonal system stands for a maximal orthonormal set; its existence follows from the axiom of choice.} He also noted that it is not clear whether a normed linear space \( \mathcal{R} \) always possesses a fundamental set. Even if we consider a well ordering on the space \( \mathcal{R} \) and the set of elements not belonging to the closed linear span of their predecessors, we do not necessarily obtain a fundamental set.

\begin{quote}
Ob jeder lineare metrische Raum eine Grundmenge im Sinne unserer Definition 4 besitzt, konnte nicht entschieden werden. (Man könnte von einer Wohlordnung des Raumes ausgehen und in dieser die Menge aller derjenigen Elemente betrachten, welche nicht der abgeschlossenen linearen Hülle der Menge ihrer Vorgänger angehören. Die so ausgesonderte Menge muss aber keine Grundmenge des Raumes im Sinne unserer Definition 4 sein.) ([L7], p. 22)
\end{quote}

The paper [L7] ends with a remark that every normed linear space \( \mathcal{R} \) can be extended into a complete normed linear space \( \mathcal{R}^* \), which coincides with the closed linear span of the space \( \mathcal{R} \) in \( \mathcal{R}^* \) (moreover, \( \mathcal{R} \) and \( \mathcal{R}^* \) have the same metric dimension).

\begin{quote}
\* \* \*
\end{quote}

Georg Aumann wrote a short report on Lōwig’s work [L7] for the journal \textit{Jahrbuch über die Fortschritte der Mathematik}.

\textit{Jahrbuch über die Fortschritte der Mathematik}.

In 1936, Teichmüller cited the work [L7] in his paper \textit{Operatoren im Wachsenden Raum}.

In 1939, Vitold L. Šmul’jan (Shmulyan, 1914–1944) cited the work [L7] in his article \textit{O nekotorych geometričeskich svojstvach ediničnoj sfery prostranstva tipa (B)} [Š].\footnote{See JFM 60.1229.01.}

\begin{footnotesize}
\footnote{The term Euclidean space refers to a real or complex linear inner product space, while a complete normed orthogonal system stands for a maximal orthonormal set; its existence follows from the axiom of choice.} \footnote{See [Š], p. 86, 93.}

George Whitelaw Mackey (1916–2006) cited Löwig’s work [L7] in his 1943 paper *On infinite dimensional linear spaces* [Mac1], and also in the paper [Mac2] published under the same name two years later:

*It has been shown by Löwig [15] that any two Hamel bases for the same linear space have the same cardinal number. This cardinal number we shall call the dimension of the space. And he added the following footnote: This is what Löwig calls the affine dimension. ([Mac2], pp. 157–158)\*\* \*Löwig [15] has shown that whenever $X$ is a linear space with more than $C$ elements then its dimension is equal to the number of its elements. ([Mac2], p. 159)*

In 1948, Schiek cited the work [L7] in his paper *Mengen mif affiner Anordnung* [Sc].

In 1950, Victor L. Klee (1925–2007) made a reference to Löwig’s work [L7] in his paper *Decomposition of an infinite-dimensional linear system into ubiquitous convex sets* [Kl1]; he recalled Löwig’s result on the existence of a (Hamel) basis of a linear space $L$ and the invariance of its cardinality (i.e., the dimension of the space $L$).

In 1951, Halmos included the work [L7] in his textbook *Introduction to Hilbert space and the theory of spectral multiplicity* [H] (2nd edition: 1957); [L7] is the 27th item in his list of references containing 52 works. He wrote:

*The elegant proof (in §16) of the uniqueness of dimension in the infinite case is due to Löwig, [27]. ([H], p. 110)*\*\* \*59\*\*

In 1954, the work [L7] was mentioned by D. T. Finkbeiner and Otton Martin Nikodým in their paper *On convex sets in abstract linear spaces where no topology is assumed (Hamel bodies and linear boundedness)* [FN], and by Nikodým in his paper *Limit-representation of linear, even discontinuous, linear functionals in Hilbert spaces* [Ni].

Taylor cited Löwig’s work [L7] in his textbook *Introduction to functional analysis* [T] from 1958; many subsequent editions followed.\*\*\*60\*\*


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59 In the above mentioned 16th paragraph entitled Dimension (pp. 29–31), Halmos presented the following three theorems:
1. Any two bases of a subspace $M$ have the same power.
2. A linear transformation $U$ from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{R}$ is an isomorphism if and only if it is an isometry, mapping $\mathcal{H}$ onto $\mathcal{R}$.
3. Two Hilbert spaces are isomorphic if and only if they have the same dimension.

60 Taylor makes a reference to Löwig’s theorem on the relation between cardinality and dimension (see p. 46).

(8) If $M$ is a complete linear metric space with distance function $d$ and if a new and always smaller [or always larger] distance function $d'$ is introduced under which $M$ becomes a complete linear metric space $M'$, the identity operator is an isomorphism of $M$ with $M'$.

