Topological spaces

Generation of topological spaces (31-35)

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CHAPTER VI

GENERATION OF TOPOLOGICAL SPACES

(Sections 31 – 35)

Section 31 is concerned with a development of some order properties of the ordered class of all closure operations and of some of its ordered subclasses. The results obtained are applied to the projective generation in Section 32 and the inductive generation in Section 33. Sections 34 and 35 are closely related to 33. In Section 34 the theory of upper and lower semi-continuous correspondences ("multivalued mappings", "set-valued mappings") is developed and the results obtained are applied to two particular kinds of quotient mappings, namely to mutually continuous and inversely lower or upper semi-continuous mappings. In Section 35 the theory of convergence is developed; particular attention is given to spaces whose closure structure can be described by means of convergent sequences. The topological results obtained will be applied to topologized algebraic structs, particularly to topological groups, rings and modules; while the projective generation can be given in 32 D, the inductive generation requires a special kind of quotient mappings (each quotient-homomorphism of a topological group is open) and therefore is not treated until 34 D. In Section 35 sequentially continuous groups (more generally, K-continuous groups) are introduced.

It should be remarked that the projective and inductive generation for semi-uniform spaces and proximity spaces will be studied in Chapter VII; there the projective (inverse) and inductive (direct) limits of presheaves of closure spaces, semi-uniform spaces and proximity spaces will be also introduced and discussed.

The delopment of projective and inductive generation is rather lengthy and therefore a detailed introduction illustrated by many examples seems to be appropriate.

Let P be a set and let $\{f_a\}$ be a family, each f_a being a mapping of P into a closure space $\mathcal{Q}_a$. Let us consider the set Γ of all closures u for P such that all the mappings

$$(*) \qquad f_a : \langle P, u \rangle \to \mathcal{Q}_a$$

are continuous. The set Γ contains the discrete closure for P because a mapping of a discrete space is continuous, whatever the closure structure of the range carrier, and if $u \in \Gamma$ then each closure finer than u also belongs to Γ because the composite of two continuous mappings is a continuous mapping. It turns out that Γ has a greatest element, say u. Thus u is the coarsest closure for P such all the mappings $(*)$ are continuous. This closure u and also the space $\langle P, u \rangle$ are said to be projectively generated by the family $\{f_a\}$, and the family $\{f_a : \langle P, u \rangle \to \mathcal{Q}_a\}$ is said to be a projective generating family. E.g. the product $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of closure spaces is projectively generated by the family of all projections $\mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a$, a subspace $\mathscr{Q}$ of a space $\mathscr{P}$ is projectively generated by $\mathrm{J} : \mathscr{Q} \to \mathscr{P}$, the greatest lower bound of a family $\{u_a\}$ of closure operations for a set P is projectively generated by the family $\{\mathrm{J} : P \to \to \langle P, u_a \rangle\}\}$.

Let P be a set and let $\{f_a\}$ be a family, each f_a being a mapping of a closure space $\mathscr{Q}_a$ into P, and let us consider the set Γ of all closures u for P such that all the mappings

$$(**) \qquad f_a : \mathscr{Q}_a \to \langle P, u \rangle$$

are continuous. The set Γ contains the accrete closure because a mapping into an accrete space is continuous, whatever the closure structure of the domain carrier, and each closure coarser than an element of Γ belongs to Γ. It turns out that Γ has a least element u which is the smallest (i.e., the finest) closure for P such that all the mappings $(**)$ are continuous. This closure u and also the space $\langle P, u \rangle$ are said to be inductively generated by the family $\{f_a\}$; the family $\{f_a : \mathscr{Q}_a \to \langle P, u \rangle\}$ is said to be an inductive generating family for closure spaces. E.g. the sum $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of closure spaces is inductively generated by the family of all canonical embeddings $\{\mathrm{inj}_a : \mathscr{P}_a \to \mathscr{P}\}$, and the least upper bound of a family $\{u_a\}$ of closures for a set P is inductively generated by the family $\{\mathrm{J} : \langle P, u_a \rangle \to P\}$.

Similarly we define projective and inductive generating families for semi-uniform spaces and proximity spaces; e.g. a semi-uniform space $\langle P, \mathscr{U} \rangle$ is said to be projectively generated by a family $\{f_a\}$ if $\mathscr{U}$ is the uniformly coarsest semi-uniformity for P such that all the mappings $f_a : \langle P, \mathscr{U} \rangle \to \mathbf{E}^* f_a$ are uniformly continuous (of course, $\mathbf{E}^* f_a$ are assumed to be semi-uniform spaces). A subspace $\mathscr{Q}$ of semi-uniform (proximity) space $\mathscr{P}$ is projectively generated by $\mathrm{J} : \mathscr{Q} \to \mathscr{P}$, and the product of a family $\{\mathscr{P}_a\}$ of semi-uniform spaces is projectively generated by the family of projections $\{\mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a\}$. We have not defined the product of a family of proximity spaces. In Section 39 we shall define the product of a family $\{\mathscr{P}_a\}$ of proximity spaces as the space projectively generated by the family of projections $\{\mathrm{pr}_a : \Pi\{|\mathscr{P}_a|\} \to \mathscr{P}_a\}$.

In supplementary Notes a general definition of a "continuous structure", including the closure operations, semi-uniformities and proximities, is given, and the concepts of a projective and an inductive generating family are introduced.

Before proceeding to further examples we shall state the main theorems for projective and inductive generation of closure spaces. It is to be noted that similar results hold for semi-uniform and proximity spaces; it is only needed to replace expressions such as e.g. continuous, fine by the corresponding expressions for semi-uniform or proximity spaces. In addition, it is shown in the Notes that, in terms of theory of categories, the notions of projective and inductive generation can be introduced in such a manner that the main theorems are carried over.

(a) *A family $\{f_a\}$ of mappings of a space $\mathscr{P}$ is a projective generating family if and only if the following condition is fulfilled:*

A mapping f into $\mathcal{P}$ is continuous if and only if all the mappings $f_a \circ f$ are continuous.

(b) *A family $\{f_a\}$ of mappings into a space $\mathcal{P}$ is an inductive generating family if and only if the following condition is fulfilled:*

A mapping f of $\mathcal{P}$ is continuous if and only if all the mappings $f \circ f_a$ are continuous.

The first theorem is a generalization of the fact that a mapping f of a space into a product $\Pi\{\mathcal{P}_a\}$ is continuous if and only if all the mappings $\mathrm{pr}_a \circ f : \mathbf{D}^*f \to \mathcal{P}_a$ are continuous. The second theorem is a generalization of the fact that a mapping f of a sum $\Sigma\{\mathcal{P}_a\}$ into a space is continuous if and only if all the mappings $f \circ \mathrm{inj}_a :$ $: \mathcal{P}_a \to \mathbf{E}^*f$ are continuous.

From theorems (a) and (b) one can deduce the following results which state a certain associativity property of projective and inductive generations:

(c) *If $\{f_a\}$ is a projective generating family and $\mathbf{E}^*f_a$ is projectively generated by $\{g_{ab} \mid b \in B_a\}$, then $\{g_{ab} \circ f_a\}$ is a projective generating family.*

(d) *If $\{f_a\}$ is an inductive generating family and each $\mathbf{D}^*f_a$ is inductively generated by a family $\{g_{ab}\}$ then $\{f_a \circ g_{ab}\}$ is an inductive generating family.*

Of course, (c) and (d) are generalizations of the facts that $\Pi\{\mathcal{P}_{ab}\}$ is homeomorphic to $\Pi_a\Pi_b\mathcal{P}_{ab}$ and $\Sigma\{\mathcal{P}_{ab}\}$ is homeomorphic to $\Sigma_a\Sigma_b\mathcal{P}_{ab}$.

The projective (inductive) progeny of a class K of spaces, denoted by proj K (ind K) is defined to be the class of all spaces projectively (inductively) generated by mappings with range carriers (domain carriers) in K. It follows from (c) and (d) that

$$\text{proj proj } K = \text{proj } K\,, \quad \text{ind ind } K = \text{ind } K\,,$$

If proj $K = K$ (ind $K = K$) then K is said to be projective-stable (inductive-stable). The last theorems can be stated as follows:

(e) *The projective progeny of any class is projective-stable.*

(f) *The inductive progeny of any class is inductive-stable.*

It is to be noted that (e) implies that the projective progeny of any class is hereditary and completely productive.

The supplementary remark of **28** A.6 can be stated as follows: a space $\mathcal{P}$ is uniformizable if and only if $\mathcal{P}$ is projectively generated by functions; stated in other words, $\mathcal{P}$ is uniformizable if and only if $\mathcal{P}$ belongs to the projective progeny of ($\mathbf{R}$). Now it follows from (e) that the class of all uniformizable spaces is hereditary and completely productive. Another example: the class T of all topological spaces is projective-stable, moreover, T is the projective progeny of any two-point non-discrete and non-accrete space. Consequently, T is hereditary and completely productive. An example on inductive generation may be in place. By 16 ex. 6 a net N converges to a point x in a space $\mathcal{P}$ if and only if the mapping associated with $\langle N, x\rangle$ is continuous. Let $\mathcal{N}$ be a collection of pairs $\langle N, x\rangle$ such that N is a net with limit points x in $\mathscr{P}$. It turns out that the collection of associated mappings inductively generates $\mathscr{P}$ if and only if $\mathscr{N}$ uniquely determines the closure structure of $\mathscr{P}$ in the sense that $x \in \overline{X} - X$ if and only if there exists a pair $\langle N, x \rangle$ in $\mathscr{N}$ such that N ranges in X.

We have noticed that the product is defined "projectively" and the sum is defined "inductively". A subspace is defined "projectively", namely $\mathscr{Q}$ is a subspace of $\mathscr{P}$ if $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ is a projective generating mapping. The corresponding "inductive" concept is the quotient of a space which is defined as follows: A space $\mathscr{Q}$ is the quotient of a space $\mathscr{P}$ under f if f is a surjective inductive generating mapping such that $\mathbf{D}^*f = \mathscr{P}$ and $\mathbf{E}^*f = \mathscr{Q}$. (Notice that each quotient of a discrete space is discrete.) It turns out that quotients inherit very few of the properties of the original spaces, e.g. every space is a quotient of a paracompact space (in particular, a quotient of a topological space need not be topological). There are two important additional assumptions on the mappings f which guarantee preservation of some properties, namely inverse upper semi-continuity and inverse lower semi-continuity which will be treated in Section 34 in a more general situation.

If we wish to restrict our attention to a certain class K of spaces (e.g. topological, uniformizable) then it is natural to introduce the concepts of a K-projective generating family and a K-inductive generating family; e.g. $\{f_a : \langle P, u \rangle \to \mathscr{P}_a\}$ is a K-projective generating family if u is the coarsest closure such that all the mappings are continuous and $\langle P, u \rangle \in K$.

It turns out that basic theorems (a) and (b) are not true in general, (a) is true if and only if K is inductive-stable, (b) is true if and only if K is projective-stable.

The theory of K-inductive generation is outlined in 33 B (in connection with the fact that the quotient space of a topological space need not be topological) and the theory of K-projective generation is outlined in 35 D.

The main results will be proved independently of each other, e.g. statements (c) and (d) will be proved without any reference to statements (a) and (b) although (c) or (d) immediately follow from (a) or (b), respectively. The projective generation in a given class K (35 D) will be treated without such repetitions.

31. ORDERED SETS OF CLOSURE OPERATIONS

The results of this section will be applied in Sections 32 and 33 to two fundamental constructions of spaces and continuous algebraic structs, namely to projective and inductive constructions, which generalize the construction of the product and the sum.

This subsection is concerned with the development of order properties of the class **C** of all closure operations ordered by the relation $\{u \to v \mid u \text{ is finer than } v\}$, and of its important subclasses. A great deal of the results will be formulated for ordered subsets **C**(P) of **C** consisting of all closures for P, P being an arbitrary set, instead of for **C**. The advantage of this lies in the fact that closures for different sets are not comparable and therefore, while **C**(P) is order-complete, the ordered class **C** is not order-complete. We leave to the reader as a simple task the statement for **C** of all the results formulated and proved for **C**(P).

The ordered class **C** will be considered to be ordered "upwards" (see 10 D.2) but not from left to right and therefore we shall say upper bound, a greatest lower bound, upper saturated, etc., but not a right bound, etc. On the other hand, instead of greatest, a lower bound, upper saturated, etc., we shall occasionally say coarsest, a fine bound, coarse saturated, etc. Finally, we shall often employ lattice-theoretical terminology, e.g. meet instead of infimum, join instead of supremum and meet-stable, completely lattice-stable, etc.

In subsection A we shall prove that every **C**(P) is order-complete and we shall describe suprema and infima in **C**(P) by means of neighborhoods and the convergence of nets. Particularly significant is theorem 31 A.7 asserting that the mappings f: $\langle P, \sup \{w_a\}\rangle \to \langle Q, \sup \{v_a\}\rangle$ and f: $\langle P, \inf \{w_a\}\rangle \to \langle Q, \inf \{v_a\}\rangle$ are continuous provided that all the mappings $f : \langle P, w_a\rangle \to \langle Q, v_a\rangle$ are continuous.

In subsection B we shall examine properties of the classes of all topological and uniformizable closure operations. A particularly significant statement is given in the non-topological lemma 31 B.2, which enables us to reduce the order properties of sets of topological and uniformizable closures to those of **C**.

Subsection C is devoted to an investigation of the order properties of the set of all closures rendering continuous or inductively continuous a given internal composition or a given external composition over a closure space. As a corollary we obtain the order properties of the ordered set of all closures admissible for a given group, ring, or module over a topological ring.

The closing subsection deals with various classes of closure operations introduced earlier. Particular attention is given to separated closures.

A. ORDERED CLASS **C**

Recall that a closure u is finer than a closure v if and only if both closures are for the same set, say P, and $uX \subset vX$ for each $X \subset P$; often we shall need various descriptions of the relation $\{u \to v \mid u \text{ is finer than } v\}$ proved earlier, and therefore, for convenience, we shall summarize them in the proposition which follows.

31 A.1. Let u and v be closures for a set P. By definition 16 A.1 of continuity, u is finer than v if and only if

(1) the identity mapping of $\langle P, u\rangle$ onto $\langle P, v\rangle$ is continuous.

According to description (1) and characterizations 16 A.8 and 16 A.4 of continuity by means of convergence of nets and neighborhoods, we obtain the following two necessary and sufficient conditions:

(2), (3) if x is a limit point (an accumulation point) of a net N in $\langle P, u\rangle$, then x is a limit (accumulation) point of N in $\langle P, v\rangle$.

(4) for each $x \in P$, each neighborhood of x in $\langle P, v\rangle$ is a neighborhood of x in $\langle P, u\rangle$.

Next, by 16 A.6 the following condition is necessary, and by 16 A.10, if v is topological, then it is also sufficient, for u to be finer than v:

(5) each v-open set is u-open; and in addition, the word open can be replaced by the word closed.

Finally, sometimes it is convenient to make use of the following form of conditions (4) and (5). Let $\{\mathscr{V}_x \mid x \in P\}$ be a family such that $\mathscr{V}_x$ is a local sub-base at x in $\langle P, v\rangle$ for each x. Then the following condition is necessary and sufficient for u to be finer than v:

(4′) every element of $\mathscr{V}_x$ is a neighborhood of x in $\langle P, u\rangle$ for each x in P.

If v is topological and $\mathscr{V}$ is an open (closed) sub-base for $\langle P, v\rangle$ then also the following condition is also necessary and sufficient for u to be finer than v:

(5′) Each element of $\mathscr{V}$ is open (closed) in $\langle P, u\rangle$.

Now we proceed to the proper object of this section. We begin with the basic theorem which asserts that $\mathbf{C}(P)$ is order-complete, and describes suprema.

31 A.2. Theorem. *Let P be a set. The ordered set $\mathbf{C}(P)$ is order-complete, the discrete closure for P (= the identity relation on* exp P*) is the least (= the finest) element of $\mathbf{C}(P)$ and the accrete closure for P (the closure of each non-void set is P) is the greatest (= coarsest) element of $\mathbf{C}(P)$. If $\{u_a \mid a \in A\}$ is a non-void family in $\mathbf{C}(P)$, then*

(6) $$(\sup \{u_a \mid a \in A\})X = \bigcup\{u_aX \mid a \in A\}$$

for each $X \subset P$.

Proof. The facts that the discrete and the accrete closures for P are the finest and the coarsest elements of $\mathbf{C}(P)$ are self-evident, and moreover they have been verified at the beginning of Section 14. Since $\mathbf{C}(P)$ possesses a greatest and least element, to prove that $\mathbf{C}(P)$ is order-complete it is enough to show that each non-void family $\{u_a\}$ in $\mathbf{C}(P)$ has a least upper bound (= supremum). Thus the proof of the theorem will be accomplished if we prove that, for each non-void family $\{u_a\}$ in $\mathbf{C}(P)$, the relation

(7) $$u = \{X \to \bigcup\{u_a X\} \mid X \subset P\}$$

on $\exp P$ ranging in $\exp P$ is the least upper bound of $\{u_a\}$ in $\mathbf{C}(P)$. The proof of the fact that u is a closure operation, that is, the verification of conditions (cl 1), (cl 2) and (cl 3), is straightforward and may be left to the reader. Next, since $\{u_a\}$ is a non-void family, we have $uX \supset u_a X$ for each X and each a, which means that the closure u is coarser than u_a for each index a. Thus u is an upper bound of $\{u_a\}$. To prove that u is the least upper bound, let us consider an upper bound v; since v is coarser than each u_a, we have $vX \supset u_a X$ for each index a and $X \subset P$ and hence $vX \supset \bigcup\{u_a X\} = uX$ for each $X \subset P$ which means that v is coarser than u.

Remark. If $\{u_a\}$ is a void family, then $uX = \emptyset$ for each $X \subset P$, where u is given by (7), and hence u is not a closure operation whenever P is non-void. It is to be noted that

(8) $$v = \{X \to \bigcap\{u_a X\} \mid X \subset P\}$$

need not be a closure operation for P. If $\{u_a\}$ is an empty family, then $vX = P$ for each $X \subset P$ and hence v is not a closure operation for P whenever $P \neq \emptyset$ because the closure of the empty set is always empty. If the relation v is a closure operation then v is the greatest lower bound of $\{u_a\}$ (this can be proved easily as in the proof of 31 A.2, formula (7)). However, v need not be a closure operation even if the family $\{u_a\}$ is non-void; in fact, v need not be additive, that is, condition (cl 3), $v(X \cup Y) = (vX \cup vY)$, need not be fulfilled. Of course, we always have $v(X \cup Y) \supset (vX \cup vY)$ which follows from the fact that $vX \subset vX_1$ if $X \subset X_1$. The converse inclusion need not be fulfilled; for example, let P be the three-point set $(1, 2, 3)$ and let u_i, $i = 1, 2$, be the closures for which $u_1(1) = (1,3)$, $u_2(2) = (2,3)$ and $u_i(j) = (j)$ in the remaining cases. Then $v(j) = (j)$ for each $j = 1, 2, 3$ (where v is given by (8)) but $v(1,2) = (1, 2, 3)$ and hence $v(1) \cup v(2) \neq v(1,2)$.

Nevertheless, there is an important type of families in $\mathbf{C}(P)$ such that the greatest lower bound is given by (8).

31 A.3. *If P is a set and $\{u_a\}$ is a range down-directed family in $\mathbf{C}(P)$, then the v from (8) is the greatest lower bound of $\{u_a\}$, that is,*

(9) $$(\inf\{u_a\})\, X = \bigcap\{u_a X\}$$

for each $X \subset P$.

Proof. Of course we say that $\{u_a\}$ is range down-directed if the set of all u_a is down-directed. In view of the foregoing remark it is enough to show that $vX_1 \cup vX_2 \supset$

$\supset v(X_1 \cup X_2)$ for each $X_1, X_2 \subset P$. Suppose $x \notin vX_1 \cup vX_2$. By the definition of v there exist a_i, $i = 1, 2$, such that $x \notin u_{a_i}X_i$. Since $\{u_a\}$ is range down-directed, we can pick an a so that u_a is finer than both u_{a_i}. Clearly $x \notin u_a X_i$, $i = 1, 2$, and hence $x \notin u_a(X_1 \cup X_2) \supset v(X_1 \cup X_2)$.

For the construction of $(\inf\{u_a\})X$ in the general case see 31 ex. 2. Now we proceed to a description of suprema and infima in terms of neighborhoods. We begin with the description of suprema; the proof will depend on Theorem 31 A.2, formula (6).

31 A.4. *Let x be a point of a set P and let $\{u_a \mid a \in A\}$ be a non-void family in* $\mathbf{C}(P)$. *For each a in A let $\mathscr{U}_a$ be a local base at x in $\langle P, u_a\rangle$. Let $\mathscr{U}$ be the collection of all sets of the form $\bigcup\{U_a \mid a \in A\}$ where $U_a \in \mathscr{U}_a$ for each a in A. Then $\mathscr{U}$ is a local base at x in $\langle P, \sup\{u_a\}\rangle$. If, in addition, $\mathscr{U}_a$ is the neighborhood system at x in $\langle P, u_a\rangle$ for each a in A, then $\mathscr{U}$ is the neighborhood system at x in $\langle P, \sup\{u_a\}\rangle$, and*

$$(9) \qquad \mathscr{U} = \bigcap\{\mathscr{U}_a \mid a \in A\}\,.$$

Proof. Let u stand for $\sup\{u_a\}$. In view of 14 B.7, to prove $\mathscr{U}$ is a local base at x in $\langle P, u\rangle$ it is sufficient to show that, for each $X \subset P$, $x \in uX$ if and only if $U \cap X \neq \emptyset$ for each U in $\mathscr{U}$. First suppose $x \in uX$ and $U \in \mathscr{U}$. We must prove $U \cap X \neq \emptyset$. According to (6) the relation $x \in uX$ implies $x \in u_\alpha X$ for some α in A. By definition of $\mathscr{U}$, the set U is the union of a family $\{U_a \mid a \in A\}$ where $U_a \in \mathscr{U}_a$. Since $x \in u_\alpha X$, by 14 B.7 we have $U_\alpha \cap X \neq \emptyset$ and hence $U \cap X \neq \emptyset$ because $U_\alpha \subset U$. Now suppose $x \notin uX$. We must find a U in $\mathscr{U}$ so that $U \cap X = \emptyset$. According to (6) we have $x \notin u_a X$ for each a in A. Since $\mathscr{U}_a$ are local bases at x we can choose a family $\{U_a \mid a \in A\}$ so that $U_a \cap X = \emptyset$ and $U_a \in \mathscr{U}_a$ for each a in A. If U is the union of $\{U_a\}$, then clearly $U \cap X = \emptyset$ and, by definition of $\mathscr{U}$, U belongs to $\mathscr{U}$. Thus $\mathscr{U}$ is a local base at x in $\langle P, u\rangle$. Now let $\mathscr{U}_a$ be the neighborhood systems at x in $\langle P, u_a\rangle$. Since $\mathscr{U}_a$ are filters, if $X \subset Y \subset P$ and $X \in \mathscr{U}_a$ then also $Y \in \mathscr{U}_a$, and consequently $\mathscr{U} \subset \mathscr{U}_a$ for each a in A. Thus $\mathscr{U} \subset \bigcap\{\mathscr{U}_a\}$. Conversely, if $U \in \bigcap\{\mathscr{U}_a\}$, then $U = \bigcup\{U_a \mid a \in A\}$ where $U = U_a$ for each a in A and hence $U \in \mathscr{U}$. Thus $\mathscr{U}$ is the intersection of $\{\mathscr{U}_a\}$. Since all $\mathscr{U}_a$ are filters, their intersection $\mathscr{U}$ is also a filter. But a local base is the neighborhood system if and only if it is a filter. Thus $\mathscr{U}$ is the neighborhood system at x in $\langle P, u\rangle$ (and (9) holds as we have already shown).

Corollaries. *Let $\{u_a \mid a \in A\}$ be a non-void family in* $\mathbf{C}(P)$ *(where P is a set). Then*

(a) *the neighborhood system of a set $X \subset P$ in $\langle P, \sup\{u_a\}\rangle$ is the intersection of the neighborhood systems of X in $\langle P, u_a\rangle$, $a \in A$; and*

(b) *a set $X \subset P$ is open (closed) in $\langle P, \sup\{u_a\}\rangle$ if and only if X is open (closed) in $\langle P, u_a\rangle$ for each a in A.*

Proof. For each x in P and a in A let $\mathscr{U}_a(x)$ be the neighborhood system at x in $\langle P, u_a\rangle$, and for each x in P let $\mathscr{U}(x)$ be the neighborhood system of x in $\langle P, \sup\{u_a\}\rangle$.

By 14 B.2 a set U is a neighborhood of a set X in a space if and only if U is a neighborhood of each point of X. In consequence, if $X \subset P$, then $\bigcap\{\mathscr{U}_a(x) \mid x \in X\}$ is the neighborhood system of X in $\langle P, u_a\rangle$, and $\bigcap\{\mathscr{U}(x) \mid x \in X\}$ is the neighborhood system of X in $\langle P, \sup\{u_a\}\rangle$. According to (9) we have $\mathscr{U}(x) = \bigcap\{\mathscr{U}_a(x) \mid a \in A\}$ for each x in P and hence

$$\bigcap\{\mathscr{U}(x) \mid x \in X\} = \bigcap\{\mathscr{U}_a(x) \mid x \in X, a \in A\} = \bigcap\{\bigcap\{\mathscr{U}_a(x) \mid x \in X\} \mid a \in A\}.$$

The left side of the preceding equality is the neighborhood system of X in $\langle P, \sup\{u_a\}\rangle$ and the right side is the intersection of neighborhood systems of X in $\langle P, u_a\rangle$, $a \in A$. The proof of (a) is complete. To prove the assertion of (b) concerning open sets it is sufficient to keep in mind that X is open if and only if X is a neighborhood of itself and to apply (a). The assertion concerning closed sets follows from that concerning open sets and the fact that X is closed if and only if $P - X$ is open.

The proof of the description of infima which follows does not depend upon the preceding results.

31 A.5. Theorem. *Let P be a set and let $\{u_a \mid a \in A\}$ be a non-void family in $\mathbf{C}(P)$. For each x in P and a in A let $\mathscr{U}_a(x)$ be a local sub-base at x in $\langle P, u_a\rangle$. Then, for each x in P, the union $\mathscr{U}(x)$ of $\{\mathscr{U}_a(x) \mid a \in A\}$ is a local sub-base at x in $\langle P, \inf\{u_a\}\rangle$. If $\mathscr{U}_a(x)$ are local bases and the family $\{u_a\}$ is range down-directed, then $\mathscr{U}(x)$ are local bases.*

Proof. I. Since $\mathscr{U}(x)$ are, obviously, filter sub-bases and $x \in \bigcap\mathscr{U}(x)$ for each x in P, by 14 B.11 (b) there exists a closure operation v for P such that $\mathscr{U}(x)$ is a local sub-base at x in $\langle P, v\rangle$ for each x in P. It will be shown that $v = \inf\{u_a\}$. First it is evident that v is a lower bound of $\{u_a\}$ (use, for instance, (4')) and consequently, v is finer than $\inf\{u_a\}$. Conversely, every u_a is coarser than $\inf\{u_a\}$, and consequently every set from $\mathscr{U}_a(x)$ is a neighborhood of x in $\langle P, \inf\{u_a\}\rangle$ for each a in A; it follows that every set from $\mathscr{U}(x)$ is a neighborhood of x in $\langle P, \inf\{u_a\}\rangle$ for each x in P. But $\mathscr{U}(x)$ is a local sub-base at x in $\langle P, v\rangle$. By (4') the closure v is coarser than $\inf\{u_a\}$; this completes the proof of the first assertion. – II. Now suppose that $\mathscr{U}_a(x)$ are local bases and the family $\{u_a\}$ is range down-directed. Fix x in P and pick U_1 and U_2 in $\mathscr{U}(x)$. We muts find a U in $\mathscr{U}(x)$ so that $U \subset U_1 \cap U_2$. There exist a_i, $i = 1, 2$, in A such that $U_i \in \mathscr{U}_{a_i}(x)$. If u_a is finer than both u_{a_i} then U_i are neighborhoods of x in $\langle P, u_a\rangle$ and, $\mathscr{U}_a(x)$ being a local base at x in $\langle P, u_a\rangle$, $\mathscr{U}_a(x)$ is a filter base and hence there exists a U in $\mathscr{U}_a(x)$ such that $U \subset U_1 \cap U_2$; clearly $U \in \mathscr{U}(x)$.

Corollary. *Let $\{u_a \mid a \in A\}$ be a non-void family in $\mathbf{C}(P)$. For each x in P let fx be the element $\{x_a \mid a \in A\}$ of the product space $\Pi\{\langle P, u_a\rangle \mid a \in A\}$ such that $x_a = x$ for each a in A. Then the mapping $f = \{x \to fx\}$ of $\langle P, \inf\{u_a\}\rangle$ into the product space $\Pi\{\langle P, u_a\rangle\}$ is an embedding.*

From 31 A.5 one can deduce at once the following description of infima in terms of convergence of nets.

31 A.6. Theorem. *Let $\{u_a\}$ be a family (not necessarily non-void) in $\mathbf{C}(P)$. A point $x \in P$ is an ~~accumulation point~~ (a limit point) of a net N in $\langle P, \inf \{u_a\}\rangle$ if and only if, for each a, the point x is ~~an accumulation point~~ (a limit point) of the net N in $\langle P, u_a\rangle$.*

Remark. Let u be the least upper bound of a non-void family $\{u_a\}$ in $\mathbf{C}(P)$, N be a net in P and x be a point of P. If x is an accumulation or a limit point of N in $\langle P, u_a\rangle$ for some a, then x is, respectively, an accumulation point or a limit point of N in $\langle P, u\rangle$ because u is coarser than u_a. Nevertheless, if x is an accumulation point or a limit point of N in $\langle P, u\rangle$, then x may be an accumulation point of N in $\langle P, u_a\rangle$ for no a. This may be shown by examples.

Summarizing, we can say that the least upper bound admits "natural" simple descriptions in terms of closures and neighborhoods, but not in terms of convergent nets; and the greatest lower bound admits "natural" simple descriptions in terms of neighborhoods and convergent nets but not in terms of closures.

The first part of the section is concluded with the following simple but very important result, the straightforward proof of which is left to the reader.

31 A.7. Theorem. *Let f be a mapping of a set P into another one Q. If A is a set, $\{u_a \mid a \in A\}$ is a family in $\mathbf{C}(P)$ and $\{v_a \mid a \in A\}$ is a family in $\mathbf{C}(Q)$ such that all mappings $f: \langle P, u_a\rangle \to \langle Q, v_a\rangle$ are continuous, then the mappings*

$$f: \langle P, \inf \{u_a\}\rangle \to \langle Q, \inf \{v_a\}\rangle$$

and

$$f: \langle P, \sup \{u_a\}\rangle \to \langle Q, \sup \{v_a\}\rangle$$

are also continuous.

B. TOPOLOGICAL AND UNIFORMIZABLE CLOSURES

If u is a closure for a set P then there exists a finest topological closure for P coarser than u, the so-called topological modification of u (16 B.1), and, similarly, there exists a finest uniformizable closure for P coarser than u, the so-called uniformizable modification of u (24 B.13, 28 C). Using these facts one can reduce some order properties of the ordered set of all topological closures for a set P or uniformizable closures for a set P to those of $\mathbf{C}(P)$. Since this situation occurs frequently it will be convenient to derive some results for general ordered sets. We begin with a definition.

31 B.1. Definition. Let $\langle X, \leqq\rangle$ be an ordered set and let Y be an ordered subset of $\langle X, \leqq\rangle$. The *upper modification of an $x \in X$ in Y* is the least element of Y greater than or equal to x, that is the element y of Y with the following property: $x \leqq y$, and if $y_1 \in Y$ and $x \leqq y_1$, then $y \leqq y_1$. Similarly, the *lower modification of $x \in X$ in Y* is the greatest element of Y less than or equal to x, that is, the element y of Y with the following property: $y \leqq x$, and *if* $y_1 \in Y$ and $y_1 \leqq x$, then $y_1 \leqq y$.

For example the topological modification of a closure u for a set P is the upper modification of u in the set of all topological closures for P. Of course, the upper and lower modification of an $x \in X$ in an ordered subset Y of X need not exist, for instance if $Y = \emptyset$. If $X = \mathsf{R}$ and $Y =]\!]\, 0, 1\, [\![$, then the elements 0 and 1 possess neither an upper modification nor a lower modification in Y. If $X = \mathsf{R}$ and $Y = \mathsf{Q}$, then no $x \in (X - Y)$ possesses an upper or lower modification in Y, while each $x \in Y$, as always, coincides with its upper modification as well as its lower modification.

Before stating the main lemma we review some definitions and proposition about ordered sets (see Section 10). Let Y be an ordered subset of an ordered set $\langle X, \leqq \rangle$, $Z \subset Y$. The following cases can appear (where the infima in X are denoted by inf and the infima in Y by $\inf_Y$): (a) there exists $\inf Z$ but not $\inf_Y Z$; (b) there exists $\inf_Y Z$ but $\inf Z$ does not exist; (c) there exists neither $\inf Z$ nor $\inf_Y Z$; (d) both infima exist but $\inf Z \neq \inf_Y Z$ (of course, $\inf_Y Z \leqq \inf Z$); (e) both infima exist and they are equal. If X is order-complete, then cases (b) and (c) must be omitted but all remaining cases may appear. If $\inf Z$ exists and belongs to Y, then $\inf Z$ is the infimum of Z in Y (case (e)). Of course, all assertions remain true if inf is replaced by sup. The set Y is said to be completely meet-preserving in $\langle X, \leqq \rangle$ if $\inf_Y \{y_a\} = \inf \{y_a\}$ for each non-void family $\{y_a\}$ in Y such that the infimum in Y exists (thus $\emptyset$ is completely meet-preserving), and Y is said to be completely meet-stable in X if $\inf \{y_a\} \in Y$ for each non-void family $\{y_a\}$ in Y such that the infimum exists, i.e., $\inf_Y \{y_a\} = \inf \{y_a\}$ whenever $\{y_a\}$ is a non-void family in Y such that the infimum in X exists. A mapping of an ordered set into another ordered set is said to be completely meet-preserving if it preserves infima of non-void families; thus $Y \subset X$ is completely meet-preserving in $\langle X, \leqq \rangle$ if and only if the identity mapping of Y into X is completely meet-preserving. Replacing inf by sup we obtain the definition of completely join-preserving and completely join-stable sets. Finally, Y is completely lattice-preserving (lattice-stable) if it is simultaneously completely join-preserving (completely join-stable) and completely meet-preserving (completely meet-stable). A mapping f is said to be idempotent if $f \circ f = f$.

31 B.2. Lemma. *Let Y be an ordered subset of an ordered set $\langle X, \leqq \rangle$. The following two conditions are equivalent:*

(a) *for each x in X there exists the upper modification of x in Y;*

(b) *there exists an order-preserving idempotent mapping v of $\langle X, \leqq \rangle$ into itself such that $v[X] = Y$ and $x \leqq vx$ for each x in X.*

If the equivalent conditions (a) *and* (b) *are fulfilled, then*

(1) $vx = \inf_Y \{y \mid y \in Y, x \leqq y\}$

for each x, i.e., v is uniquely determined by Y, and

(2) $\inf_Y \{y_a\} = \inf_X \{y_a\}$

whenever $\mathbf{E}\{y_a\} \subset Y$ *and one of the infima exists; hence Y is completely meet-stable and completely meet-preserving in X, and*

(3) $\sup_Y \{vx_a\} = v \sup_X \{x_a\}$

whenever the supremum in X exists; in particular, if X is order-complete or boundedly order-complete then Y has the same property. Finally, if $\langle X, \leqq \rangle$ *is order-complete, then the equivalent conditions* (a) *and* (b) *are equivalent to the following condition (which is always necessary):*

(c) *Y is completely meet-stable in X, and the maximal elements of X belong to Y.*

Similar results hold for the lower modification.

Proof. I. Suppose (a). If vx denotes the upper modification of x in Y, then clearly the mapping $v = \{x \to vx\} : \langle X, \leqq \rangle \to \langle X, \leqq \rangle$ fulfils the condition (b), and furthermore (1) holds.

II. Conversely, suppose (b). We shall prove that vx is the upper modification of x in X. Given $x \in X$, if $x \leqq y$, $y \in Y$, then $vx \leqq vy$, because v is order-preserving, and $vy = y$ because $y = vz$ for some z (since $v[X] = Y$); hence $vy = vvz = vz$ (v is idempotent) and finally $vy = y$. Thus $vx \leqq y$ which shows that vx is indeed the upper modification of x in Y.

III. The mapping v is uniquely determined by Y because the upper modification of an x is unique.

IV. Now suppose that the equivalent conditions (a) and (b) are fulfilled and v is a mapping satisfying (b). Let $\{y_a\}$ be a non-void family in Y. We know (see II) that Y is the set of all $y \in X$ such that $vy = y$. If y is the infimum of $\{y_a\}$ in Y, then y is the infimum of $\{y_a\}$ in X. Indeed, y is a lower bound of $\{y_a\}$ in X and if x is any lower bound of $\{y_a\}$ in X, then $vx \leqq vy_a = y_a$ for each a, and hence vx is a lower bound of $\{y_a\}$ in Y which implies $vx \leqq y$ and thus $x \leqq y$. If x is the infimum of $\{y_a\}$ in X, then $vx \leqq vy_a = y_a$ for each a and hence $vx \leqq x$ which implies $vx = x$; thus $x \in Y$ and hence x is the infimum of $\{y_a\}$ in Y.

Finally, let x be the supremum of a family $\{x_a\}$ in X and let $y = vx$; we shall prove that y is the supremum of $\{vx_a\}$ in Y. Evidently y is an upper bound of $\{vx_a\}$, and if z is any upper bound of $\{vx_a\}$ in Y, then $x \leqq z$ and hence $y = vx \leqq vz = z$, i.e. $y \leqq z$.

V. It remains to show that if $\langle X, \leqq \rangle$ is order-complete then (c) is equivalent to conditions (a) and (b). It has already been shown that conditions (a) and (b) imply (c) (without the assumption that $\langle X, \leqq \rangle$ is order-complete). Conversely suppose that (c) holds and X is order-complete. The ordered set X has the greatest element, which is, of course, a maximal element and hence belongs to Y by (c). Now if x is any element of X and z is the infimum in X of all $y \in Y$, $x \leqq y$, then $z \in Y$ by (c) (because this set is non-void) and clearly z is the upper modification of x in Y. Thus (a) holds.

VI. It is to be noted that the lemma for the lower modification is obtained by applying the lemma for the upper modification to the inversely ordered set.

31 B.3. Remark. Given a set P, the ordered set of all topological closures for P will be denoted by $\tau\mathbf{C}(P)$ and the ordered set of all uniformizable closures will be denoted by $\upsilon\mathbf{C}(P)$.

Recall that the topological modification is denoted by τ and hence $\tau[\mathbf{C}(P)]$ is the set of all τu, $u \in \mathbf{C}(P)$, i.e., the set of all topological closures for P. Similarly, υ is the uniformizable modification and hence $\upsilon[\mathbf{C}(P)]$ is the set of all υu, $u \in \mathbf{C}(P)$, i.e., the set of all uniformizable closures for P.

31 B.4. Theorem. *Let P be a set. The ordered sets $\tau\mathbf{C}(P)$ and $\upsilon\mathbf{C}(P)$ are order-complete, the sets $\tau\mathbf{C}(P)$ and $\upsilon\mathbf{C}(P)$ are completely meet-stable in $\mathbf{C}(P)$ and the mappings $\tau : \mathbf{C}(P) \to \tau\mathbf{C}(P)$ and $\upsilon : \mathbf{C}(P) \to \upsilon\mathbf{C}(P)$ are surjective and completely meet-preserving.*

Proof. By 16 B.3 the topological modification τu of u is the upper modification of u in the set of all topological closures for P and τu exists for each u. From lemma 31 B.2 we obtain all statements for topological closures. Similarly, by 24 B.13 υu is the upper modification of u in the set of all uniformizable closures for P and υu exists for each u.

31 B.5. The least upper bound in $\mathbf{C}(P)$ of a family of uniformizable closures for P need not be topological. For example, let $P = (0, 1, 2)$ and let us consider the closure u_i, $i = 1, 2$, for P such that $u_i(0) = (0, i) = u_i(i)$ and $u_i(j) = j$ for $0 \neq j \neq i$. Evidently, both u_i are uniformizable closures for P. On the other hand the supremum u of $\{u_i \mid i = 1, 2\}$ is not topological. Indeed, $u(1) = u_1(1) \cup u_2(1) = (0, 1)$ and $u(0, 1) \supset u(0) = u_1(0) \cup u_2(0) = (0, 1, 2)$ and hence $uu(1) \neq u(1)$. Now it follows from lemma 31 B.2 that neither the "lower" topological modification nor the "lower" uniformizable modification need exist; more precisely, if a set P has at least three elements, then there exists a closure u for P which has no lower modification in the set of all topological (uniformizable) closures.

Because of the great importance of topological and uniformizable closures it will be convenient to describe suprema and infima in the ordered set $\tau\mathbf{C}(P)$ of all topological closures for P and in the set $\upsilon\mathbf{C}(P)$ of all uniformizable closures for P directly.

31 B.6. Theorem. *Let $\{u_a\}$ be a non-void family in $\tau\mathbf{C}(P)$ where P is a set, and let $\mathcal{U}_a$ be the collection of all open sets in $\langle P, u_a\rangle$ for each a. Then the intersection $\mathcal{U}$ of $\{\mathcal{U}_a\}$ is the collection of all open sets in the set P endowed with the supremum of $\{u_a\}$ in $\tau\mathbf{C}(P)$ and the union $\mathcal{V}$ of $\{\mathcal{U}_a\}$ is an open sub-base for the set P endowed with the infimum of $\{u_a\}$ in $\tau\mathbf{C}(P)$.*

Proof. This is an immediate consequence of 31 B.4 and the description of open sets with respect to the supremum taken in $\mathbf{C}(P)$ (Corollary (b) of 31 A.4) and the description of neighborhoods with respect to the infimum (see Theorem 31 A.5). It may be in place to give a proof which does not depend on properties of $\mathbf{C}(P)$. It is evident that $\mathcal{U}$ is the collection of all open sets for a topological space $\langle P, u\rangle$ (15 A.6) and $\mathcal{V}$ is an open sub-base for a topological space $\langle P, v\rangle$ (15 A.9). Since

$\mathscr{V} \supset \mathscr{U}_a \supset \mathscr{U}$ for each a, v is a lower bound of $\{u_a\}$ in $\tau\mathbf{C}(P)$ and u is an upper bound of $\{u_a\}$ in $\tau\mathbf{C}(P)$. If w is an upper bound (lower bound) of $\{u_a\}$ in $\tau\mathbf{C}(P)$ and $\mathscr{W}$ is the set of all open sets in $\langle P, w\rangle$, then necessarily $\mathscr{U}_a \supset \mathscr{W}(\mathscr{U}_a \subset \mathscr{W})$ for each a and hence $\bigcap\{\mathscr{U}_a\} = \mathscr{U} \supset \mathscr{W}$ $(\bigcup\{\mathscr{U}_a\} = \mathscr{V} \subset \mathscr{W})$; this implies that w is coarser (finer) than $u(v)$.

Corollary. *Let P be a set and let ω be the single-valued relation on $\tau\mathbf{C}(P)$ which assigns to each u the collection of all u-open sets. Then the mapping $\omega : \tau\mathbf{C}(P) \to$ $\to \langle \exp\exp P, \supset\rangle$ is one-to-one, order-preserving and completely join-preserving (but it need not be meet-preserving).*

31 B.7. Theorem. *Let P be a set and let $\{u_a\}$ be a non-void family in $\mathfrak{v}\mathbf{C}(P)$. For each a let $\mathscr{N}_a$ be the collection of all exact open sets in $\langle P, u_a\rangle$. Then the intersection $\mathscr{N}$ of $\{\mathscr{N}_a\}$ is the collection of all exact open sets in $\langle P, u\rangle$ where u is the supremum of $\{u_a\}$ in $\mathfrak{v}\mathbf{C}(P)$ ($\mathscr{N}$ is an open base for $\langle P, u\rangle$), and the union $\mathscr{M}$ of $\{\mathscr{N}_a\}$ is an open sub-base for $\langle P, v\rangle$, where v is the infimum of $\{u_a\}$ in $\mathfrak{v}\mathbf{C}(P)$.*

Proof. We know that a space is uniformizable if and only if it is topological and the collection of all exact open sets is an open base. Since $\mathfrak{v}\mathbf{C}(P)$ is completely meet-stable in $\mathbf{C}(P)$ (31 B.4) the statement concerning infima follows easily from Theorem 31 A.5. To prove the statement concerning suprema it is sufficient to show that a function $f : \langle P, u\rangle \to \mathsf{R}$ is continuous if and only if the function $f_a : \langle P, u_a\rangle \to \mathsf{R}$ is continuous for each a. "Only if" is evident, and to prove "if" let us consider the supremum u' of $\{u_a\}$ in $\mathbf{C}(P)$. By 31 B.4 u is the uniformizable modification of u' and hence, by 28 C, $f : \langle P, u\rangle \to \mathsf{R}$ is continuous if (and only if) $f : \langle P, u'\rangle \to \mathsf{R}$ is continuous. By 31 A.7, if a mapping $g : \langle P, u_a\rangle \to \mathscr{Q}$ is continuous for each a, then $g : \langle P, u'\rangle \to \mathscr{Q}$ is also continuous. The statement follows.