(9) The theorem of Löwig that every separable, infinite-dimensional, Banach space has a vector basis of the cardinal number of the continuum, and the fact that two such spaces may fail to be isomorphic (say $l^1(\omega)$ and $l^2(\omega)$), shows that some relation between the metric is needed in (8). ([Da], 1973, p. 42)

In 1960, Klee cited Löwig’s paper [L7] in his article *Mappings into normed linear spaces* [Kl2].

In 1982, Manuel Valdivia cited the work [L7] in his article *A class of locally convex spaces without C-webs* [V].


In 1997, Herbert Schröder cited the work [L7] in his book *Funktionalanalyse* [Sch]:


Paul Howard and Jean E. Rubin cited the work [L7] in their 1998 monograph *Consequences of the axiom of choice* [HR]; the following theorem is mentioned in two places – Löwig’s Theorem: If $B_1$ and $B_2$ are both bases for the vector space $V$ then $|B_1| = |B_2|$. ([HR], p. 40, 78)

In 1999, Jürg Rätz cited the work [L7] in his article *Comparison of inner products* [Rä].


It is also cited in the recent work of Wolfgang Arendt (born 1950) and Robin Nittka entitled *Equivalent complete norms and positivity* [AN] and published in 2009:

*Let $\mathcal{B}$ be a Hamel basis of a vector space $E$. Then \text{card}(E) = \max\{\text{card}(\mathcal{B}), c\}$, see Löwig [16].* ([AN], p. 425)


Let us finally quote from Ralph-Hardo Schulz’s textbook *Repetitorium Bach-}

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61 See [P], p. 5 and p. 44.

Der grundlegende Satz ist der Satz von Löwig über die Gleichmächtigkeit aller Basen eines Vektorraums: Ist \( V \) ein VR, und sind \( B \) und \( C \) Basen von \( V \), so gibt es eine Bijektion von \( B \) auf \( C \); folglich gilt \( |B| = |C| \). ([Scu], p. 8)

∗ ∗ ∗

The short paper Über allgemeine Spektralfunktionen [L8] summarises the basic results of Löwig’s presentation at the second congress of mathematicians from Slavic countries, which took place in Prague from 22nd to 28th September 1934.\(^{62}\) The congress was attended by numerous Czech and German mathematicians from universities, high schools and other institutions of the Czechoslovak Republic, but also by many foreign mathematicians (Pavel Sergeyevich Aleksandrov, Dan Barbilian, Wilhelm Blaschke, Samuel Dickstein, Maurice Fréchet, Guido Fubini, Heinz Hopf, Witold Hurewicz, Jovan Karamata, Solomon Lefschetz, Stefan Mazurkiewicz, Karl Menger, Kyrille Popoff, Petre Sergescu, Waclaw Sierpiński, Alfred Tarski, Lubomir Tchakaloff, Vladimir Varicak, George Neville Watson, Tadeusz Ważewski, Stanisław Zaremba, and others). The chairman of the organising committee was Karel Petr (1868–1950).

One of the most active attendees was Fréchet, who delivered the following three lectures:

- Détermination de la classe la plus générale d’espaces vectoriels distanciés applicables vectoriellement sur l’espace concret de Hilbert,
- Sur deux relations simples entre le “coefficient“ de corrélation et le “rapport“ de corrélation,
- Sur les précisions comparées de la valeur moyenne et de la valeur médiane.\(^{63}\)

Apart from that, he also gave an opening speech on 23rd September as well as a closing speech on 28th September.\(^{64}\)

It is unclear whether Stefan Banach was present at the congress. His name is included in the list of congress delegates from the Uniwersytet Jana Kazimierza (Lwów), but not in the list of members and attendees.

Together with other contributed papers and additional congress materials, Löwig’s short article [L8] was published in the journal Časopis pro pěstování matematiky a fysiky 64 (1935), pp. xxxv, xlii–xliii.

\(^{62}\) The first congress of mathematicians from Slavic countries took place in Warsaw in September 1929.

\(^{63}\) In 1935, they were published in the journal Časopis pro pěstování matematiky a fysiky on pages 176–177, 209–210, 210–211. See also the independent publication Zprávy o druhém sjezdu matematiků zemí slovanských, JCMF, Praha, 1935.

\(^{64}\) Časopis pro pěstování matematiky a fysiky 64 (1935), pp. xxxv, xlii–xliii.
It contained a generalisation of earlier results by Ernst Hellinger (1883–1950) [Hel] from 1907 and Hans Hahn [Hah] from 1912 concerning the orthogonal equivalence of two quadratic forms of infinitely many variables.

* * *

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Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstandings, resentment, and raw prejudice.

The hostility towards lattice theory began when Dedekind published the two fundamental papers that brought the theory to life well over one hundred years ago. Kronecker in one of his letters accused Dedekind of “losing his mind in abstractions,” or something to that effect. 

It is a miracle that families of sets closed under unions and intersections can be characterized solely by the distributive law and by some simple identities. Jaded as we are, we tend to take Birkhoff’s discovery for granted and to forget that it was a fundamental step forward in mathematics.

G.-C. Rota ([Ro], pp. 1440, 1441)