Corollary. *Let P be a set and let ϱ be a single-valued relation on $\mathfrak{v}\mathbf{C}(P)$ which assigns to each u the collection of all exact open sets in $\langle P, u\rangle$. Then the mapping ϱ: $\mathbf{v}\mathbf{C}(P) \to \langle \exp\exp P, \supset\rangle$ is one-to-one, order-preserving and completely join-preserving (but it need not be meet-preserving).*

C. CLOSURES FOR ALGEBRAIC STRUCTS

The purpose of this subsection is to prove that, given a group, ring, module over a topological ring or an algebra over a topological ring, say $\mathscr{G}$, then the set of all closures admissible for $\mathscr{G}$ is completely meet-stable in the set $\mathbf{C}(|\mathscr{G}|)$ of all closures for $|\mathscr{G}|$ (31 C.11, 31 C.16). It is to be noted that projective and inductive constructions of topological algebraic structs will be based on the results of this subsection.

The proof will be given in a sequence of propositions, each of which will be of interest in itself.

We begin with a theorem which will not be needed for 31 C.11 or 31 C.16 but which completes the results of this subsection and which will be needed for a general theorem on internal algebraic structs 31 C.10.

31 C.1. Theorem. *Suppose that $\{\sigma_a\}$ is a family of internal compositions for a set P. Let Γ be the ordered set of all closures u for P such that each topologized internal composition $\langle\sigma_a, u\rangle$ is inductively continuous. Then the set Γ is completely lattice-stable in $\mathbf{C}(P)$ and contains the accrete and the discrete closure for P, and hence Γ is order-complete and every closure for P has an upper modification as well as a lower modification in Γ.*

Proof. By 19 A.3 a topologized internal composition $\langle\sigma, u\rangle$ on P is inductively continuous if and only if each left translation $\{x \to y\sigma x\} : \langle P, u\rangle \to \langle P, u\rangle$, $y \in P$, and also each right translation $\{x \to x\sigma y\} : \langle P, u\rangle \to \langle P, u\rangle$, $y \in P$, is continuous. Hence Theorem 31 C.1 is an immediate consequence of the following proposition.

31 C.2. *If $\{f_b\}$ is a family of mappings of a set P into itself and Γ is the ordered set of all closures u for P such that each mapping $f_b : \langle P, u\rangle \to \langle P, u\rangle$ is continuous, then Γ is completely lattice-stable in $\mathbf{C}(P)$ and contains the accrete and the discrete closure for P, and hence Γ is order-complete and every closure for P has an upper and lower modification in Γ.*

Proof. Since $\mathbf{C}(P)$ is order-complete (31 A.2), by lemma 31 B.2 it is sufficient to show that Γ is completely lattice-stable in $\mathbf{C}(P)$ and contains the accrete and discrete closures for P, i.e. that if u is the supremum (infimum) in $\mathbf{C}(P)$ of a family $\{u_a\}$ in Γ, then $u \in \Gamma$. However, this follows immediately from theorem 31 A.7 asserting that if $f_b : \langle P, u_a\rangle \to \langle P, u_a\rangle$ is continuous for each a, then $f_b : \langle P, u\rangle \to \langle P, u\rangle$ is also continuous (even if the index set is empty).

The case of the continuity of a topological internal composition is not too simple. By definition, a topologized internal composition $\langle\sigma, u_a\rangle$ on P is continuous if the mapping $\sigma : \langle P, u_a\rangle \times \langle P, u_a\rangle \to \langle P, u_a\rangle$ is continuous. We shall write the product $\langle P, u_a\rangle \times \langle P, u_a\rangle$ as $\langle P \times P, u_a \times u_a\rangle$. If each $\langle\sigma, u_a\rangle$ is continuous then, by theorem 31 A.7, the mappings $\sigma : \langle P \times P, \sup\{u_a \times u_a\}\rangle \to \langle P, \sup\{u_a\}\rangle$ and $\sigma : \langle P \times P, \inf\{u_a \times u_a\}\rangle \to \langle P, \inf\{u_a\}\rangle$ are also continuous; but we do not know whether $\sup\{u_a \times u_a\} = \sup\{u_a\} \times \sup\{u_a\}$ and $\inf\{u_a \times u_a\} = \inf\{u_a\} \times \times \inf\{u_a\}$. It turns out that the latter equality holds but the former is false as shown in the example which follows.

31 C.3. The mapping of the ordered set $\mathbf{C}(P)$ into $\mathbf{C}(P \times P)$ which assigns to each closure u for P the product closure $u \times u$ is not join-preserving. Perhaps the simplest example may be obtained as follows. Let $P = (0, 1, 2)$ and let u_i, $i = 1, 2$ be the closure for P such that $u_i(i) = (i, 0) = u_i(0)$ and $u_i(j) = (j)$ for $i \neq j \neq 0$. If $u = \sup(u_1, u_2)$ (in $\mathbf{C}(P)$), then the product closure $u \times u$ is strictly coarser than the supremum v of the product closures $u_1 \times u_1$ and $u_2 \times u_2$ in $\mathbf{C}(P \times P)$. Indeed, P is the only neighborhood of 0 in $\langle P, u\rangle$ and hence $P \times P$ is the only neighborhood of $\langle 0, 0\rangle$ in $\langle P \times P, u \times u\rangle$. On the other hand $(0, i)$ is a neighborhood of 0 in $\langle P, u_i\rangle$ and hence $(0, i) \times (0, i)$ is a neighborhood of $\langle 0, 0\rangle$ in $\langle P \times P, u_i \times u_i\rangle$, and finally, $\bigcup\{(0, i) \times (0, i) \mid i = 1, 2\} \neq P \times P$ is a neighborhood of $\langle 0, 0\rangle$ in

$\langle P \times P, v \rangle$. It is to be noted that this example may be used to show that the mapping in question is not join-preserving whenever the cardinal of P is at least 3. In 31 C.13 we shall show that the mapping $\{u \to u \times u\} : \mathbf{C}(P) \to \mathbf{C}(P \times P)$ is not countably monotone join-preserving, i.e. the equality $\sup \{u_a \times u_a\} = \sup \{u_a\} \times \sup \{u_a\}$ need not be true for a range-monotone countable family (whereas evidently this equality holds for every finite range-monotone family).

Now we shall prove that the mapping $\{u \to u \times u\}$ of $\mathbf{C}(P)$ into $\mathbf{C}(P \times P)$ is completely meet-preserving and in fact we shall prove essentially more. First recall that the product $\Pi\{\langle X_a, \leqq_a \rangle\}$ of ordered sets is defined to be the ordered set $\langle \Pi\{X_a\}, \leqq \rangle$, where $\leqq$, the so-called product order, is defined by letting $x \leqq y$ if and only if $\mathrm{pr}_a x \leqq_a \mathrm{pr}_a y$ for each a, i.e. $\leqq$ is the relational product of the family $\{\leqq_a\}$.

31 C.4. Theorem. *Let $\{P_a \mid a \in A\}$ be a non-void family of non-void sets. The mapping $f = \{\{u_a\} \to \Pi\{u_a\}\} : \Pi\{\mathbf{C}(P_a)\} \to \mathbf{C}(\Pi\{P_a\})$ is completely meet-preserving, in particular, order-preserving.*

Proof. Let $\{v_a \mid a \in A\}$ be the infimum of a non-void family $\{\{u_{ba} \mid a \in A\} \mid b \in B\}$ in $\Pi\{\mathbf{C}(P_a)\}$, $u_b = \Pi\{u_{ba} \mid a \in A\}$, $v = \Pi\{v_a \mid a \in A\}$. We shall prove that v is the infimum of $\{u_b\}$ in $\mathbf{C}(\Pi\{P_a\})$. It is sufficient to show (by 31 A.6) that a net N converges to x with respect to v if and only if N converges to x with respect to each u_b. Next, since u_b and v are product closures, by 17 C.9 N converges to x with respect to $u_b(v)$ if and only if $\mathrm{pr}_a \circ N$ converges to $\mathrm{pr}_a x$ with respect to $u_{ba}(v_a)$ for each $a \in A$. Finally, by definition of the product order, v_a is the infinum of $\{u_{ba} \mid b \in B\}$ in $\mathbf{C}(P)$ for each a in A, and hence, by 31 A.6, $\mathrm{pr}_a \circ N$ converges to $\mathrm{pr}_a x$ with respect to v_a if and only if $\mathrm{pr}_a \circ N$ converges to $\mathrm{pr}_a x$ with respect to u_{ba} for each $b \in B$. The statement follows.

Remark. The foregoing theorem will become a corollary of the associativity of projective generation (32 A.9).

31 C.5. *Let σ be an internal composition on a set P and let Γ be the ordered set of all closures u for P such that the topologized composition $\langle \sigma, u \rangle$ is continuous. Then the accrete and the discrete closures for P belongs to Γ and Γ is completely meet-stable, and hence Γ is order-complete and every closure for P has an upper modification in Γ.*

Proof. If $P = \emptyset$ then the statement is trivial. Suppose $P \neq \emptyset$. Evidently the accrete and the discrete closures belong to Γ. If $\{u_a\}$ is a non-void family in Γ, then each mapping $\sigma : \langle P \times P, u_a \times u_a \rangle \to \langle P, u_a \rangle$ is continuous, by theorem 31 A.7 the mapping $\sigma : \langle P \times P, \inf \{u_a \times u_a\} \rangle \to \langle P, \inf \{u_a\} \rangle$ is continuous, and by the foregoing theorem 31 C.4 $\inf \{u_a \times u_a\} = \inf \{u_a\} \times \inf \{u_a\}$ and hence also $\langle \sigma, \inf \{u_a\} \rangle$ is continuous, i.e. $\inf \{u_a\} \in \Gamma$. The remaining statements follow from lemma 31 B.2.

It will be shown in 31 C.12 that Γ need not be join-stable. For an examination of the continuity of the inversion of an internal composition we shall need the following simple result.

31 C.6. Theorem. *Let Q be a subset of a set P. The mapping f of $\mathbf{C}(P)$ into $\mathbf{C}(Q)$ which assigns to each u its relativization to Q, is surjective and completely lattice-preserving. The mapping $f: \tau\mathbf{C}(P) \to \tau\mathbf{C}(Q)$ is surjective and completely lattice-preserving.*

Proof. I. Let $\{u_b\}$ be a non-void family in $\mathbf{C}(P)$. If X is any subset of Q, then $(f \sup \{u_b\}) X = Q \cap \bigcup\{u_b X\}$ and $(\sup \{f u_b\}) X = \bigcup\{f u_b X\} = \bigcup\{Q \cap u_b X\} = Q \cap \bigcup\{u_b X\}$, and hence $f \sup \{u_b\} = \sup \{f u_b\}$. Let $x \in Q$ and let $\mathcal{U}_b$ be the neighborhood system at x in $\langle P, u_b\rangle$ for each b. Then the union $\mathcal{U}$ of $\{\mathcal{U}_b\}$ is a local sub-base at x in $\langle P, \inf \{u_b\}\rangle$, $[\mathcal{U}_b] \cap Q = \mathcal{V}_b$ is the neighborhood system at x in $\langle Q, f u_b\rangle$ and the union $\mathcal{V}$ of $\{\mathcal{V}_b\}$ is a local sub-base at x in $\langle Q, \inf \{f u_b\}\rangle$. But clearly $\mathcal{V} = [\mathcal{U}] \cap Q$ and hence $\inf \{f u_b\}$ is a relativization of $\inf \{u_b\}$. Clearly f is surjective. — II. Let $\{u_b\}$ be a non-void family in $\tau\mathbf{C}(P)$ and $\mathcal{U}_b$ be the collection of all open sets of $\langle P, u_b\rangle$; the collection $\mathcal{V}_b = [\mathcal{U}_b] \cap Q$ is the set of all open sets of $\langle Q, f u_b\rangle$. By 31 B.6 the intersection $\mathcal{U}(\mathcal{V})$ of $\{\mathcal{U}_b\}$ ($\{\mathcal{V}_b\}$) is the set of all open sets of the space $\langle P, \sup \{u_b\}\rangle$ ($\langle Q, \sup \{f u_b\}\rangle$). But clearly $\mathcal{V} = [\mathcal{U}] \cap Q$. The statement concerning infima follows from I and the fact that the infimum in $\mathbf{C}(P)$ of topological closures is a topological closure.

31 C.7. Remark. In the notation of 31 C.6, the mapping f does not commute with τ, i.e. $f\tau u \neq \tau f u$ in general. This was proved in 17 A.6. Of course, $f\tau u$ is always coarser than $\tau f u$. It follows that the second statement cannot be obtained from the first statement and properties of the topological modification τ.

31 C.8. *Suppose that P is a set and $\{f_a\}$ is a family of single-valued relations such that $\mathbf{D}f_a \subset P$, $\mathbf{E}f_a \subset P$ for each a, and let Γ be the ordered set of all closures u for P such that the mapping f_a of the subspace $\mathbf{D}f_a$ of $\langle P, u\rangle$ into the subspace $\mathbf{E}f_a$ of $\langle P, u\rangle$ is continuous for each a. Then the set Γ is completely lattice-stable in $\mathbf{C}(P)$, the accrete and the discrete closures for P belong to Γ, and hence, Γ is order-complete and each closure for P has a lower and upper modification in Γ.*

Proof. Evidently the discrete and the accrete closures belong to Γ. If $\{u_b\}$ is a non-void family in Γ and $u = \sup \{u_b\}$ ($u = \inf \{u_b\}$) in $\mathbf{C}(P)$, then by 31 C.6, for each $Q \subset P$ the relativization of u to Q is the supremum (the infimum) in $\mathbf{C}(Q)$ of $\{v_b\}$, where v_b is the relativization of u_b to Q. Applying 31 A.7 we obtain $u \in \Gamma$. The remaining statements follow from lemma 31 B.2.

31 C.9. Corollary. *If P is a set, $\{\sigma_a\}$ is a family of semi-group structures on P and Γ is the set of all $u \in \mathbf{C}(P)$ such that the inversion of each $\langle \sigma_a, u\rangle$ is continuous, then Γ is completely lattice-stable in $\mathbf{C}(P)$, Γ contains the discrete and the accrete closure, Γ is order-complete and every closure for P has its lower and upper modification in Γ.*

Proof. The inversion of each σ_a is a single-valued relation in P ranging in P.

Now we are prepared to state the main result concerning internal algebraic structs.

31 C.10. Theorem. *Let P be a set, let $\{\sigma_a\}$, $\{\varrho_b\}$ and $\{\mu_c\}$ be families of internal compositions on P and let each μ_c be a semi-group structure. Let Γ be the ordered set of all closures u for P such that each $\langle\sigma_a, u\rangle$ is continuous, each $\langle\varrho_b, u\rangle$ is inductively continuous and the inversion of each $\langle\mu_c, u\rangle$ is continuous. Then the discrete and the accrete closures for P belong to Γ, Γ is completely meet-stable in $\mathbf{C}(P)$, and hence Γ is order-complete and every closure u for P has an upper modification in Γ. If the family $\{\sigma_a\}$ is empty, then Γ is completely lattice-stable and every closure u for P has a lower modification in Γ.*

Proof. Apply 31 C.1, 31 C.5, and 31 C.9.

31 C.11. Corollary. *Let $\mathscr{G}$ be a group (ring, field) and let Γ be the ordered set of all closures admissible for $\mathscr{G}$ (i.e., compatible with the structure of $\mathscr{G}$). Then the discrete and the accrete closures for $|\mathscr{G}|$ belong to Γ, Γ is completely meet-stable in $\mathbf{C}(P)$, and hence Γ is order-complete and every closure for P has an upper modification in Γ.*

31 C.12. Example. We shall show that Γ from 31 C.11 need not be join-stable. Let $\langle G, \sigma\rangle$ be any commutative group containing at least two elements, let u be the discrete closure for P and v be a non-discrete closure admissible for $\langle G, \sigma\rangle$, e.g. the accrete closure for G. Consider the product group $\langle H, \varrho\rangle = \langle G, \sigma\rangle \times \times \langle G, \sigma\rangle$. Let 0 be the neutral element of $\langle G, \sigma\rangle$; thus $\langle 0, 0\rangle$ is the neutral element of $\langle H, \varrho\rangle$.

(a) $\langle H, \varrho, u \times v\rangle$ and $\langle H, \varrho, v \times u\rangle$ are topological groups by 19 A.12 because $\langle H, \varrho, u \times v\rangle = \langle G, \sigma, u\rangle \times \langle G, \sigma, v\rangle$, $\langle H, \varrho, v \times u\rangle = \langle G, \sigma, v\rangle \times \langle G, \sigma, u\rangle$.

(b) If $w = \sup(u \times v, v \times u)$, then $\langle H, \varrho, w\rangle$ is an inductively continuous group with continuous inversion (by 31 C.10).

(c) The closure w is the inductive product of v and v, i.e., $w = \operatorname{ind}(v \times v)$. Indeed, if V is a neighborhood of 0 in $\langle G, v\rangle$, then $(0) \times V$ is a neighborhood of $\langle 0, 0\rangle$ in $\langle H, u \times v\rangle$, $V \times (0)$ is a neighborhood of $\langle 0, 0\rangle$ in $\langle H, v \times u\rangle$ and hence $W = ((0) \times V) \cup (V \times (0))$ is a neighborhood of $\langle 0, 0\rangle$ in $\langle H, w\rangle$, and clearly the sets W form a local base at $\langle 0, 0\rangle$ in $\langle H, w\rangle$.

(d) Since v is not discrete and $w = \operatorname{ind}(v \times v)$, the closure w is not topological and consequently $\langle H, \varrho, w\rangle$ is not a topological group because every topological group is topological. It may be appropriate to prove directly that $\langle \varrho, w\rangle$ is not a continuous internal composition.

By 19 A.4 it is sufficient to show that $[W]\,\varrho[W]$ is contained in $(G \times (0)) \cup \cup ((0) \times G)$ for no $W = ((V \times (0)) \cup ((0) \times V))$, where V is any neighborhood of 0 in $\langle G, v\rangle$. However, this is almost evident because we can choose an x in $(V - (0))$ and then $\langle 0, x\rangle$, $\langle x, 0\rangle \in W$ and $\langle 0, x\rangle\, \varrho \langle x, 0\rangle = \langle x, x\rangle$ belongs to the set $J_G - \langle 0, 0\rangle$ which is disjoint with $(G \times (0)) \cup ((0) \times G)$.

(e) If $\langle G, v\rangle$ is separated, then the diagonal J_G of $G \times G$ is closed in $\langle G \times G)$ ind $(v \times v)\rangle$ and hence in $\langle G \times G, \tau w\rangle$. It follows that $U = (H - J_G) \cup (\langle 0, 0\rangle,$

is a neighborhood of $\langle 0, 0\rangle$ in $\langle H, \tau w\rangle$. The proof of the second part of (d) can be used to show that $\langle \varrho, \tau w\rangle$ is not a continuous internal composition. As a consequence, the set Γ is not join-stable in $\tau\mathbf{C}(H)$.

31 C.13. Example. We shall construct an increasing sequence $\{u_n\}$ of admissible closures for the additive group of reals, which will be denoted by $\langle G, +\rangle$, such that the closure $u = \sup\{u_n\}$ is not admissible for $\langle G, +\rangle$ (it will follow that $\sup\{u_n \times u_n\} \neq u \times u$; indeed, if $\sup\{u_n \times u_n\} = u \times u$ then the proof of 31 C.5, applied to sup instead of to inf, yields that $\langle +, u\rangle$ is a continuous composition, and since the inversion of $\langle +, u\rangle$ is continuous by 31 C.9, u must be admissible for the group $\langle G, +\rangle$). For each positive integer n let G_n be the subgroup of $\langle G, +\rangle$ generated by the element n^{-1}, i.e. G_n consists of all $k \cdot n^{-1}$ with k varying over Z, and let $G_0 = (0)$. Let v be the usual closure for $\langle G, +\rangle$, i.e. v is the order closure, and let $\mathcal{U}$ be the local base at 0 in $\langle G, +, v\rangle$. For each n let $\mathcal{U}_n$ be the set of all $[U] + [G_n]$ $(= \mathbf{E}\{x + y \mid x \in U, y \in G_n\})$, $U \in \mathcal{U}$. It is easily seen that $\mathcal{U}_n$ is a filter base fulfilling conditions (gnb i) of 19 B.7, and consequently there exists a unique closure u_n admissible for $\langle G, +\rangle$ such that $\mathcal{U}_n$ is a local base at 0. Clearly $\{u_n\}$ is an increasing sequence in $\mathbf{C}(G)$, (the filter generated by $\mathcal{U}_n$ always contains $\mathcal{U}_{n+1}$) and $u_0 = v$. Let $u = \sup\{u_n\}$. The set $H = \bigcup\{G_n\}$ is dense in $\langle G, v\rangle$ and hence in $\langle G, u\rangle$. But each neighborhood of 0 in $\langle G, u\rangle$ contains H, and consequently each neighborhood of 0 in $\langle G, u\rangle$ is dense in $\langle G, u\rangle$, and hence the closure of each neighborhood of 0 is G. Thus, if u were admissible for $\langle G, +\rangle$ then necessarily u would be the accrete closure for G; but u is not accrete. Indeed, if $x, y \in G$, $y \notin x + [H]$ (such x and y exist because card $H = \aleph_0$ and card $G > \aleph_0$) then there exists a neighborhood U of y such that $x \notin U$, i.e., $y \notin u(x)$. Indeed, the distance (in R) of y to each $x + [G_n]$ is positive, say r_n, and if V_n is the open r_n-sphere about 0 in R, then $[V_n] \cup [G_n]$ is a neighborhood of 0 in $\langle G, u_n\rangle$, and $W_n = y + [V_n] + [G_n]$ is a neighborhood of y in $\langle G, u_n\rangle$ which does not contain x; it follows that $\bigcup\{W_n\}$ is a neighborhood of y in $\langle G, u\rangle$ (by 31 A.4) which evidently does not contain x.

Remark. If $\langle G, +\rangle$ is a finite group, then the set $\mathbf{C}(G)$ is finite and therefore the set Γ of all closures compatible for $\langle G, +\rangle$ is countably monotonically join-stable. On the other hand, if G is infinite, then there exists a group structure for G such that the set Γ is not countably monotonically join-stable. Actually, in 31 C.13 it is enough to take for $\langle G, +\rangle$ any subgroup of the additive group of reals, containing a H such that the quotient group G/H has at least three elements. Of course the group G can be taken countable.

Now we proceed to external compositions. We shall need the following simple result the proof of which is elementary and therefore left to the reader.

31 C.14. *Let P be a non-void set and $\mathcal{R} = \langle R, v\rangle$ be a closure space. The mapping*

$$f : \{u \to v \times u\} : \mathbf{C}(P) \to \mathbf{C}(R \times P)$$

is injective and completely lattice-preserving.

31 C.15. Theorem. *Let P be a set, $\{\sigma_a\}$, $\{\varrho_b\}$ and $\{\mu_c\}$ families of internal compositions on P, each μ_c being a semi-group structure, and $\{\langle\kappa_d, v_d\rangle\}$ and $\langle\lambda_e, w_e\rangle$ families of domain-topologized external compositions on P. Let Γ be the ordered set of all closures u for P such that each $\langle\sigma_a, u\rangle$ is continuous, each $\langle\varrho_b, u\rangle$ is inductively continuous, the inversion of each $\langle\mu_c, u\rangle$ is continuous, each topologized external composition $\langle u, \kappa_d, v_d\rangle$ is continuous, and finally, every $\langle u, \lambda_e, w_e\rangle$ is inductively continuous. Then the set Γ is completely meet-stable in $\mathbf{C}(P)$, the accrete closure belongs to Γ, and hence every closure u for P has an upper modification in $\mathbf{C}(P)$.*

The proof based on 31 C.10 and 31 A.7, 31 C.14 is left to the reader.

31 C.16. Corollary. *Let $\mathscr{L}$ be a module (algebra) over a topological ring $\mathscr{R}$ and let Γ be the ordered set of all closures admissible for $\mathscr{L}$. Then Γ is completetely meet-stable in $\mathbf{C}(|\mathscr{L}|)$, the accrete closure for $|\mathscr{L}|$ belongs to Γ, and hence every closure for $|\mathscr{L}|$ has an upper modification in Γ.*

Remark. It is easily seen that Γ need not be join-stable (take $\mathscr{R}$ discrete and use the method from 31 C.12).

D. EXAMPLES

Here we shall investigate the properties of the ordered sets of all semi-separated, locally connected, quasi-discrete, semi-separated and separated closures for a given set P. Particular attention is given to separated closures. We shall examine maximal elements of the ordered class of all (topological) separated closures. It turns out that maximal elements of the class Γ of all separated closures are just the compact elements of Γ (31 D.8). Characterization of maximal elements of the ordered class of all topological separated closures is more complicated (31 D.9). It should be noted that although compact spaces will be studied in Section 41, they have already been introduced in 29 B.2 and the exercises to Section 17.

31 D.1. Semi-uniformizable closures. *Let P be a set. The ordered set Γ of all semi-uniformizable closures for P is completely lattice-stable in $\mathbf{C}(P)$, and the discrete closure for P and the accrete closure for P belong to Γ. As a consequence, Γ is order-complete, every closure for P has a lower modification and an upper modification in Γ and Γ is a completely lattice-preserving subset of $\mathbf{C}(P)$.*

Proof. Clearly the discrete closure for P and the accrete closure for P are semi-uniformizable and hence belong to Γ. Now by virtue of lemma 31 B.2 it is sufficient to show that Γ is completely lattice-stable in $\mathbf{C}(P)$. Since every closure for P has an upper modification in Γ (by 23 B.2), again by 31 B.2 the set Γ is necessarily completely meet-stable in $\mathbf{C}(P)$. To show that Γ is completely join-stable, take any non-void family $\{u_a\}$ in Γ and let us prove that the supremum u of $\{u_a\}$ in $\mathbf{C}(P)$ belongs to Γ. Remember that a closure w for P is semi-uniformizable if and only if

$x \in w(y)$ implies $y \in w(x)$. Assuming $x \in u(y)$ we shall prove $y \in u(x)$. Since $x \in u(y)$, by 31 A.2 there exists an index a so that $x \in u_a(y)$. Since $u_a \in \Gamma$, necessarily $y \in u_a(x)$, and u being coarser than u_a, we obtain $y \in u(x)$. Thus $x \in u(y)$ implies $y \in u(x)$ and hence $u \in \Gamma$.

Remark. In the proof of the preceding theorem we used the fact that each closure for P has an upper modification in Γ, and this follows from the existence of a fine semi-uniformity of a closure space. It is to be noted that one can prove directly that Γ is completely meet-stable. Indeed, if u is the infimum in $\mathbf{C}(P)$ of a family $\{u_a\}$, then $x \in u_a X$ for each a does not imply $x \in uX$, but if X is finite, then the implication does hold (see ex. 2).

31 D.2. Locally connected closures. *The ordered set Γ of all locally connected closures for a set P is completely join-stable in $\mathbf{C}(P)$, and the accrete closure for P as well as the discrete closure belong to Γ. As a consequence, each closure for P has a lower modification in Γ and Γ is completely join-preserving in $\mathbf{C}(P)$.*

Proof. Since the discrete closure and the accrete closure obviously belong to Γ, by lemma 31 B.2 it is enough to show that Γ is completely join-stable in $\mathbf{C}(P)$. Let u be the supremum in $\mathbf{C}(P)$ of a non-void family $\{u_a\}$ in Γ. We must prove $u \in \Gamma$, i.e. given a neighborhood U of a point x in $\langle P, u\rangle$ we must find a connected neighborhood V of x in $\langle P, u\rangle$ such that $V \subset U$. Since U is a neighborhood of x in each space $\langle P, u_a\rangle$ we can choose a family $\{V_a\}$ such that V_a is a connected neighborhood of x in $\langle P, u_a\rangle$ and $V_a \subset U$. Put $V = \bigcup\{V_a\}$. Clearly $V \subset U$ and by 31 A.4 V is a neighborhood of x in $\langle P, u\rangle$. Finally, since u is coarser than each u_a, each set V_a is connected in $\langle P, u\rangle$, and consequently V is connected in $\langle P, u\rangle$ as the union of a family of connected sets containing a common point, namely x.

Remark. It is easy to see that the set Γ need not be meet-preserving.

31 D.3. Quasi-discrete closures. *Let P be a set. The ordered set Γ of all quasi-discrete closures for P is completely join-stable in $\mathbf{C}(P)$, the discrete and the accrete closures for P belong to Γ, and consequently Γ is an order-complete set, each closure for P has a lower modification in Γ and Γ is completely join-preserving in $\mathbf{C}(P)$. Next, Γ is order-dense in $\mathbf{C}(P)$, more precisely, each closure for P is the infimum in $\mathbf{C}(P)$ of a family in Γ. Finally, $\Gamma = \mathbf{C}(P)$ if and only if P is a finite set.*

Proof. I. Clearly the discrete and the accrete closures for P belong to Γ. If $\{u_a\}$ is a non-void family in Γ and $u = \sup\{u_a\}$ in $\mathbf{C}(P)$, then $uX = \bigcup\{u_a X\}$ for each $X \subset P$ (by 31 A.2), and hence if $x \in uX$ then $x \in u_a X$ for some a; since $u_a \in \Gamma$, $x \in u_a Y$ for some finite subset Y of X and hence $x \in uY$. Thus u is quasi-discrete, i.e. $u \in \Gamma$. Now the remaining statements of the first part of theorem follow from lemma 31 B.2. — II. Let u be a closure for P. We shall construct a family in Γ so that u will be its infimum. For each family $\{U_x \mid x \in P\}$ such that U_x is a neighborhood of x in $\langle P, u\rangle$ let $U = \Sigma\{U_x \mid x \in P\}$ and $v_U = \{X \to U^{-1}[X] \mid X \subset P\}$. Since U is a reflexive relation ($x \in U_x$ for each x), each v_U is a quasi-discrete closure for P (see 26 A.2), and if $x \in uX$ then $U_x \cap X \neq \emptyset$ and hence $x \in v_U(X \cap U_x) \subset v_U X$. Thus each v_U is coarser then u. We shall prove that u is the infimum of $\{v_U\}$ in $\mathbf{C}(P)$. It is enough to show that if $y \notin uX$ then $y \notin v_U X$ for some U. Suppose $y \notin uX$ and take $\{U_x \mid x \in P\}$ with $U_y \cap X = \emptyset$ and $U_x = P$ for $x \neq y$. If $U = \Sigma\{U_x\}$, then clearly $y \notin v_U X$. – III. The last statement is evident.

31 D.4. Semi-separated closures. *Let P be a set and Γ the set of all semi-separated closures for P. The set Γ is the closed interval $[\![u, v]\!]$ in $\mathbf{C}(P)$ where u is the discrete closure for P and v is the topological closure for P such that $X \subset P$ is closed if and only if X is finite or $X = P$. It follows that Γ is down-saturated, completely lattice-stable, completely lattice-preserving and every closure for P has a lower modification in Γ. The ordered set Γ is order-complete.*

Proof. By 26 B.8 v is the coarsest semi-separated closure for P and by 26 B.7 Γ is down-saturated (i.e. fine-saturated). Thus Γ indeed is the indicated closed interval. The remaining statements follows from lemma 31 B.2 (and order-completeness of $\mathbf{C}(P)$).

31 D.5. *The set Γ of all separated closures for a set P is down-saturated (i.e. fine-saturated). If P is finite, then Γ coincides with the set of all semi-separated closures for P.* – Obvious.

31 D.6. Example. Let P be an infinite set and Γ the set of all separated closures for P. Then Γ is not join-stable, and the supremum of an increasing sequence of separated closures need not be separated.

Proof. I. Fix two distinct elements x_1 and x_2 of P and let us consider the closure u_i, $i = 1, 2$ for P such that each $x \in P$, $x \neq x_i$ is isolated in $\langle P, u \rangle$ and a subset U of P is a neighborhood of x_i in $\langle P, u_i \rangle$ if and only if $x_i \in U$ and $P - U$ is finite. Clearly u_i are separated closures for P but the supremum u of (u_1, u_2) in $\mathbf{C}(P)$ is not separated; indeed, the points x_1 and x_2 are not separated. – II. To prove the second statement we can assume P to be countable and hence, for convenience, we can take for P the set of all rational numbers. Choose a sequence $\{y_i\}$ in $\mathsf{R} - P$ such that the range of $\{y_i\}$ is dense in the space R of reals, and let us define a sequence $\{u_n\}$ of closures for P as follows: if $x \in (P - (0))$, then U is a neighborhood of x in $\langle P, u_n \rangle$ if and only if $U \cup (\mathsf{R} - P)$ is a neighborhood of x in R and U is a neighborhood of 0 in $\langle P, u_n \rangle$ if and only if $U \cup (\mathsf{R} - P)$ is a neighborhood of the set $(0, y_0, \ldots, y_n)$ in R. Clearly each u_n is a separated closure for P. On the other hand, the supremum u of $\{u_n\}$ is not separated. Indeed, it is easily seen that each neighborhood of 0 in $\langle P, u \rangle$ is dense in $\langle P, u \rangle$ because it is dense in R.

Remark. If $\{u_n\}$ is the sequence from the second part of the proof of 31 D.6, then $\sup \{u_n\} \times \sup \{u_n\} \neq \sup \{u_n \times u_n\}$. Indeed, since each u_n is separated, the diagonal J_P of $P \times P$ is closed in $\langle P \times P, u_n \times u_n \rangle$ for each n, and consequently the diagonal is closed in $\langle P \times P, \sup \{u_n \times u_n\} \rangle$; on the other hand the diagonal is not closed in $\langle P \times P, \sup \{u_n\} \times \sup \{u_n\} \rangle$, because $\sup \{u_n\}$ is not separated.

31 D.7. Definition. A *coarse separated closure* (a *coarse separated topological closure*) is a separated closure u (a separated topological closure u) such that no separated closure (separated topological closure) is strictly coarser than u. Thus coarse separated closures (coarse separated topological closures) are maximal elements of the ordered class of all separated (separated topological) closures.

It turns out that there exists a separated closure (topological separated closure) which is not finer than any coarse separated closure (coarse separated topological closure). Evidently every topological closure which is a coarse separated closure is a coarse separated topological closure, but the converse is not true. First we shall give a characterization of coarse separated (topological) closures.

31 D.8. Theorem. *A separated closure u for a set P is a coarse separated closure if and only if the following condition is fulfilled:*

(a) *If $\mathscr{X}$ is a proper filter of sets on P, then $\bigcap\{uX \mid X \in \mathscr{X}\} \neq \emptyset$.*

Remark. The closures satisfying the condition of 31 D.8 are said to be compact. Thus a separated closure u is a coarse separated closure if and only if u is a compact closure. It should be remarked that compact closures will be investigated in Section 41.

Proof. I. First suppose that there exists a separated closure v which is strictly coarser than u. There exists an element x of P such that the neighborhood system $\mathscr{V}$ of x in $\langle P, v\rangle$ is strictly smaller than the neighborhood system $\mathscr{U}$ of x in $\langle P, u\rangle$. Choose a U in $\mathscr{U} - \mathscr{V}$ and consider the set $\mathscr{X}$ of all $V - U$ with $V \in \mathscr{V}$. By our assumption $\mathscr{X}$ is a filter base of sets in P. Since $\langle P, v\rangle$ is separated, the intersection of the collection $v[\mathscr{V}]$ is (x), hence the intersection of the collection $u[\mathscr{V}]$ is (x) (because $uX \subset vX$ for each $X \subset P$ and each $V \in \mathscr{V}$ contains x). As a consequence, $\bigcap u[\mathscr{X}] \subset (x)$. But clearly $x \in uX$ for no $X \in \mathscr{X}$ and hence $\bigcap u[\mathscr{X}] = \emptyset$. — II. Now suppose that condition (a) is not fulfilled and take a filter $\mathscr{X}$ on P such that $\bigcap u[\mathscr{X}] = \emptyset$. For each y in P let $\mathscr{U}_y$ be the neighborhood system of y in $\langle P, u\rangle$. Fix an element x in P and let us consider the closure v for P such that $\mathscr{U}_y$ is the neighborhood system at y in $\langle P, v\rangle$ for each $y \in P$, $y \neq x$, and $\mathscr{V} = [\mathscr{U}_x] \cup [\mathscr{X}]$ $(= \mathsf{E}\{U \cup X \mid U \in \mathscr{U}_x, X \in \mathscr{X}\})$ is the neighborhood system at x in $\langle P, v\rangle$ (such a closure v exists by 14 B.11 (a) because each $\mathscr{U}_y$ as well as $\mathscr{V}$ is a filter). It is almost evident that $\mathscr{V} \subset \mathscr{U}_x$ but $\mathscr{V} \neq \mathscr{U}_x$ (there exist $X \in \mathscr{X}$ and $U \in \mathscr{U}_x$ such that $U \cap X = \emptyset$). Thus v is strictly coarser than u. We shall prove that v is separated. It is enough to show that, for each $y \in P$, the intersection of closures of neighborhoods of y in $\langle P, v\rangle$ is (y). If $y \in P - (x)$, then $\mathscr{U}_y$ is the neighborhood system of y in the space $\langle P, v\rangle$, and we shall prove that then necessarily $\bigcap v[\mathscr{U}_y] = (y)$. If $z \in P - (y)$, then there exists a U in $\mathscr{U}_y$ such that $z \in P - uU$, i.e. $P - U$ is a neighborhood of z in $\langle P, u\rangle$; if $z \neq x$, then $P - U$ is also a neighborhood of z in $\langle P, v\rangle$ and hence $z \notin \bigcap v[\mathscr{U}_y]$. If $z = x$, then we can choose a X in $\mathscr{X}$ such that $y \in P - uX$ and clearly $U_1 = (P - X) \cap U$ is a neighborhood of y in $\langle P, u\rangle$ and hence in $\langle P, v\rangle$, and $x \notin vU_1$. It remains to take the case $y = x$. However, if $V \in \mathscr{V}$, then obviously $uV = vV$, and consequently $\bigcap v[\mathscr{V}] = \bigcap u[\mathscr{V}] \subset \bigcap u[\mathscr{X}] \cup \bigcap u[\mathscr{U}_x] = \emptyset \cup (x)$. The proof is complete. One may notice that v is a topological closure whenever u is a topological closure and the filter $\mathcal{X}$ has a base consisting of open subsets of $\langle P, u\rangle$.

31 D.9. Theorem. *A topological separated closure for a set P is a coarse separated topological closure if and only if the following two conditions are fulfilled:*

(a) *If $\mathcal{U}$ is a filter base of sets in P consisting of open sets in $\langle P, u\rangle$, then $\bigcap u[\mathcal{U}] \neq \emptyset$.*

(b) *For each point x of $\langle P, u\rangle$ the regular open neighborhoods of x form a local base at x.*

Remark. Topological spaces satisfying condition (a) of 31 D.9 are said to be H-closed spaces; this term will be clear from proposition 31 D.10 (remember that a separated space is called a Hausdorff space). The topological spaces satisfying condition (b) are said to be semi-regular. It is to be noted that, evidently, every regular topological space is semi-regular. On the other hand, a semi-regular topological space need not be regular. This will be shown in 31 D.13 (d).

Proof. I. Given a topological space $\langle P, u\rangle$, the collection v of all regular open sets in $\langle P, u\rangle$ is a base for the open sets of a topological space $\langle P, v\rangle$ (see 14 C.10 and 15 A.5) which is separated if and only if the space $\langle P, u\rangle$ is separated; moreover, v is always coarser than u and $u = v$ if and only if condition (b) is fulfilled. It follows that condition (b) is necessary. If condition (a) is not fulfilled and if $\mathcal{U}$ is a filter base of sets in P consisting of all open subsets of $\langle P, u\rangle$ such that $\bigcap u[\mathcal{U}] = \emptyset$, and finally, if $\mathcal{X}$ is the smallest filter containing $\mathcal{U}$, then the construction of the second part of the proof of 31 D.8 leads to a topological closure v for P. Thus condition (a) is necessary.

II. Now suppose that u fulfils conditions (a) and (b) and let v be a separated topological closure coarser than u. We shall show that $v = u$. Since, by (b), the regular open subsets of $\langle P, u\rangle$ form an open base for $\langle P, u\rangle$, it is enough to prove that each regular open subset of $\langle P, u\rangle$ is open in $\langle P, v\rangle$. Let U be a regular open subset of $\langle P, u\rangle$, i.e. $U = \operatorname{int}_u uU = P - u(P - uU)$. The set $P - uU = G$ is open in $\langle P, u\rangle$ and $U = P - uG$. Thus it is sufficient to show that $uG = vG$ for each open subset G of $\langle P, u\rangle$. Since v is coarser than u, we have $uG \subset vG$, and it remains to prove $vG \subset uG$. Suppose $x \in vG - uG$ and take the collection $\mathcal{V}$ of all open neighborhoods of x in $\langle P, v\rangle$; thus $V \cap G \neq \emptyset$ for each V in $\mathcal{V}$. It follows that $\mathcal{U} = [\mathcal{V}] \cap G$ is a filter base of sets in P and moreover, each element of $\mathcal{U}$ is an open subset of $\langle P, u\rangle$ as the intersection $V \cap G$ where G is open in $\langle P, u\rangle$ by our assumption and V is open in $\langle P, u\rangle$ because V is open in $\langle P, v\rangle$ and v is coarser than u. By condition (a) we obtain $\bigcap u[\mathcal{U}] \neq \emptyset$. On the other hand clearly $\bigcap u[\mathcal{U}] \subset \bigcap v[\mathcal{V}]$; but $\bigcap v[\mathcal{V}] = (x)$ because v is a separated closure, and $x \notin uG$ by our assumption, and hence $\bigcap u[\mathcal{U}] = \emptyset$ which is a contradiction. The proof is complete.

31 D.10. *A separated closure space $\langle P, u\rangle$ is a coarse separated closure space (i.e. $\langle P, u\rangle$ fulfils condition* (a) *of* 31 D.8*) if and only if the following condition is satisfied: if $\langle Q, v\rangle$ is a separated closure space such that $\langle P, u\rangle$ is a subspace of $\langle Q, v\rangle$, then P is closed in $\langle Q, v\rangle$. A separated topological space $\langle P, u\rangle$ fulfils con-*

dition (a) *of* 31 D.9 *if and only if the following condition is satisfied: if* $\langle Q, v\rangle$ *is a topological separated space such that* $\langle P, u\rangle$ *is a subspace of* $\langle Q, v\rangle$, *then* P *is closed in* $\langle Q, v\rangle$.

Proof. I. Let $\langle P, u\rangle$ be a subspace of a separated space $\langle Q, v\rangle$ such that P is not closed in $\langle Q, v\rangle$, and let us choose an x in $vP - P$. If $\mathscr{V}$ is a local base at x in $\langle Q, v\rangle$, then $\bigcap v[\mathscr{V}] = (x)$ because v is separated and $\mathscr{U} = [\mathscr{V}] \cap P$ is a filter base because $x \in vP$. Clearly $\bigcap u[\mathscr{U}] \subset P \cap \bigcap v[\mathscr{V}] = P \cap (x) = \emptyset$. Thus condition (a) of 31 D.8 is not fulfilled and $\langle P, u\rangle$ is not a coarse separated closure. If, in addition, $\langle Q, v\rangle$ is topological, then we can take the collection of all open neighborhoods of x in $\langle Q, v\rangle$ as $\mathscr{V}$; then $\mathscr{U}$ is a collection of open subsets of $\langle P, u\rangle$, and consequently $\langle P, u\rangle$ does not fulfil condition (a) of 31 D.9.

II. Conversely, let $\mathscr{U}$ be a filter base of sets in $\langle P, u\rangle$ such that $\bigcap u[\mathscr{U}] = \emptyset$. Let Q be a set consisting of all elements of P and a further point, say x. Let us define a closure v for Q such that $\langle P, u\rangle$ is an open subspace of $\langle Q, v\rangle$ and $\mathscr{V} = (x) \cup [\mathscr{U}]$ is a local base at x in $\langle Q, v\rangle$. It is almost self-evident that $\langle Q, v\rangle$ is separated whenever $\langle P, u\rangle$ is separated, and $\langle Q, v\rangle$ is topological whenever $\langle P, u\rangle$ is topological and the sets of $\mathscr{U}$ are open in $\langle P, u\rangle$. Since clearly $x \in vP - P$, the sufficiency of the conditions in both statements follows.

Remark. By 31 D.10, roughly speaking, a separated closure space (separated topological space) $\langle P, u\rangle$ fulfils condition (a) of 31 D.8 (31 D.9) if and only if $\langle P, u\rangle$ is closed in every separated (separated topological) space.

The next proposition clarifies the relationship between topological coarse separated closures and coarse separated topological closures.

31 D.11. *A coarse separated topological closure is a coarse separated closure if and only if it is regular.*

Proof. I. To prove "if" we shall prove somewhat more: every regular topological space $\langle P, u\rangle$ satisfying condition (a) of 31 D.9 satisfies condition (a) of 31 D.8. Remember that in a regular topological space each closed set is the intersection of closures of its open neighborhoods. Thus, in a regular topological space, if $\mathscr{X}$ is a filter base of sets in P and if $\mathscr{U}$ is the set of all open U such that $U \supset X$ for some $X \in \mathscr{X}$, then $\bigcap u[\mathscr{U}] = \bigcap u[\mathscr{X}]$. The statement follows.

II. To prove "only if" we must show that a topological separated space $\langle P, u\rangle$ satisfying condition (a) of 31 D.8 is regular. If $\langle P, u\rangle$ is not regular then there exists a point x of $\langle P, u\rangle$ and an open neighborhood U of x such that the closure of each neighborhood of x intersects $P - U$; thus, if $\mathscr{V}$ is the neighborhood system at x in $\langle P, u\rangle$, then $\mathscr{X} = \mathsf{E}\{uV - U \mid V \in \mathscr{V}\}$ is a filter base in P and $uX \subset P - U$ for each X in $\mathscr{X}$ (U is open), and hence $\bigcap u[\mathscr{X}] \subset (P - U) \cap \bigcap u[\mathscr{V}] = \emptyset$ because $\bigcap u[\mathscr{V}] = (x)$ (since u is separated) and $x \in U$. The proof is complete.

31 D.12. *No non-void countable separated topological space* $\langle P, u\rangle$ *without isolated points fulfils condition* (a) *of* 31 D.9.

Corollary. *If $\langle P, u\rangle$ is a non-void countable separated closure space without isolated points, then u is finer than no coarse separated topological closure.*

Proof of 31 D.12. Let $\langle P, u\rangle$ be a non-void separated topological space without isolated points. Evidently each non-void open set is infinite. There exists a single-valued relation ϱ which assigns to each pair $\langle x, X\rangle$, where $x \in P$ and X is a non-void open subset of P, an open set V such that $x \notin uV$ and $V \cap X \neq \emptyset$. Indeed, given $\langle x, X\rangle$, we can choose a y in $X - (x)$ (because X is infinite) and an open neighborhood V of y such that $x \notin uV$ (because u is topological and separated). Now if, in addition, P is countable, then there exists a sequence $\{x_n\}$ ranging on P. Put $U_0 = \varrho\langle x_0, P\rangle$, $U_n = U_{n-1} \cap \varrho\langle x_n, U_{n-1}\rangle$ for $n > 0$. Clearly $\{U_n\}$ is a decreasing sequence of non-void open sets and $\bigcap\{uU_n\} = \emptyset$ because $x_n \notin uU_n$.

In closing we shall give an example of a separated topological space satisfying condition (a) of 31 D.9 but not condition (b) of 31 D.9, and an example of a space satisfying both condition (a) and (b) of 31 D.9 but not condition (a) of 31 D.8.

31 D.13. (a) Every bounded closed interval of the reals fulfils condition (a) of 31 D.8. Indeed, by 17 ex. 5, every order-complete ordered space fulfils condition (a) of 31 D.8.

(b) Let $\langle P, v\rangle$ be the closed interval $[\![0, 1]\!]$ of the reals endowed with the order-closure and let u be the closure for P such that the subspace $P - (0)$ of $\langle P, u\rangle$ coincides with the subspace $P - (0)$ of $\langle P, v\rangle$, $P - (0)$ is open in $\langle P, u\rangle$ and U is a neighborhood of 0 in $\langle P, u\rangle$ if and only if $U \cup \mathsf{E}\{n^{-1} \mid n \in \mathsf{N}, n \neq 0\}$ is a neighborhood of 0 in $\langle P, v\rangle$.

It is easily seen that $\langle P, u\rangle$ is a topological space satisfying condition (a) but not (b) of 31 D.9.

(c) Let $P = ([\![0, 1 [\![\times [\![-1, 1]\!]) \cup (\langle 1, 1\rangle) \cup (\langle 1, -1\rangle)$ and let u be the closure for P such that u agrees on $[\![0, 1 [\![\times [\![-1, 1]\!]$ with the relativization of the closure structure of $\mathsf{R} \times \mathsf{R}$, the one-point sets $(\langle 1, 1\rangle)$ and $(\langle 1, -1\rangle)$ are closed in $\langle P, u\rangle$ and the collection of all sets of the form $\langle 1, 1\rangle \cup (]\!] r, 1 [\![\times]\!] 0, 1]\!])$ $(\langle 1, -1\rangle \cup (]\!] r, 1 [\![\times [\![-1, 0 [\![))$, $0 \leqq r < 1$, is a local base at $\langle 1, 1\rangle$ $(\langle 1, -1\rangle)$ in $\langle P, u\rangle$.

It is easily seen that $\langle P, u\rangle$ satisfies condition (b) of 31 D.9 and does not satisfy condition (a) of 31 D.8 (consider the collection of all the sets $]\!] r, 1 [\![\times (0)$, $0 \leqq r < 1$). It is more difficult to show that $\langle P, u\rangle$ fulfils condition (a) of 31 D.9. First notice that if U is an open set the closure of which contains neither $\langle 1, -1\rangle$ nor $\langle 1, 1\rangle$, then $U \subset [\![0, r]\!] \times [\![-1, 1]\!]$ for some $r < 1$. Now if $\mathscr{U}$ is a filter base of sets in P consisting of open subsets of $\langle P, u\rangle$ and if the closure of some $U \in \mathscr{U}$ contains neither $\langle 1, -1\rangle$ nor $\langle 1, 1\rangle$, then there exists an r, $0 \leqq r < 1$, such that $[\mathscr{U}] \cap Q$ is a filter base of sets in Q, where $Q = [\![0, r]\!] \times [\![-1, 1]\!]$. However, the subspace Q of $\langle P, u\rangle$ coincides with the product of intervals $[\![0, r]\!]$ and $[\![-1, 1]\!]$ endowed with the order closure, which fulfil condition (a) of 31 D.9 (as we needed in (a)). By 17 ex. 5, the subspace Q fulfils condition (a) of 31 D.9 and hence $Q \cap \bigcap u[[\mathscr{U}] \cap Q] \neq \emptyset$ and hence $\bigcap u[\mathscr{U}] \neq \emptyset$.

(d) *The space $\langle P, u\rangle$ of* (c) *is semi-regular but not regular.*

32. PROJECTIVE GENERATION FOR CLOSURE SPACES

Let $\{f_a\}$ be a family, each f_a being a mapping of a set P into a closure space $\mathcal{Q}_a$. It turns out that there exists a coarsest closure u such that the mappings $f_a : \langle P, u \rangle \to \mathcal{Q}_a$ are continuous; this closure is said to be projectively generated by the family $\{f_a\}$. For example, the product closure is projectively generated by the family of all projections (17 C.6), and a closure u for a set P is projectively generated by the collection of all continuous functions if and only if u is uniformizable (this requires proof). This section is devoted to the investigation of projectively generated closures.

In the first subsection we shall be concerned with various descriptions of projectively generated closures by means of closure operations, neighborhoods and the convergence of nets in the range carriers of the generating mappings, and with general theorems on the projective construction which generalize the corresponding results for product closures. We shall also prove that the projective construction can be reduced to the construction of the product closure and the construction of the closure projectively generated by a single mapping.

In subsection B we shall study, for a given class of spaces K, the class proj K of all spaces projectively generated by a family of mappings with range carriers in K. Here we shall see that the general theorems of Sections 31 and 32 are rather profound and that many theorems of chapters III, IV and V are their immediate consequences. In subsection C we shall examine projectively generated algebraic structs; all results will be consequences of the theorems of 32 A, B and 31 C. The closing subsection D is devoted to examples.

It should be noted that projective constructions will also be provided for semi-uniform spaces and proximity spaces and therefore the terminology might seem to be somewhat complicated at this stage. Finally, it should be pointed out that all the results of Section 31 are assumed known.

A. GENERALITIES

32 A.1. Definition. A *projective family of mappings*) with a common domain carrier* $\mathcal{P}$ is a family $\{f_a\}$ such that each f_a is a mapping of $\mathcal{P}$ into a struct; if the

*) In the theory of categories such a family is sometimes said to be coinital. A family with a common range carrier (in categorial terminology with a common end-object) is called cofinal; in our exposition such a family is termed an inductive family.

range carrier of each f_a belongs to a class K, then $\{f_a\}$ is said to be a *projective family of mappings for K with a common domain carrier* $\mathscr{P}$. If we say that $\{f_a\}$ is a *projective family of mappings for* K then it is to be understood that $\{f_a\}$ is a projective family for K with a common domain carrier $\mathscr{P}$ which either belongs to K or is a set. We shall see that this ambiguity does not lead to any confusion.

In this section we shall be concerned with projective families for closure spaces, i.e. families $\{f_a\}$ such that each f_a is a mapping into a closure space and all the mappings f_a have a common domain carrier which is a set or a closure space. For example, if $\langle P, u\rangle$ is the product of a family $\{\langle P_a, u_a\rangle\}$ of closure spaces then the family $\{\mathrm{pr}_a : \langle P, u\rangle \to \langle P_a, u_a\rangle\}$ as well as the family $\{\mathrm{pr}_a : P \to \langle P_a, u_a\rangle\}$ is a projective family for closure spaces.

Now we are prepared to introduce those concepts which are basic to the proper subject of the section.

32 A.2. Definition. A *closure u for a set P* is said to be *projectively generated by a family of mappings* $\{f_a \mid a \in A\}$ if $\{f_a\}$ is a projective family of mappings for closure spaces with a common domain carrier P or $\langle P, u\rangle$ and u is the coarsest closure for P such that all the mappings $f_a : \langle P, u\rangle \to \mathbf{E}^* f_a$ are continuous; the family $\{f_a\}$ is said to be a *projective generating family for* $\langle P, u\rangle$. A *closure space* $\langle P, u\rangle$ is said to be *projectively generated by a family of mappings* $\{f_a\}$ if $\{f_a\}$ is a projective generating family for $\langle P, u\rangle$ and $\langle P, u\rangle$ is the common domain carrier of all f_a. The definitions just stated will be carried over to collections of mappings and single mappings as follows: a collection $\mathscr{F}$ has a property $\mathfrak{P}$ if and only if the family $\{f \mid f \in \mathscr{F}\}$ has the property $\mathfrak{P}$, and a mapping f has a property $\mathfrak{P}$ if and only if the singleton (f) has the property $\mathfrak{P}$. Thus, if we say that f is a projective generating mapping (for closure spaces, for a closure space) it is to be understood that the family $\{f \mid f \in (f)\}$ has the corresponding property.

32 A.3. Examples. (a) A closure space projectively generated by an empty family of mappings is an accrete space. – Obvious.

(b) A closure space projectively generated by a family of constant mappings is an accrete space. Indeed, a constant mapping of a closure space $\mathscr{P}$ into another closure space is always continuous (disregarding the closure structures in question).

(c) A closure space projectively generated by a family of mappings into accrete spaces is an accrete space. Indeed, a mapping of any closure space $\mathscr{P}$ into an accrete closure space is continuous.

(d) A homeomorphism is a projective generating mapping. – Obvious.

(e) If $\{u_a\}$ is a family of closure operations for a set P, then $\inf\{u_a\}$ is projectively generated by the family of mappings $\{\mathrm{J} : P \to \langle P, u_a\rangle\}$; stated in other words, the family $\{\mathrm{j} : \langle P, \inf\{u_a\}\rangle \to \langle P, u_a\rangle\}$ is a projective generating family for closure spaces. – Obvious.

(f) The product $\langle P, u\rangle$ of a family $\{\langle P_a, u_a\rangle\}$ of closure spaces is projectively generated by the family $\{\mathrm{pr}_a : \langle P, u\rangle \to \langle P_a, u_a\rangle\}$ of all projections; stated in other words, the family of all projections of a product space is a projective generating family for closure spaces. – This is a restatement of Theorem 17 C.6.

Now we proceed to the general theory. We begin with a description of projectively generated closures.

32 A.4. Theorem. *Every projective family in the class* **C** *with a common domain* P *generates exactly one closure operation for* P. *If* u *is projectively generated by a single mapping* $f: P \to \langle Q, v\rangle$, *then*

$$uX = f^{-1}[vf[X]] \tag{1}$$

for each $X \subset P$. *If* u *is projectively generated by a family of mappings* $\{f_a \mid a \in A\}$ *and* u_a *is the closure projectively generated by the mapping* f_a, $a \in A$, *then* u *is the greatest lower bound of the family* $\{u_a\}$.

Proof. I. Let $\{f_a\}$ be a projective family in **C** and let a set P be the common domain of all f_a. Obviously there exists at most one closure for P generated by $\{f_a\}$. We shall prove the existence. Let us consider the collection Ψ of all closures w for P such that all mappings $f_a : \langle P, w\rangle \to \mathbf{E}^*f_a$ are continuous. According to 31 A.7 $\sup \Psi$ (in $\mathbf{C}(P)$) belongs to Ψ. By definition 32 A.2 the closure $\sup \Psi$ is projectively generated by $\{f_a\}$, that is, $\sup \Psi$ is the coarsest closure for P making all mappings f_a continuous. – II. Now let u be the closure for a set P projectively generated by a family of mappings $\{f_a\}$ (thus P is the common domain of all the f_a) and, for each a, let u_a be the closure projectively generated by the mapping f_a. We shall prove $u = \inf\{u_a\}$. For each a let us consider the set Ψ_a of all closures w for P such that the mapping $f_a : \langle P, w\rangle \to \mathbf{E}^*f_a$ is continuous. By the first part of the proof we have $\sup \Psi = u$ and $\sup \Psi_a = u_a$ for each a. Obviously $\Psi = \bigcap\{\Psi_a\}$. It follows that $\sup \Psi = \inf\{\sup \Psi_a\}$ which is precisely the equality $u = \inf\{u_a\}$. – III. It remains to prove (1). Let u be projectively generated by a mapping $f : P \to \langle Q, v\rangle$, and let us consider the single-valued relation w on $\exp P$ ranging in $\exp P$ which assigns to each X the set $f^{-1}[vf[X]]$. The reader will find no difficulty in verifying that w is a closure operation for P. Clearly $f[wX] \subset vf[X]$ and hence the mapping $f : \langle P, w\rangle \to \langle Q, v\rangle$ is continuous. To prove $u = w$ it remains to show that u is finer than w. The mapping $f : \langle P, u\rangle \to \langle Q, v\rangle$ is continuous and hence $f[uX] \subset vf[X]$ for each $X \subset P$, and consequently $uX \subset f^{-1}[f[uX]] \subset f^{-1}[vf[X]] = wX$ for each $X \subset P$; this implies that u is finer than w and concludes the proof.

Corollary. *If* f *is an embedding of a space* $\mathscr{P}$ *into a space* $\mathscr{Q}$, *then the space* $\mathscr{P}$ *is projectively generated by* f. *If a space* $\mathscr{P}$ *is projectively generated by a mapping* f *and if* f *is a one-to-one mapping, then* f *is an embedding.*

It may also be in place to notice that, in view of the preceeding theorem, Theorem 28 A.9 can be restated as follows:

32 A.5. *Each of the following two conditions is necessary and sufficient for a closure space* $\mathscr{P}$ *to be uniformizable:*

(a) $\mathscr{P}$ *is projectively generated by a mapping into some uniformizable space.*

(b) $\mathscr{P}$ *is projectively generated by a mapping into some cube* $[\![0, 1]\!]^{\aleph}$.

From 32 A.4 and the results of subsection 31 A we shall derive descriptions of projectively generated closures in terms of neighborhoods and convergent nets.

32 A.6. Theorem. *Each of the following two conditions* (a) *and* (b) *is necessary and sufficient for a closure u for a set P to be projectively generated by a non-void family of mappings* $\{f_a \mid a \in A\}$. *Condition* (b) *is necessary and sufficient even if* $A = \emptyset$.

(a) *if* $x \in P$ *and, for each a,* $\mathcal{U}_a$ *is a local sub-base at* $f_a x$ *in* $\mathbf{E}^* f_a$, *then the collection of all* $f_a^{-1}[U]$, $a \in A$, $U \in \mathcal{U}_a$, *is a local sub-base at x in* $\langle P, u \rangle$.

(b) *A point x of P is a limit point of a net N in* $\langle P, u \rangle$ *if and only if the point* $f_a x$ *is a limit point of the net* $f_a \circ N$ *in* $\mathbf{E}^* f_a$ *for each a in A.*

Proof. Write $\mathcal{P} = \langle P, u \rangle$. If $A = \emptyset$ and $\mathcal{P}$ is projectively generated by $\{f_a\}$, then $\mathcal{P}$ is an accrete space (as has already been noted), and the condition (b) is fulfilled because, in an accrete space, each point is a limit point of each net. If $A = \emptyset$ and condition (b) is fulfilled, then each point of $\mathcal{P}$ must be a limit point of each net in $\mathcal{P}$ which implies that $\mathcal{P}$ is an accrete space. Thus condition (b) is both necessary and sufficient whenever $A = \emptyset$. To prove the theorem for the case $A \neq \emptyset$ we shall verify these descriptions for spaces generated by a single mapping.

32 A.7. Lemma. *Each of the following two conditions is necessary and sufficient for a closure u for a set P to be projectively generated by a mapping* $f: P \to \langle Q, v \rangle$:

(a) *if* $\mathcal{U}$ *is a local base (sub-base) at fx in* $\langle Q, v \rangle$, *then* $f^{-1}[\mathcal{U}]$ *is a local base (sub-base) at x in* $\langle P, u \rangle$;

(b) *a point x of P is a limit (accumulation) point of a net N in* $\langle P, u \rangle$ *if and only if the point fx is a limit (accumulation) point of the net* $f \circ N$ *in* $\langle Q, v \rangle$.

Proof. First it is easy to see that the two statements of (a) and the two statements of (b) are equivalent. We shall prove that the statement of (a) concerning bases is necessary and implies the statement of (b) concerning limit points, and the statement of (b) concerning limit points is sufficient. — I. First suppose u is projectively generated by f (thus (1) holds, by 32 A.4) and $\mathcal{U}$ is a local base at fx in $\langle Q, v \rangle$. In view of 14 B.7 to show that $f^{-1}[\mathcal{U}]$ is a local base at x in $\langle P, u \rangle$ it is enough to prove: $x \in uX$ if and only if $f^{-1}[U] \cap X \neq \emptyset$ for each U in $\mathcal{U}$. Since $\mathcal{U}$ is a local base at fx in $\langle Q, v \rangle$, we again have by 14 B.7 that $fx \in vY$ if and only if $U \cap Y \neq \emptyset$ for each U in $\mathcal{U}$. Since obviously $U \cap f[X] \neq \emptyset$ if and only if $f^{-1}[U] \cap X \neq \emptyset$, formula (1) implies that $x \in uX$ if and only if $X \cap f^{-1}[U] \neq \emptyset$. — II. Now assume the statement (a). Clearly a net N in P is eventually in each $f^{-1}[U]$, $U \in \mathcal{U}$, if and only if $f \circ N$ is eventually in each $U \in \mathcal{U}$. It follows that x is a limit point of N in $\langle P, u \rangle$ if and only if fx is a limit point of $f \circ N$ in $\langle Q, v \rangle$. — III. Finally, assume the statement (b); we shall prove that (1) holds. Suppose $X \subset P$. If $x \in uX$, then x is a limit point of a net N in X and by condition (b), fx is a limit point of $f \circ N$. Since clearly $f \circ N$ is in $f[X]$, we have $fx \in vf[X]$. Thus $uX \subset f^{-1}[vf[X]]$. Conversely, if $x \in f^{-1}[vf[X]]$, then fx is a limit point of a net M in $f[X]$. Let us choose a net N in X such that $M = f \circ N$. Since fx is a limit point of $f \circ N$, x is a limit point of N. Since N is in X, $x \in uX$.

Proof of 32 A.6 for the case $A \neq \emptyset$. For each a in A let u_a be the closure projectively generated by f_a. By virtue of 32 A.4, inf $\{u_a\}$ is the closure projectively generated by the family $\{f_a\}$. Combining Lemma 32 A.7 with the descriptions of local sub-bases and convergent nets relative to the infimum of a family of closures (31 A.5 and 31 A.6) we obtain the theorem.

Remark. From the description 17 C.9 of convergent nets in the product spaces and the foregoing theorem we can obtain a new proof of the fact that the product closure is projectively generated by the family of all projections (32 A.3 (f)).

Now we are prepared to prove two fundamental properties of projectively generated closures (32 A.8 and 32 A.9). The remaining statements will be corollaries of these two theorems.

32 A.8. Theorem. *If a space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a\}$, then a mapping f of a space $\mathscr{Q}$ into $\mathscr{P}$ is continuous if and only if all composites $f_a \circ f$ are continuous.*

Proof. If f is continuous then each mapping $f_a \circ f$ is continuous as the composite of two continuous mappings. Conversely, suppose that all composites $f_a \circ f$ are continuous and let x be a limit point of a net N in $\mathscr{Q}$. We must show that fx is a limit point of $f \circ N$ in $\mathscr{P}$. Each mapping $f_a \circ f$ being continuous, the point $(f_a \circ f)\, x$ $(= f_a(fx))$ is a limit point of the net $(f_a \circ f) \circ N$ $(= f_a \circ (f \circ N))$ in $\mathbf{E}^*f_a$ $(= \mathbf{E}^*f_a \circ f)$ for each a and consequently, by 32 A.6, the point fx is a limit point of $f \circ N$.

It may be noted that the last theorem is a generalization of Theorem 17 C.10 which states that a mapping f of a space into a product is continuous if and only if the composites of f with all projections are continuous.

32 A.9. Theorem on associativity. *Let us suppose that $\{\mathscr{Q}_a \mid a \in A\}$ is a family of closure spaces and, for each $a \in A$, the space $\mathscr{Q}_a$ is projectively generated by a family of mappings $\{g_{ab} \mid b \in B_a\}$. Then a family $\{f_a\}$, each f_a being a mapping of a given space $\mathscr{P}$ into $\mathscr{Q}_a$, projectively generates the space $\mathscr{P}$ if and only if the family*

$$\{g_{ab} \circ f_a \mid a \in A\,, \quad b \in B_a\}$$

projectively generates $\mathscr{P}$.

Proof. According to Theorem 32 A.6, condition (b), it is enough to prove that the following two statements (2) and (3) are equivalent for each point x of $\mathscr{P}$ and each net N in $\mathscr{P}$:

(2) for each a in A, $f_a x$ is a limit point of the net $f_a \circ N$ in $\mathscr{Q}_a$;

(3) for each a in A and b in B_a the point $g_{ab} \circ f_a x$ is a limit point of the net $(g_{ab} \circ f_a) \circ N$ in $\mathbf{E}^*(g_{ab} \circ f_a) = \mathbf{E}^*(g_{ab})$.

Fix an a in A. The space $\mathscr{Q}_a$ being projectively generated by the family $\{g_{ab} \mid b \in B_a\}$, again by 32 A.6 (b), $f_a x$ is a limit point of the net $f_a \circ N$ if and only if, for each b in B_a, the point $g_{ab}(f_a x)$ $(= (g_{ab} \circ f_a)x)$ is a limit point of the net $g_{ab} \circ (f_a \circ N)$ $(= (g_{ab} \circ f_a) \circ N)$ in the range of g_{ab}. Since the index a was chosen arbitrarily in A, the equivalence of (2) and (3) follows.

It may be noted that the preceding theorem is a generalization of Theorem 17 C.19 which asserts that the canonical mapping of the product space $\Pi\{P_a \mid a \in A\}$ onto the product $\Pi\{\Pi\{P_b \mid b \in B_c\} \mid c \in C\}$ is a homeomorphism where A is the union of $\{B_c\}$, and $\{B_c\}$ is a disjoint family consisting of non-void sets.

Now we shall prove that the property of projective generating families stated in 32 A.8 is characteristic.

32 A.10. Theorem. *A projective family $\{f_a\}$ of mappings for closure spaces with a common domain carrier $\mathscr{P}$ is a projective generating family if and only if the following condition is fulfilled:*

A mapping f of a closure space into the closure space $\mathscr{P}$ is continuous if and only if all the mappings $f_a \circ f$ are continuous.

Proof. The condition is necessary by 32 A.8. Conversely suppose that the condition is fulfilled. Write $\mathscr{P} = \langle P, u\rangle$. If v is any closure for P then, by the condition, the identity mapping $\mathsf{J} : \langle P, v\rangle \to \langle P, u\rangle$ is continuous, i.e. v is finer than u, if and only if all the mappings $f_a : \langle P, v\rangle \to \mathbf{E}^* f_a$ are continuous. As a consequence, u is projectively generated by the family $\{f_a\}$.

32 A.11. Projective factorization. *For every projective family $\{f_a\}$ of mappings for closure spaces with common domain carrier $\mathscr{P}$, which is a space, there exists a unique projective generating family $\{g_a\}$ for closure spaces with common range carrier $\mathscr{Q}$ such that $|\mathscr{P}| = |\mathscr{Q}|$ and $f_a = g_a \circ h$ for each a where h is the identity mapping of $\mathscr{P}$ into $\mathscr{Q}$. The mapping h is continuous if and only if all the mappings f_a are continuous.*

Proof. Write $\mathscr{P} = \langle P, u\rangle$. Take the closure v projectively generated by the family of mappings $f_a : P \to \mathbf{E}^* f_a$ and put $g_a = f_a : \langle P, v\rangle \to \mathbf{E}^* f_a$, $h = \mathsf{J} : \langle P, u\rangle \to$ $\to \langle P, v\rangle$.

Remark. In accordance with the general rule regarding the use of square brackets we can write

$$\{f_a\} = [\{g_a\}] \circ h \,(= \{g_a \circ h\})$$

and this formula is sometimes named the projective factorization of the projective family $\{f_a\}$ (of course f_a, g_a and h are the mappings from 32 A.11).

It has already been shown that the product closure is projectively generated by the family of all projections, that is, the construction of the product closure is a special case of the projective construction. Now we will show that the construction of a projectively generated closure can be reduced to the construction of a product closure and a closure projectively induced by a single mapping. If a space $\mathscr{P}$ is projectively generated by an empty family, then $\mathscr{P}$ is an accrete space, and consequently $\mathscr{P}$ is projectively generated by any constant mapping. If the family is non-void, then the reduction is described in the theorem which follows.

32 A.12. Theorem. *A non-void projective family $\{f_a\}$ of mappings for closure spaces is a projective generating family if and only if the reduced product f of the family $\{f_a\}$ is a projective generating mapping for closure spaces.*

Proof. Let $\mathscr{P}$ be the common domain carrier of all the f_a and let $\mathscr{Q}$ be the product of the family $\{\mathbf{E}^*f_a\}$ of closure spaces. Recall that f is the mapping of $\mathscr{P}$ into $\mathscr{Q}$ which assigns to each x the point $\{f_a x\}$. If g_a is the projection of $\mathscr{Q}$ into $\mathbf{E}^*f_a$, then $f_a = g_a \circ f$ for each a. Since $\{g_a\}$ is a projective generating family, by 32 A.9, f is a projective generating mapping if and only if $\{f_a\}$ is a projective generating family.

32 A.13. Theorem. *If a closure space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a\}$ and $\mathscr{Q}$ is a subspace of $\mathscr{P}$, then $\mathscr{Q}$ is projectively generated by the family $\{g_a\}$ where each g_a is the domain-restriction of f_a to $\mathscr{Q}$.*

Proof. If h is the identity mapping of $\mathscr{Q}$ into $\mathscr{P}$ then $g_a = f_a \circ h$ for each a. Since h is a projective generating mapping and $\{f_a\}$ is a projective generating family, $\{g_a\}$ is a projective generating family by 32 A.9.

The preceding theorem states that the projective construction commutes with the operation of taking of subspaces. In conclusion we shall prove the following corollary to 32 A.9 (associativity).

32 A.14. Theorem. *Let P be a set, $\{Q_a \mid a \in A\}$ a family of sets and $\{f_a\}$ a family of single-valued relations such that $\mathbf{D}f_a = P$ and $\mathbf{E}f_a \subset Q_a$ for each a. Let κ be the mapping of the product ordered set $\Pi\{\mathbf{C}(Q_a)\}$ into the ordered set $\mathbf{C}(P)$ which assigns to each $\{u_a\}$ the closure projectively generated by the family $\{f_a : P \to \langle Q_a, u_a\rangle\}$. Then the mapping κ is completely meet-preserving.*

Remark. Before proceding with the proof let us notice that this theorem is a generalization of Theorem 31 C.4 asserting that the mapping of the product ordered set $\Pi\{\mathbf{C}(Q_a)\}$ into the ordered set $\mathbf{C}(\Pi\{Q_a\})$, which assigns to each $\{u_a\}$ the product closure $\Pi\{u_a\}$, is completely meet-preserving. Indeed, it suffices to take $P = \Pi\{Q_a\}$ and $f_a = \mathrm{pr}_a \cap (P \times Q_a)$.

Proof. Suppose that $u = \{u_a\}$ is the infimum of a non-void family $\{v_b \mid b \in B\}$ in $\Pi\{\mathbf{C}(Q_a)\}$, $v_b = \{v_{ba} \mid a \in A\}$. According to the definition of the product order, for each a in A, the closure u_a is the infimum of the family $\{v_{ba} \mid b \in B\}$ in $\mathbf{C}(Q_a)$ and hence u_a is projectively generated by the family of mappings $\{\mathrm{J} : Q_a \to \to \langle Q_a, v_{ba}\rangle \mid b \in B\}$. Since κu is projectively generated by the family $\{f_a : P \to \to \langle Q_a, u_a\rangle \mid a \in A\}$, by theorem 32 A.9 the closure κu is projectively generated by the family of mappings $\{f_a : P \to \langle Q_a, v_{ba}\rangle \mid a \in A, b \in B\}$. Next, $\inf\{\kappa v_b \mid b \in B\}$ is projectively generated by the family of mappings $\{\mathrm{J} : P \to \langle P, \kappa v_b\rangle \mid b \in B\}$ and each space $\langle P, \kappa v_b\rangle$ is projectively generated by the family $\{f_a : \langle P, \kappa v_b\rangle \to \langle Q_a, v_{ba}\rangle \mid a \in \in A\}$ and hence, by 32 A.9, $\inf\{\kappa v_b \mid b \in B\}$ is projectively generated by the family of composites, i.e. by the family $\{f_a : P \to \langle Q_a, v_{ba}\rangle \mid a \in A, b \in B\}$. Thus the closures κu and $\inf\{\kappa v_b \mid b \in B\}$ are projectively generated by the same family and hence $\kappa u = \inf\{\kappa v_b \mid b \in B\}$. The proof is complete.

Remark. The mapping κ in 32 A.14 need not be join-preserving, e.g., in the particular case described in the remark folloving 32 A.14 (see 31 C.3).

B. PROJECTIVE-STABLE CLASSES

We shall investigate properties of the class of all spaces projectively generated by a family of mappings with range carriers in a given class of spaces. The theory will be applied to the classes of all topological, uniformizable and pseudometrizable spaces. Further examples will be given in the closing subsection 32 D. For convenience we shall agree on some special notation which will be used only in this section.

32 B.1. Definition. If K is a class of closure spaces and $\aleph$ is a cardinal, then the symbol $\operatorname{proj}_{\aleph} K$ will stand for the class of all closure spaces projectively generated by families of mappings with range carriers in K such that the cardinal of the index set is at most $\aleph$, and the symbol $\operatorname{proj} K$ will stand for the class of all closure spaces projectively generated by families of mappings with range carriers in K; thus $\operatorname{proj} K$ consists of all spaces belonging to at least one $\operatorname{proj}_{\aleph} K$. As usual, this notation will be applied to classes of closures, i.e. if L is a class of closure operations and if K is the class of all closure spaces $\langle P, u \rangle$ such that $u \in L$, then $\operatorname{proj}_{\aleph} L$ ($\operatorname{proj} L$) stands for the class consisting of the closure structures of all $\mathscr{P} \in \operatorname{proj}_{\aleph} K$ ($\mathscr{P} \in \operatorname{proj} K$). The class $\operatorname{proj} K$ is called the *projective progeny of* K. If $K = \operatorname{proj} K$, then K is said to be *projective-stable.*

32 B.2. *For any class of spaces* K *and* $\aleph = 0$ *the class* $\operatorname{proj}_{\aleph} K$ *consists of all the accrete spaces. If* K *and* K' *are classes of closure spaces and if* $\aleph$ *and* $\aleph'$ *are cardinals, then*

(a) $\aleph \leq \aleph'$, $K \subset K'$ *implies*

$$\operatorname{proj}_{\aleph} K \subset \operatorname{proj}_{\aleph'} K' \subset \operatorname{proj} K' \supset \operatorname{proj} K\,,$$

(b) *if* $\aleph \neq 0$, *then* $\operatorname{proj}_{\aleph} K \supset K$,

(c) *if* $\aleph \leq 1$ *or* $\aleph \geq \aleph_0$, *then* $\operatorname{proj}_{\aleph} \operatorname{proj}_{\aleph} K = \operatorname{proj}_{\aleph} K$,

(d) $\operatorname{proj} \operatorname{proj} K = \operatorname{proj} K$,

(e) *the class* $\operatorname{proj}_{\aleph} K$ *is hereditary,*

(f) *if* $\aleph \leq 1$ *or* $\aleph \geq \aleph_0$, *then the class* $\operatorname{proj}_{\aleph} K$ *is closed under products of families of cardinal* $\leq \aleph$,

(g) *the class* $\operatorname{proj} K$ *is hereditary and completely productive,*

(h) *if* $\aleph \geq \aleph_0$, *then* $\operatorname{proj}_1 \operatorname{proj}_{\aleph} K = \operatorname{proj}_{\aleph} K$.

Proof. Every accrete space is projectively generated by the empty family and a space projectively generated by the empty family is accrete. Thus $\operatorname{proj}_0 K$ is the class of all accrete closure spaces. Statements (a) and (b) are evident. Statements (c) and (d) follow immediately from 32 A.9. Statement (e) follows from 32 A.13.

Statement (f) follows from the fact that the product space is projectively generated by the family of projections. Statement (g) follows from (e) and (f). Finally, if f is a projective generating mapping and $\mathbf{E}^*f$ is projectively generated by a non-void family $\{f_a\}$, then clearly $\{f_a \circ f\}$ is a projective generating family. The last statement follows.

Remark. It is to be noted that $\text{proj}_2\, \text{proj}_2\, K \neq \text{proj}_2\, K$ in general. For example, if K is a singleton $(\mathscr{P})$ where $\mathscr{P}$ is a two-point discrete space, then every discrete space consisting of eight points belongs to $\text{proj}_2\, \text{proj}_2\, K$ but not to $\text{proj}_2\, K$.

32 B.3. Theorem. *Let K be a class of closure spaces and let $\aleph \geqq 1$ be a cardinal. A space $\mathscr{P}$ belongs to* $\text{proj}_\aleph K$ *if and only if either $\mathscr{P}$ is an accrete space or $\mathscr{P}$ is homeomorphic with a subspace of a product space $\mathscr{Q} \times \mathscr{R}$ where $\mathscr{Q}$ is an accrete space (which can be chosen so that $|\mathscr{Q}| = |\mathscr{P}|$) and $\mathscr{R}$ is the product of a family in K of cardinality at most $\aleph$.*

First we shall prove

32 B.4. *If $\mathscr{Q}$ is a non-void accrete space and $\mathscr{R}$ is any space, then the projection of the product space $\mathscr{Q} \times \mathscr{R}$ onto $\mathscr{R}$ is a projective generating mapping.*

Proof. Let $\{f_a \mid a \in A\}$ be a projective generating family for closure spaces and let B be a subset of A such that the range carrier of each f_a, $a \in A - B$ is an accrete space. Then clearly $\{f_a \mid a \in B\}$ is a projective generating family. Since the product space is projectively generated by projections, the statement follows.

Proof of 32 B.3. I. By 32 B.2 the class $\text{proj}_\aleph K$ is hereditary and contains all accrete spaces. Since $\text{proj}_1\, \text{proj}_\aleph K = \text{proj}_\aleph K$ (by 32 B.2 (h)), every homeomorph of a space from $\text{proj}_\aleph K$ belongs to $\text{proj}_\aleph K$, and if $\mathscr{Q}$ is an accrete space and $\mathscr{R}$ is a space from $\text{proj}_\aleph K$, then the product space $\mathscr{Q} \times \mathscr{R}$ also belongs to $\text{proj}_\aleph K$ (by 32 B.4). Finally, if $\mathscr{R}$ is the product of a family in K of a cardinal at most $\aleph$, then $\mathscr{R}$ belongs to $\text{proj}_\aleph K$ because the product space is projectively generated by projections. Thus all spaces described in the theorem belong to $\text{proj}_\aleph K$.

II. Conversely, let $\langle P, u \rangle \in \text{proj}_\aleph K$. If $\langle P, u \rangle$ is not an accrete space, then $\langle P, u \rangle$ is projectively generated by a non-void family of mappings $\{f_a \mid a \in A\}$. Let f be the reduced product of the family $\{f_a\}$ and let $\mathscr{R}$ be the range carrier of f (thus $\text{gr}\, f = \{x \to \{f_a x\} \mid x \in P\}$ and $\mathscr{R}$ is the product of $\{\mathbf{E}^*f_a\}$). By 32 A.12 the mapping f is a projective generating mapping. Let $\mathscr{Q}$ be the set P endowed with the accrete closure and let g be the identity mapping of $\langle P, u \rangle$ onto $\mathscr{Q}$. Clearly g is continuous and hence (g, f) is a projective generating collection of mappings (because (f) is such a collection); by 32 A.12 the reduced product $h = g \times_{\text{red}} f \; (= \{x \to \langle x, fx \rangle\} :$ $: \langle P, u \rangle \to \mathscr{Q} \times \mathscr{R})$ is a projective generating mapping. But clearly h is injective and hence h is an embedding. The proof is complete.

As an immediate consequence we obtain the following theorem:

32 B.5. Theorem. *Let K be a class of closure spaces. A space $\mathscr{P}$ belongs to the class* proj K *if and only if either $\mathscr{P}$ is an accrete space or $\mathscr{P}$ is homeomorphic with* *a subspace of a product space $\mathcal{Q} \times \mathcal{R}$, where $\mathcal{Q}$ is an accrete space (which can be chosen so that $|\mathcal{Q}| = |\mathcal{P}|$) and $\mathcal{R}$ is the product of a family in K.*

32 B.6. Example. If $\mathcal{S}$ is the empty space, then the class $\text{proj}_{\aleph}(\mathcal{S})$ as well as the class $\text{proj}(\mathcal{S})$ coincide with the class of all accrete spaces.

32 B.7. Theorem. *Let $\mathcal{S}$ be a non-void space and let $\aleph \geqq 1$ be a cardinal. Then a closure space $\mathcal{P}$ belongs to $\text{proj}_{\aleph} \mathcal{S}$ if and only if $\mathcal{P}$ is homeomorphic with a subspace of $\mathcal{Q} \times \mathcal{S}^{\aleph}$ where $\mathcal{Q}$ is an accrete space (which can be chosen so that $|\mathcal{Q}| = |\mathcal{P}|$). A space $\mathcal{P}$ belongs to* proj $\mathcal{S}$ *if and only if $\mathcal{P}$ is homeomorphic with a subspace of a space $\mathcal{Q} \times \mathcal{S}^{\aleph}$ for some accrete space $\mathcal{Q}$ (which can be chosen so that $|\mathcal{Q}| = |\mathcal{P}|$) and for some cardinal $\aleph$.*

Proof. The second statement follows from the first one and the first statement follows from 32 B.3; it is enough to show that every accrete space is a subspace of a product space in the theorem, but this is evident since $\mathcal{S}$ is non-void, and $\mathcal{S}^{\aleph'}$ with $\aleph' \leqq \aleph$ is homeomorphic with a subspace of $\mathcal{S}^{\aleph}$.

32 B.8. Examples. (a) Suppose that we know that the class $\tau\mathbf{C}$ of all topological spaces is hereditary, completely productive and contains all accrete spaces (all this has already been proved). By 32 B.5 we obtain $\text{proj}\, \tau\mathbf{C} = \tau\mathbf{C}$. (b) Suppose that we know that the class $\mathfrak{v}\mathbf{C}$ of all uniformizable spaces is hereditary, completely productive and contains all accrete spaces. By 32 B.5 we obtain $\text{proj}\, \mathfrak{v}\mathbf{C} = \mathfrak{v}\mathbf{C}$. (c) Let K be the class of all pseudometrizable spaces. If we know that K is hereditary, countably productive and K contains all accrete spaces, then Theorem 32 B.3 yields $\text{proj}_{\aleph_0} K = K$. (d) By 32 B.7 the class proj (R) consists of all spaces which are homeomorphic with a subspace of a space of the form $\mathcal{Q} \times \mathsf{R}^{\aleph}$ where $\mathcal{Q}$ is an accrete space and $\aleph$ is an appropriate cardinal. If the class of all uniformizable spaces is defined as proj (R), then theorem 32 B.2 states that the class of all uniformizable spaces is hereditary, completely productive and contains all accrete spaces, and theorem 32 B.7 gives a description of uniformizable spaces.

32 B.9. Theorem. *Let K be a class of spaces and let L be the class consisting of closure structures of all spaces from K. In order that* $\text{proj}\, K = K$ *it is necessary and sufficient that* (a) $\text{proj}_1 K \subset K$, *and* (b) *the class L is completely meet-stable in the ordered class* **C**.

Remarks. Evidently condition (a) can be replaced by the following condition: K contains a non-void space, and if f is a projective generating mapping with $\mathbf{E}^*f \in K$, then also $\mathbf{D}^*f \in K$. Next, it follows from (a) and (b) that, if $\text{proj}\, K = K$, then every closure has an upper modification in L; in particular, L is order-complete and completely meet-preserving in **C**.

Theorem 32 B.9 is an immediate consequence of the following more general result:

32 B.10. *Let $\aleph \geqq 1$ be a cardinal, K a class of closure spaces and L the class consisting of closure structures of spaces of K. In order that* $\text{proj}_{\aleph} K = K$ *it is* *necessary and sufficient that* (a) $\operatorname{proj}_1 K \subset K$ *and* (b) *if* u *is the infimum in* $\mathbf{C}$ *of a non-void family* $\{u_a \mid a \in A\}$ *in* L *and* card $A \leqq \aleph$, *then* $u \in L$.

Proof. Clearly both conditions are necessary (for (b) remember that $\inf \{u_a\}$ is projectively generated by the family of mappings $\{J : \langle P, \inf \{u_a\}\rangle \to \langle P, u_a\rangle\}$, where P is the set such that all the u_a are closures for P). Conversely, assuming conditions (a) and (b) let us consider any space $\langle P, u\rangle \in \operatorname{proj}_\aleph K$, and take a family of mappings $\{f_a \mid a \in A\}$ which projectively generates $\langle P, u\rangle$ and such that card $A \leqq \aleph$, $\mathbf{E}^*f_a \in K$ for each a in A. If the cardinal of A is at most 1, then $\langle P, u\rangle$ belongs to $\operatorname{proj}_1 K$ and hence to K by condition (a). If the cardinal of A is at least 1, let us consider the family $\{u_a\}$ such that each u_a is projectively generated by the mapping $f_a : P \to \mathbf{E}^*f_a$. By 32 A.4 the closure u is the infimum of $\{u_a\}$, by condition (a) each closure u_a belongs to L, and by condition (b) the infimum of $\{u_a\}$ also belongs to L.

Remark. Evidently if $\operatorname{proj}_\aleph K = K$, then L is $\aleph$-meet-stable in $\mathbf{C}$ and every accrete closure belongs to L; on the other hand L need not be completely meet-stable in $\mathbf{C}$ and the upper modification of an element of $\mathbf{C}$ in L need not exist. For example, if K is the class of all pseudometrizable spaces, then $\operatorname{proj}_{\aleph_0} K = K$ (32 D.2) but $\operatorname{proj} K$ is the class of all uniformizable spaces, and the upper modification of an element of $\mathbf{C}$ in K need not exist. Finally, note that it follows from 32 B.9 that $\operatorname{proj}_1 K \subset \subset K$, $\operatorname{proj} K \neq K$ imply that at least one closure has no upper modification in L.

32 B.11. The class $\tau\mathbf{C}$ of all topological spaces. *One has that* $\operatorname{proj} \tau\mathbf{C} = \tau\mathbf{C}$ *and if* $\mathscr{P}$ *is a two-point non-discrete and non-accrete space (equivalently, non-discrete and feebly semi-separated) then* $\operatorname{proj}(\mathscr{P}) = \tau\mathbf{C}$. *Stated in other words, if a space* $\mathscr{Q}$ *is projectively generated by a family of mappings into topological spaces, then* $\mathscr{Q}$ *is a topological space, and moreover* $\mathscr{Q}$ *is projectively generated by a family of mappings into the space* $\mathscr{P}$ *described above.*

Proof. I. The class of all topological closures is completely meet-stable and contains all accrete closures (31 B.4); hence, by 32 B.9, to prove $\operatorname{proj} \tau\mathbf{C} = \tau\mathbf{C}$ it is enough to show that a closure space $\langle Q, v\rangle$ projectively generated by a mapping f into a topological space $\langle R, w\rangle$ is topological. If $X \subset Q$, then $vX = f^{-1}[wf[X]]$ by 32 A.4, and the set $wf[X]$ being closed in $\langle R, w\rangle$ (w is topological) and the mapping f being continuous, we find that the set vX is closed in $\langle Q, v\rangle$. Thus $\langle Q, v\rangle$ is topological. — II. To prove $\operatorname{proj}(\mathscr{P}) = \tau\mathbf{C}$ it is sufficient to show that $\operatorname{proj}(\mathscr{P}) \supset \tau\mathbf{C}$, because $\mathscr{P}$ is topological (every two-point space is topological) and hence by I $\operatorname{proj}(\mathscr{P}) \subset \operatorname{proj} \tau\mathbf{C} = \tau\mathbf{C}$. Under an appropriate notation of points of $\mathscr{P}$ by x and y we have $\overline{(x)} = (x)$ and $\overline{(y)} = (x, y) = |\mathscr{P}|$. Let $\langle Q, v\rangle$ be a topological space and $\mathscr{B}$ an open base for $\langle Q, v\rangle$. The reader will find no difficulty in showing that the space $\langle Q, v\rangle$ is projectively generated by the family $\{f_U \mid U \in \mathscr{B}\}$, where f_U is the mapping of $\langle Q, v\rangle$ into $\mathscr{P}$ which assigns to each point $z \in U$ the point y and to each point $z \in Q - U$ the point x (compare with 26 B.9(b)).

32 B.12. Pseudometrizable spaces. *Suppose that we know that the class of all pseudometrizable closures is countably meet-stable (see* 31 ex. 4*). Since clearly* *every accrete space is pseudometrizable, to prove that the class of all pseudometrizable spaces is invariant under countable projective construction, by* 32 B.10 *it is enough to show that if f is a projective generating mapping and the range carrier $\langle Q, v\rangle$ of f is pseudometrizable, then the domain carrier $\langle P, u\rangle$ of f is also pseudometrizable;* this can be proved as follows: if a pseudometric d induces v, then $D = d \circ (\operatorname{gr} f \times \operatorname{gr} f)$ is a pseudometric for P inducing the closure u. But this is evident because the mapping $f : \langle P, D\rangle \to \langle Q, d\rangle$ is distance-preserving (i.e. $d\langle fx, fy\rangle = D\langle x, y\rangle$; that D induces u may be shown by noting that a net $D\langle x_a, x\rangle$ converges to zero if and only if the net $d\langle fx_a, fx\rangle$ converges to zero). It should be remarked that other proofs will be given in 32 D.

Every class $\operatorname{proj}_{\aleph} K$ contains all accrete spaces. Sometimes it is convenient to omit from $\operatorname{proj}_{\aleph} K$ those accrete spaces which are obtained trivially, i.e. as projectively generated by the empty family or by constant mappings.

32 B.13. Definition. A *distinguishing projective family of mappings* is a projective family $\{f_a\}$ with a common domain $\mathscr{S}$ such that for all distinct elements x and y of $|\mathscr{S}|$ there exists an index a such that $f_a x \neq f_a y$.

Now given a class K of spaces and a cardinal $\aleph$ one can consider the class, say $\operatorname{Proj}_{\aleph} K$, of all spaces projectively generated by a distinguishing family $\{f_a \mid a \in A\}$ with range carriers in K such that the cardinal of A is at most $\aleph$, and the class Proj K defined similarly. Then the accrete spaces mentioned above are avoided, except for the void space.

32 B.14. *A distinguishing projective family $\{f_a\}$ of mappings for closure spaces is a projective generating family if and only if the reduced product f of $\{f_a\}$ is an embedding.* – 32 A.4 Corollary, 32 A.12.

We leave to the reader as a simple task the formulation and proof of propositions for the classes $\operatorname{Proj}_{\aleph}$ and Proj similar to those for $\operatorname{proj}_{\aleph}$ and proj.

C. TOPOLOGIZED ALGEBRAIC STRUCTS

We shall investigate projective constructions for topologized algebraic structs. Roughly speaking, this subsection is related to subsections A and B as subsection 31 C is to 31 A and 31 B.

32 C.1. *Let σ be an internal composition on a set P, μ be an internal composition on a set Q and f be a single-valued homomorphism-relation under σ and μ such that* $\mathbf{D}f = P$. *Let v be a closure for Q and let u be the closure projectively generated by the mapping $f : P \to \langle Q, v\rangle$. Then, if $\langle \mu, v\rangle$ is a continuous or inductively continuous composition, then $\langle \sigma, u\rangle$ has the same property.*

Proof. Since f is a homomorphism-relation we have $f \circ \sigma = \mu \circ (f \times f)$ and hence $f' \circ \sigma' = \mu \circ (f' \times f')$ where $f' = f : \langle P, u\rangle \to \langle Q, v\rangle$, $\sigma' = \sigma : \langle P, u\rangle \times \langle P, u\rangle \to \langle P, u\rangle$ and $\mu' = \mu : \langle Q, v\rangle \times \langle Q, v\rangle \to \langle Q, v\rangle$. By our assumption f' is a projective generating mapping, and hence $f' \times f'$ is continuous. Now if $\langle \mu, v \rangle$ is continuous, i.e. μ' is a continuous mapping, then $\mu' \circ (f' \times f')$ is continuous and hence $f' \circ \sigma'$ is continuous, and finally, f' being a projective generating mapping, σ' is continuous by 32 A.8, i.e. $\langle \sigma, u \rangle$ is continuous. Similarly, if μ' is inductively continuous, then $\mu' \circ (f' \times f')$ and hence $f' \circ \sigma'$ is inductively continuous, and f' being a projective generating mapping, σ' is inductively continuous by 32 A.8, i.e. $\langle \sigma, u \rangle$ is an inductively continuous composition.

32 C.2. *Under the notation and assumptions of* 32 C.1, *if σ and μ are semi-group structures, then the inversion of $\langle \sigma, u \rangle$ is continuous whenever the inversion of $\langle \mu, v \rangle$ is continuous.*

Proof. If g is the inversion of σ and h is the inversion of μ, then $f \circ g = h \circ f$. Let $\mathscr{R}$ be the subspace $\mathbf{D}g$ of $\langle P, u \rangle$, $g' = g : \mathscr{R} \to \mathscr{R}$, $\mathscr{S}$ the subspace $\mathbf{D}h$ of $\langle Q, v \rangle$, $h' = h : \mathscr{S} \to \mathscr{S}$ and finally, $f' = f : \mathscr{R} \to \mathscr{S}$. By 32 A.13 f' is a projective generating mapping. Clearly $f' \circ g' = h' \circ f'$. If h' is continuous, then $h' \circ f'$ and hence $f' \circ g'$ is continuous, and f' being a projective generating mapping, g' is continuous by 32 A.8.

As an immediate corollary of 32 C.1 and 32 C.2 we obtain the following important theorem.

32 C.3. Theorem. *Let f be a homomorphism of a group (ring) $\mathscr{G}$ into a group (ring) $\mathscr{H}$. If v is a closure admissible for $\mathscr{H}$, and u is the closure projectively generated by $f : |\mathscr{G}| \to \langle |\mathscr{H}|, v \rangle$, then u is admissible for $\mathscr{G}$.*

Using the theorems of 31 C and Theorem 32 A.4 we obtain at once from 32 C.1 – 32 C.3 the following important result.

32 C.4. Theorem. *Let $\{\mathscr{H}_a \mid a \in A\}$ be a family, and let $\mathscr{G}$ and all the $\mathscr{H}_a$ be semi-groups, groups or rings. Let $\{f_a\}$ be a family, each f_a being a homomorphism of $\mathscr{G}$ into $\mathscr{H}_a$. Finally, let $\{v_a\}$ be a family such that v_a is a closure for $|\mathscr{H}_a|$ and let u be the closure projectively generated by the family of mappings $\{f_a : |\mathscr{G}| \to \to \langle |\mathscr{H}_a|, v_a \rangle\}$. Then, if all the $\langle \mathscr{H}_a; v_a \rangle$ are either continuous or inductively continuous semi-groups, or topological groups, or topological rings, then $\langle \mathscr{G}; u \rangle$ has the same property.*

Proof. By 32 C.1—3 the theorem is true for the case where the cardinal of the index set is one. In the general case let u_a be the closure generated by $f_a : |\mathscr{G}| \to \langle |\mathscr{H}_a|, v_a \rangle$; by 32 A.4 the closure u is the infimum of $\{u_a\}$. Finally, by the theorem 31 C.10, the set of all closures making $\mathscr{G}$ a continuous or inductively continuous semi-group or topological group or topological ring, respectively, is completely meet-stable in $\mathbf{C}(|\mathscr{G}|)$ and contains the accrete closure.

32 C.5. *Let $f : \langle P, u \rangle \to \langle Q, v \rangle$ be a projective generating mapping and let $\langle u, \varrho, w \rangle$ and $\langle v, \varrho_1, w \rangle$ be topologized external compositions such that* gr f *is a homomorphism-relation under ϱ and ϱ_1. If $\langle v, \varrho_1, w \rangle$ is continuous or inductively continuous then $\langle u, \varrho, w \rangle$ has the same property.*

Proof. Let $\varrho' = \varrho : \langle A, w \rangle \times \langle P, u \rangle \to \langle P, u \rangle$, $\varrho'_1 = \varrho_1 : \langle A, w \rangle \times \langle Q, v \rangle \to \to \langle Q, v \rangle$ (i.e. ϱ' and ϱ'_1 are associated topologized external multiplications). Clearly

$$f \circ \varrho' = \varrho'_1 \circ ((\mathrm{J} : \langle A, w \rangle \to \langle A, w \rangle) \times f) .$$

If $\langle v, \varrho_1, w \rangle$ is continuous or inductively continuous, i.e. if the mapping ϱ'_1 is continuous or inductively continuous, then the right side of the above equality has the same property, and hence, f being a projective generating mapping, ϱ' has the same property.

32 C.6. Theorem. *Let $\mathscr{R}$ be a topological ring, $\mathscr{L}$ a module over $\mathscr{R}$ and $\{\mathscr{L}_a\}$ a family of modules over $\mathscr{R}$. Let $\{f_a\}$ be a family, each f_a being a homomorphism of $\mathscr{L}$ into $\mathscr{L}_a$. If $\{v_a\}$ is a family such that each v_a is a closure admissible for the module $\mathscr{L}_a$ over $\mathscr{R}$ and if u is the closure projectively generated by the family $\{f_a : |\mathscr{L}| \to \langle|\mathscr{L}_a|, v_a\rangle\}$, then u is admissible for the module $\mathscr{L}$ over $\mathscr{R}$. The same holds on replacing modules by algebras throughout.*

Proof. By 32 C.4 the closure operation u is admissible for the underlying group (ring) of $\mathscr{L}$ and hence it remains to show that the external structure of $\mathscr{L}$ is continuous under u. If u_a is the closure projectively generated by the mapping $f_a : |\mathscr{L}| \to \to \langle|\mathscr{L}_a|, v_a\rangle$, then $u = \inf \{u_a\}$ by 32 A.4, the external structure of $\mathscr{L}$ is continuous under each u_a by 32 C.5, and hence, by 31 C.16, under u.

In the concluding part we shall be concerned with projective constructions for topological modules. Recall that by our convention 19 E.3 all the properties defined for mappings for closure spaces are carried over to mappings for topological algebraic structs; e.g., if f is a mapping of a topological module into another one then we say that f is a projective generating mapping provided that the mapping f regarded as a mapping of the underlying closure spaces is a projective generating mapping.

32 C.7. Theorem. *Every topological real module is projectively generated by a family of homomorphisms into pseudometrizable topological real modules.*

Proof. Let $\mathscr{L}$ be a topological real module and let Γ be the set of all pseudometrizable closures compatible for the underlying module of $\mathscr{L}$ which are coarser than the closure structure of $\mathscr{L}$. By ex. 16, each neighborhood of the zero in $\mathscr{L}$ is a neighborhood of zero with respect to a closure of Γ. As a consequence $\mathscr{L}$ is projectively generated by the family $\{\mathrm{J} : \mathscr{L} \to \mathscr{L}_u \mid u \in \Gamma\}$ where $\mathscr{L}_u$ denotes the underlying module of $\mathscr{L}$ endowed with u.

A *real topological module* $\mathscr{L}$ is said to be *locally convex* if convex neighborhoods of the zero of $\mathscr{L}$ form a local base. Recall that a set X is convex in a real module if $rx + sy$ belongs to X whenever $x \in X$, $y \in X$, $r \geqq 0$, $s \geqq 0$, $r + s = 1$. For properties of convex sets see the exercises to Section 19.

32 C.8. Theorem. *A topological real module is locally convex if and only if it is projectively generated by a family of homomorphisms into normed spaces.*

Proof. Let K be the class of all locally convex real modules. Since the inverse image under a homomorphism of a convex set is a convex set, we find immediately that each space projectively generated by a family of homomorphisms whose range carriers lie in K belongs to K. Next, in a normed space the spheres about the zero are convex and hence each normed space is locally convex. It remains to show that every locally convex space is projectively generated by a family of mappings into normed spaces. Let $\mathscr{L}$ be a locally convex space. It will suffice to show that for each neighborhood U of the zero there exists a continuous norm φ for $\mathscr{L}$ such that $\varphi x < 1$ implies $x \in U$ and this follows from the following proposition the proof of which was given in 19 ex. 4.

32 C.9. *Let X be an absorbing balanced convex neighborhood of zero in a real topological module $\mathscr{L}$. For each x in $\mathscr{L}$ let A_x be the set of those positive real r such that $x \in r[X]$. Then A_x is non-void (because X is absorbing), and*

$$\varphi = \{x \to \inf A_x \mid x \in |\mathscr{L}|\}$$

is a continuous norm for $\mathscr{L}$ such that $\varphi x < 1$ implies $x \in X$, and $x \in X$ implies $\varphi x \leqq 1$.

Remark. By 19 ex. 5, there exists a metric linear space which is not locally convex.

32 C.10. Definition. A *topological $\mathscr{R}$-module $\mathscr{L}$* is said to be *weak* if $\mathscr{L}$ is projectively generated by a family of homomorphisms into the topological $\mathscr{R}$-module associated with $\mathscr{R}$.

32 C.11. Theorem. *Every weak real topological module is locally convex but no infinite-dimensional normed separated space is weak.*

Proof. The first assertion follows from 32 C.8 because R is a normed module. Let $\mathscr{L}$ be a normed real module. If $\mathscr{L}$ is infinite-dimensional and separated, then the set $\bigcap\{f_i^{-1}[0]\}$ is unbounded for each finite family $\{f_i\}$ of linear functionals (27 ex. 15), and hence a bounded neighborhood of the zero contains no set of the form $\bigcap\{f_i^{-1}[U_i]\}$ where $\{f_i\}$ is a finite family of functionals and U_i are neighborhoods of the zero in R. As a consequence (32 A.6), $\mathscr{L}$ is not projectively generated by linear functionals.

32 C.12. Remark. It follows from 19 ex. 2 that each finite dimensional topological real module is weak and normable.

32 C.13. Locally convex modification. Let $\mathscr{L}$ be a topologized real module (the norm is denoted by $|.|$), $\mathscr{L}'$ the underlying module of $\mathscr{L}$. Denote by $\mathscr{L}_\varphi$ the module $\mathscr{L}'$ endowed with a norm φ for $\mathscr{L}'$ over R. Let $\mathscr{U}$ be the set of all norms for $\mathscr{L}'$ such that the mapping $\mathsf{J} : \mathscr{L} \to \mathscr{L}_\varphi$ is continuous and let us consider the closure u projectively generated by the family of mappings $\{\mathsf{J} : \mathscr{L}' \to \mathscr{L}_\varphi \mid \varphi \in \mathscr{U}\}$. By 33 C.6 u is admissible for $\mathscr{L}'$ over R. The closure u will be called the locally convex modification of the closure structuie of $\mathscr{L}$, and the resulting topological module over R will be called the *locally convex modification of $\mathscr{L}$* and will be denoted by lc $\mathscr{L}$.

32 C.14. Theorem. *The locally convex modification* lc $\mathscr{L}$ *of a topologized real module* $\mathscr{L}$ *is the unique locally convex space with the same underlying module as* $\mathscr{L}$ *and with the following property:*

If f is a homomorphism of $\mathscr{L}$ *into a locally convex module* $\mathscr{K}$, *then f is continuous if and only if the mapping* $g = f :$ lc $\mathscr{L} \to \mathscr{K}$ *is continuous.*

Proof. Evidently there exists at most one space lc $\mathscr{L}$ satisfying the condition. We shall prove that lc $\mathscr{L}$ satisfies the condition. By definition lc $\mathscr{L}$ is locally convex and the mapping $h = \mathsf{J} : \mathscr{L} \to$ lc $\mathscr{L}$ is continuous. It follows that if g is continuous then $f = g \circ h$ is continuous. Conversely, suppose that f is continuous and let us consider the set $\mathscr{N}$ of all continuous norms for $\mathscr{K}$. If $\varphi \in \mathscr{N}$, then clearly φ gr f is a continuous norm for $\mathscr{L}$, and therefore φ gr f is a continuous norm for lc $\mathscr{L}$ (by the definition of lc $\mathscr{L}$). However, gr $f \doteq$ gr g and therefore φ gr g is a continuous norm for lc $\mathscr{L}$ for each φ in $\mathscr{N}$. Since $\mathscr{K}$ is projectively generated by $\{\mathsf{J} : \mathscr{K} \to \to \langle |\mathscr{K}|, \varphi \rangle\}$, g is continuous by 32 A.10.

32 C.15. Weak modification. *Let* $\mathscr{L}$ *be a topologized module over a topological ring* $\mathscr{R}$ *and let* $\mathscr{L}'$ *be the underlying module of* $\mathscr{L}$ *over* $\mathscr{R}$. *Let* $\mathscr{F}$ *be the set of all continuous linear forms f on* $\mathscr{L}$, *and let u be the closure projectively generated by the family of mappings* $\{f : |\mathscr{L}| \to \mathscr{R} \mid f \in \mathscr{F}\}$. *By* 32 C.6 *the closure u is compatible for* $\mathscr{L}'$ *over* $\mathscr{R}$ *(*$\mathscr{R}$ *can be considered as a topological module over* $\mathscr{R}$*). The closure u will be called the* weak modification of the closure structure of $\mathscr{L}$, *and the resulting topological* $\mathscr{R}$*-module will be called the* weak modification of $\mathscr{L}$ *and will be denoted by* weak $\mathscr{L}$.

The reader can prove without a difficulty the following characterization of weak $\mathscr{L}$.

32 C.16. Theorem. *The weak modification* $\mathscr{L}_1$ *of a topologized module* $\mathscr{L}$ *over a topological module* $\mathscr{R}$ *is the unique weak space over* $\mathscr{R}$, *with the same underlying module as* $\mathscr{L}$, *which satisfies the following condition:*

Let f be a homomorphism of $\mathscr{L}$ *into a weak topological module* $\mathscr{K}$ *over* $\mathscr{R}$. *Then f is continuous if and only if the mapping* $f : \mathscr{L}_1 \to \mathscr{K}$ *is continuous.*

D. EXAMPLES

According to 32 B.7 a class K is projective-stable if and only if K is hereditary, completely productive and contains all accrete spaces. A direct proof of proj $K = K$ is often more convenient than the proof of the facts that K is hereditary, completely productive and contains all accrete spaces. For example we shall prove:

32 D.1. *The class of all regular spaces is projective-stable.*

Proof. Let $\mathscr{P}$ be projectively generated by a family of mappings $\{f_a\}$ such that the range carriers of all the f_a are regular. If $\{f_a\}$ is empty, then $|\mathscr{P}|$ is a unique neighborhood of any point of $\mathscr{P}$ and $|\mathscr{P}|$ is closed. Suppose that $\{f_a\}$ is non-void and U is any neighborhood of a point x of $\mathscr{P}$. There exists a finite set B of indices and a family $\{U_a \mid a \in B\}$ such that U_a is a neighborhood of $f_a x$ in $\mathbf{E}^* f_a$ and $\bigcap\{f_a^{-1}[U_a] \mid a \in B\} \subset U$. Choose a family $\{V_a \mid a \in B\}$ such that V_a is a neighborhood of $f_a x$ and the closure of V_a is contained in U_a for each a, and put $V = \bigcap\{f_a^{-1}[V_a] \mid a \in B\}$. Since the closure of V_a is contained in U_a and f_a is continuous, the closure of $f_a^{-1}[V_a]$ is contained in $f^{-1}[U_a]$; consequently, the closure of V is contained in each $f^{-1}[U_a]$ and hence in U.

We gave a complete proof to show that the proof of proj $K = K$ is the same as the proof of the fact that K is completely productive, i.e. that the special properties of projections of a product space into coordinate spaces plays no important part in the proof. On the other hand, a completely productive class containing all accrete spaces need not be hereditary and therefore need not be projective-stable, e.g. the class of all compact spaces (41 A).

32 D.2. *Let M be the class of all pseudometrizable spaces. Then* $\text{proj}_{\aleph_0} M = M$, *and* $\text{proj}_{\aleph} M$, *where* $\aleph \geqq \aleph_0$, *consists of all uniformizable topological spaces each of which has an $\aleph$-locally finite open base (i.e. a base which is the union of $\aleph$ locally finite families).*

Proof. I. Suppose that a space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a \mid a \in A\}$ such that the cardinal of A is $\aleph$. By 30 B.2 each space $\mathbf{E}^* f_a$ has σ-locally finite open base $\mathscr{B}_a$. For each a let $\mathscr{C}_a$ be the collection of all $f_a^{-1}[B]$, $B \in \mathscr{B}_a$. Since f_a is continuous, $\mathscr{C}_a$ is a σ-locally finite collection in $\mathscr{P}$, and hence $\mathscr{C} = \bigcup\{\mathscr{C}_a \mid a \in A\}$ is $\aleph$-locally finite. Since $\{f_a\}$ is a projective generating family, $\mathscr{C}$ is an open sub-base for $\mathscr{P}$. If $\mathscr{C} = \bigcup\{\mathscr{D}_b \mid b \in B\}$, the cardinal of B is $\aleph$ and each $\mathscr{D}_b$ is locally finite, then the smallest multiplicative collection $\mathscr{E}$ containing $\mathscr{C}$ is an open base for $\mathscr{P}$, and clearly $\mathscr{E} = \bigcup\{\mathscr{E}_F \mid F$ is a finite subset of $B\}$, where $\mathscr{E}_F$ is the smallest multiplicative collection containing all $\mathscr{D}_b$, $b \in F$. Clearly, each $\mathscr{E}_F$ is locally finite and the set of all the sets F has cardinal $\aleph$. Thus each space of $\text{proj}_{\aleph} M$ has a $\aleph$-locally finite base. Each space of M is uniformizable and hence each space of $\text{proj}_{\aleph} M \subset \text{proj}\, M$ is uniformizable. In particular, each space of $\text{proj}_{\aleph_0} M$ has a σ-locally finite open base and hence is pseudometrizable by the pseudometrization theorem 30 B.2.

II. Assume that $\{U_a \mid a \in A\}$ is an open $\aleph$-locally finite base for a uniformizable space $\mathscr{P}$. Let $A = \bigcup\{A_b \mid b \in B\}$, where the cardinal of B is $\aleph$ and each family $\{U_a \mid a \in A_b\}$ is locally finite. For each a in A and each b in B let V_{ab} be the union of all U_c, $c \in A_b$, such that U_c and $|\mathscr{P}| - U_a$ are functionally separated. It is easily seen that the sets V_{ab} and $|\mathscr{P}| - U_a$ are functionally separated (see 28 ex. 9). From the fact that $\mathscr{P}$ is uniformizable we derive immediately that $U_a = \bigcup\{V_{ab} \mid b \in B\}$ for each a. Indeed, if G is a neighborhood of x in $\mathscr{P}$, then the sets (x) and $|\mathscr{P}| - G$ are functionally separated and hence a neighborhood of (x) and $|\mathscr{P}| - G$ are functionally separated.

For each $\langle a, b\rangle \in A \times B$ let f_{ab} be a continuous function on $\mathscr{P}$ which is 1 on V_{ab}, 0 on $|\mathscr{P}| - U_a$ and which fulfils the inequality $0 \leqq f_{ab} \leqq 1$. The family $\{f_{ab} \mid a \in A_b\}$

is locally finite. For each b and b' in B let

$$d_{bb'} = \{\langle x, y\rangle \to \Sigma\{|f_{ab}x - f_{ab}y| \mid a \in A_{b'}\} \mid x \in |\mathscr{P}|, y \in |\mathscr{P}|\}.$$

Clearly each $d_{bb'}$ is a continuous pseudometric for $\mathscr{P}$, and it follows from $U_a = \bigcup\{V_{ab} \mid b \in B\}$ that U_a contains an open 1-sphere about x with respect to some $d_{bb'}$. It follows that $\mathscr{P}$ is projectively generated by the family

$$\{\mathrm{J} : \mathscr{P} \to \langle|\mathscr{P}|, d_{bb'}\rangle \mid \langle b, b'\rangle \in B \times B\}.$$

32 D.3. Separated spaces. *If a separated space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a\}$, then $\{f_a\}$ is a distinguishing family, i.e. if $x \neq y$, then $f_a x \neq f_a y$ for some a (indeed, if $f_a x = f_a y$ for each a then the closure of (x) contains y). On the other hand, if $\{f_a\}$ is a distinguishing projective generating family for a space $\mathscr{P}$ and if the range carrier of each $\{f_a\}$ is separated, then clearly $\mathscr{P}$ is separated. Consequently, if $\{f_a\}$ is a projective generating family for $\mathscr{P}$ and the range carriers are separated, then $\mathscr{P}$ is separated if and only if the family is distinguishing.*

33. INDUCTIVE GENERATION FOR CLOSURE SPACES

In the preceding section we studied the coarsest closure for a set P making all given mappings $f_a : P \to \langle Q_a, v_a \rangle$ ($\langle Q_a, v_a \rangle$ being closure spaces) continuous. Here we shall concern ourselves with the dual situation. Let there be given a family $\{f_a\}$, each f_a being a mapping of a closure space $\mathcal{Q}_a$ into a set P which does not depend on a, and we shall study the finest closure u for P such that all mappings $f_a : \mathbf{D}^* f_a \to \to \langle P, u \rangle$ are continuous. It turns out that, roughly speaking, the operation of forming the sum of a family of closure spaces plays the same part in the theory of inductively generated closures as the operation of forming products in the theory of projectively generated closures. Fundamental theorems are proved in subsection A.

We shall also see that inductively generated closures inherit very few of the properties from the closure structures of domain carriers of the mappings of the generating family. Closures projectively generated by a family of mappings into topological, uniformizable or regular spaces are topological, uniformizable or regular, respectively; every closure operation is inductively generated by a family of mappings whose domain carriers are hereditarily paracompact spaces (33 B.2); in particular, a closure inductively generated by a family of mappings from topological spaces need not be a topological closure. Because of the great importance of topological closures we shall introduce (in 33 B) the notion of a closure topologically inductively generated by a family of mappings $\{f_a\}$ as the finest topological closure making all the f_a continuous. The main results are proved without any reference to the theory of inductively generated closures. On the other hand, evidently the closure topologically inductively generated by a family $\{f_a\}$ is the topological modification of the closure inductively generated by the family $\{f_a\}$, and this fact enables us to reduce the theory of topologically inductively generated closures to the theory of inductively generated closures. For convenience, this reduction will be given in a more general situation, namely for "*K-inductively generated closures*" where K is a projective-stable class of spaces. Thus we obtain, e.g., the theory of inductive generation for uniformizable spaces and regular spaces.

As in the case of projective generation, inductive generation can be reduced to the construction of the sum of a family of spaces and construction of the closure inductively generated by a single mapping only (namely, the corresponding reduced sum). In subsection C we shall study closures inductively generated by a single mapping and the related notions of a quotient mapping and a decomposition space, and the class of spaces stable under the inductive construction, the so-called inductive-stable classes of spaces.

The closing subsection (33 D) is devoted to various examples; e.g. we shall introduce the inductive product of a family of closure spaces which generalize the inductive product of two spaces and we shall explain "pasting" and "sewing" of closure spaces which often occur in the theory of functions.

It should be remarked that two special cases of inductive generating mappings will be considered in the next section, and various important examples related to inductive construction will be given in Section 35 devoted to the examination of convergence which can be regarded as a part of the theory of inductive generation.

A. GENERALITIES

For convenience we shall introduce the following concept (compare with 32 A.1).

33 A.1. Definition. An *inductive family of mappings with a common range carrier* $\mathscr{P}$ is a family $\{f_a\}$ such that each $\{f_a\}$ is a mapping of a struct into $\mathscr{P}$; if the domain carrier of each f_a belongs to a class K, then $\{f_a\}$ is said to be an *inductive family in K with a common range carrier* $\mathscr{P}$. If we say that $\{f_a\}$ is an *inductive family of mappings for K* then it is to be understood that $\{f_a\}$ is an inductive family in K with a common range carrier $\mathscr{P}$ which either belongs to K or is a set. We shall see that this ambiguity does not lead to any confusion.

Notice that $\emptyset$ is a projective as well as an inductive family of mappings for each class K. In this section we shall study inductive families for closure spaces, i.e. families $\{f_a\}$ such that each f_a is a mapping of a closure space and all the mappings f_a have a common range carrier which is a set or a closure space. For example, if $\langle P, u\rangle$ is the sum of a family $\{\langle P_a, u_a\rangle\}$ of closure spaces, then the family $\{\text{inj}_a : \langle P_a, u_a\rangle \to \to \langle P, u\rangle\}$ as well as $\{\text{inj}_a : \langle P_a, u_a\rangle \to P\}$ are inductive families for closure spaces.

33 A.2. Definition. A *closure operation u* for a set P is said to be *inductively generated by a family of mappings* $\{f_a \mid a \in A\}$ if $\{f_a\}$ is an inductive family of mappings for closure spaces with the common range carrier P or $\langle P, u\rangle$ and u is the finest closure for P such that all the mappings $f_a : \mathbf{D}^*f_a \to \langle P, u\rangle$ are continuous; the family $\{f_a\}$ is said to be an *inductive generating family for* $\langle P, u\rangle$. A *closure space* $\langle P, u\rangle$ is said to be *inductively generated by a family of mappings* $\{f_a\}$ if $\{f_a\}$ is an inductive generating family for $\langle P, u\rangle$ and $\langle P, u\rangle$ is the common range carrier of all f_a. The definitions just stated will be carried over to collections of mappings and single mappings as follows: a collection $\mathscr{F}$ has a property $\mathfrak{P}$ if and only if the family $\{f \mid f \in \mathscr{F}\}$ has the property $\mathfrak{P}$, and a mapping f has a property $\mathfrak{P}$ if and only if the singleton (f) has the property $\mathfrak{P}$. Thus, if we say that f is an inductive generating mapping (for closure spaces, for a space $\langle P, u\rangle$) it is to be understood that the family $\{f \mid f \in (f)\}$ has the corresponding property.

33 A.3. Examples. (a) The empty family $\emptyset$ is simultaneously a projective family and an inductive family. We know that every accrete space is projectively generated by the empty family. It is evident that every discrete space is inductively generated by the empty family.

(b) If a closure space $\mathscr{P}$ is inductively generated by a family of constant mappings, then $\mathscr{P}$ is discrete. Indeed, a constant mapping into any space is continuous.

(c) A space inductively generated by a family of mappings of discrete spaces is discrete. Indeed, a mapping of a discrete space into any space is continuous.

(d) If $\{u_a\}$ is a family in $\mathbf{C}(P)$, then $\sup \{u_a\}$ is inductively generated by the family $\{J : \langle P, u_a \rangle \to P\}$.

(e) The sum $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of closure spaces is inductively generated by the family $\{\mathrm{inj}_a : \mathscr{P}_a \to \mathscr{P}\}$ of canonical embeddings. – This is a restatement of 17 B.3.

Now we proceed to the general theory. The first theorem corresponds to an analogous result (32 A.4) for projectively generated closures.

33 A.4. Theorem. *Every non-void inductive family of mappings for the class* $\mathbf{C}$ *generates exactly one closure operation. If a closure u for a set P is inductively generated by a family of mappings* $\{f_a \mid a \in A\}$ *and if, for each a in A,* u_a *is the closure inductively generated by the mapping* f_a*, then* $u = \sup \{u_a \mid a \in A\}$*. If u is a closure for a set P inductively generated by a mapping* $f : \langle Q, v \rangle \to P$*, then*

$$(*) \qquad uX = X \cup f[vf^{-1}[X]]$$

for each $X \subset P$*. Finally, if u is inductively generated by a family of mappings* $\{\langle f_a, \langle Q_a, v_a \rangle, P \rangle \mid a \in A\}$*, then*

$$(**) \qquad uX = X \cup \bigcup \{f_a[v_a f_a^{-1}[X]] \mid a \in A\}$$

for each $X \subset P$*; stated in other words,* $x \in uX$ *if and only if* $x \in X$ *or* $f_a^{-1}[x]$ *intersects* $v_a f^{-1}[X]$ *for some a in A.*

Proof. I. The uniqueness is obvious. We shall prove the existence. Let $\{f_a \mid a \in A\}$ be an inductive family in $\mathbf{C}$ with $\mathbf{E}^* f_a$ a set P and let us consider the set Φ of all closures w for P making continuous all mappings f_a. By 31 A.7 the closure $\inf \Phi$ (in $\mathbf{C}(P)$) belongs to Φ, and from definition 33 A.2 it is obvious that $\inf \Phi$ is the closure inductively generated by $\{f_a\}$.

II. Now, for each a, let u_a be the closure inductively generated by the mapping f_a. According to I, $u_a = \inf \Phi_a$ where Φ_a is the set of all closures for P making continuous the mapping f_a. Since obviously $\Phi = \bigcap \{\Phi_a \mid a \in A\}$, we have $\inf \Phi = \sup \{\inf \Phi_a\}$ which implies the equality $u = \sup \{u_a\}$.

III. Now let u be the closure for P inductively generated by a mapping $f : \langle Q, v \rangle \to P$. We shall prove that $(*)$ is true for each $X \subset P$. Consider the single-valued relation w on $\exp P$ ranging in $\exp P$ which assigns to each $X \subset P$ the set $X \cup f[vf^{-1}[X]]$. The reader can verify without difficulty that the relation w is a closure for P. By definition of continuity, a mapping $f : \langle Q, v \rangle \to \langle P, u_1 \rangle$ is continuous if and only if $u_1X \supset f[vf^{-1}[X]]$ for each $X \subset P$, that is, if and only if u_1 is coarser than w. It follows that $w = u$.

IV. It remains to prove the formula (**). But this follows from the facts which have already been proved. Indeed, by II $u = \sup \{u_a\}$, and by III $u_aX = X \cup f[v_a f^{-1}[X]]$ for each $X \subset P$. Since $u = \sup \{u_a\}$, by 31 A.2 $uX = \bigcup\{u_aX\}$ whenever the indexed set A is non-void; this yields (**) under the assumption $A \neq \emptyset$. If $A = \emptyset$, then obviously u is the discrete closure for P, that is $uX = X$ for each $X \subset P$, and obviously (**) is also fulfilled.

Corollaries. (a) *In order that a space $\langle P, u\rangle$ be inductively generated by a mapping $f : \langle Q, v\rangle \to \langle P, u\rangle$ it is necessary and sufficient that $f[vY] = uf[Y]$ for each $Y = f^{-1}[X]$, $X \subset P$ and $uX = X$ if $X \subset P - f[Q]$.*

(b) *Let f be a mapping of $\langle Q, v\rangle$ onto $\langle P, u\rangle$. If f is a projectively generating mapping, then f is an inductive generating mapping. Stated in other words, if $f : Q \to \langle P, u\rangle$ projectively generates v, then $f : \langle Q, v\rangle \to P$ inductively generates u.*

(b′) *If f is an injective inductively generating mapping, then f is projectively generating.*

(c) *If f is an inductive generating mapping then the set $|\mathbf{E}^*f| - \mathbf{E}f$ is an open and closed discrete subset of $\mathbf{E}^*f$; in particular, each point of $|\mathbf{E}^*f| - \mathbf{E}f$ is isolated.*

(d) *If $\{u_a\}$ is a family in $\mathbf{C}(P)$ and if I_a is the identity mapping of $\langle P, u_a\rangle$ onto $\langle P, \sup \{u_a\}\rangle$, then $\{I_a\}$ is an inductive generating family for $\langle P, \sup \{u_a\}\rangle$.*

(e) *The assertion of* 33 A.3 (e).

Proof. Statement (a) is a straightforward consequence of the description (*), statement (b) follows from the description (*) and the description (1) from 32 A.4 of projectively generated closures. Statement (c) follows from (*) or perhaps more easily from (a).

The next three corollaries of description (**) of inductively generated closures express basic properties of inductively generated spaces and therefore they will be formulated as theorems. It is to be noted that these theorems are analogues of Theorems 32 A.10, 32 A.9 and 32 A.13 for projectively generated spaces. Their proofs are a matter of a relatively simple calculation based on the description (**) of generated closures. The reader will find no difficulty in providing these without reading the proofs which follow Theorem 33 A.7.

33 A.5. Theorem. *Let $\{f_a \mid a \in A\}$ be a family of mappings of closure spaces into a closure space $\mathscr{P}$. In order that the space $\mathscr{P}$ be inductively generated by the family $\{f_a\}$ it is necessary and sufficient that a mapping f of the space $\mathscr{P}$ into a space $\mathscr{R}$ be continuous if and only if all mappings $f \circ f_a$ are continuous (compare with* 32 A.10*).*

33 A.6. Theorem. *Let $\{f_a \mid a \in A\}$ be a family of mappings from closure spaces into a space $\mathscr{P}$ and the domain space $\mathbf{D}^*f_a$ of each f_a be inductively generated by a family of mappings $\{g_{ab} \mid b \in B_a\}$. Then the space $\mathscr{P}$ is inductively generated by the family $\{f_a\}$ if and only if it is inductively generated by the family $\{f_a \circ g_{ab} \mid b \in \in B_a, a \in A\}$ (compare with* 32 A.9*).*

33 A.7. Theorem on partial commutativity. *If a closure space $\mathscr{P}$ is inductively generated by a family of mappings $\{f_a\}$ and if $\mathscr{Q}$ is a subspace of $\mathscr{P}$, then $\mathscr{Q}$ is inductively generated by the family $\{g_a\}$, where each g_a is the restriction of f_a to a mapping of the subspace $f_a^{-1}[|\mathscr{Q}|]$ of $\mathbf{D}^*f_a$ into $\mathscr{Q}$, i.e. $\mathbf{D}g_a = f_a^{-1}[|\mathscr{Q}|]$, $\mathbf{E}^*g_a = \mathscr{Q}$ and $g_a x = f_a x$ for each $x \in \mathbf{D}g_a$. (Compare with* 32 A.13.*)*

Proof of 33 A.5. I. First let us suppose that $\mathscr{P}$ is inductively generated by $\{f_a\}$ and f is a mapping of $\mathscr{P}$ into a space $\mathscr{R}$. If f is continuous, then each composition $f \circ f_a$ is continuous as the composition of two continuous mappings. Conversely, suppose that all compositions $f \circ f_a$ are continuous. If $\{Y_a\}$ is any family such that $Y_a \subset \mathbf{D}f_a$ then by continuity of $f \circ f_a$

$$f \circ f_a[\overline{Y}_a] \, (= f[f_a[\overline{Y}_a]]) \subset \overline{f \circ f_a[Y_a]}^{\mathscr{R}} .$$

In particular, if $X \subset |\mathscr{P}|$ and $Y_a = f_a^{-1}[X]$ then from (**) we obtain

$$f[\overline{X}^{\mathscr{P}}] = f[X] \cup \bigcup \{f[f_a[\overline{Y}_a]]\} \subset \overline{f[X]}^{\mathscr{R}}$$

which establishes the continuity of f.

II. Suppose that the condition is fulfilled. If f is the identity mapping of $\mathscr{P}$ onto $\mathscr{P}$, then f is continuous and by assumption all $f \circ f_a$ are continuous. But $f \circ f_a = f_a$ and hence all f_a are continuous. Thus the closure structure of $\mathscr{P}$ is coarser than the closure inductively generated by the family of mappings $\{f_a\}$ considered as mappings into the underlying set $|\mathscr{P}|$ of $\mathscr{P}$. If u is any closure for the set $|\mathscr{P}|$ such that all $g_a = \langle \operatorname{gr} f_a, \mathbf{D}^*f_a, \langle |\mathscr{P}|, u\rangle\rangle$ are continuous, and if f is the identity mapping of the space $\mathscr{P}$ onto $\langle |\mathscr{P}|, u\rangle$, then $g_a = f \circ f_a$ for each a and hence each $f \circ f_a$ is continuous. Hence, by the condition, f is continuous; this means that the closure structure of $\mathscr{P}$ is finer than u. It follows that the closure structure of $\mathscr{P}$ is the finest closure for the set $|\mathscr{P}|$ making all the mappings f_a continuous; this concludes the proof.

Proof of 33 A.6. By formula (**) of 33 A.4 we have

(1) $\overline{Y}_a^{\mathbf{D}f_a} = Y_a \cup \bigcup \{g_{ab}[\overline{g_{ab}^{-1}[Y_a]}^{\mathbf{D}g_{ab}}] \mid b \in B_a\}$

for each $a \in A$ and $Y_a \subset \mathbf{D}f_a$. By formula (**) the fact that $\mathscr{P}$ is inductively generated by $\{f_a\}$ is equivalent to

(2) $X \subset |\mathscr{P}| \Rightarrow \overline{X}^{\mathscr{P}} = X \cup \bigcup \{f_a[\overline{f_a^{-1}[X]}^{\mathbf{D}f_a}] \mid a \in A\}$,

and the fact that $\mathscr{P}$ is inductively generated by $\{f_a \circ g_{ab}\}$ is equivalent to

(3) $X \subset |\mathscr{P}| \Rightarrow \overline{X}^{\mathscr{P}} = X \cup \bigcup \{f_a \circ g_{ab}[\overline{(f_a \circ g_{ab})^{-1}[X]}^{\mathbf{D}g_{ab}}] \mid a \in A,\ b \in B_a\}$.

According to (1), the conditions (2) and (3) are equivalent (put $Y_a = f_a^{-1}[X]$ and notice that $(f_a \circ g_{ab})^{-1}[X] = g_{ab}^{-1}[Y_a]$).

Proof of 33 A.7. The fact that $\mathscr{P}$ is inductively generated by $\{f_a\}$ is equivalent to (2). By the definition of relativization closures we have $\overline{X}^{\mathscr{Q}} = \overline{X}^{\mathscr{P}} \cap |\mathscr{Q}|$ for each $X \subset |\mathscr{Q}|$ and hence

$$\overline{X}^{\mathscr{Q}} = X \cup \bigcup \{g_a[\overline{g_a^{-1}[X]}^{\mathbf{D}g_a}] \mid a \in A\}$$

for each $X \subset |\mathscr{Q}|$; this means that $\mathscr{Q}$ is inductively generated by $\{g_a\}$.

In the preceding section we saw that the projective generation can be reduced to the construction of the product closure and the closure projectively generated by a single mapping. Now it will be shown that the construction of the inductively generated closure can be reduced to the construction of the sum closure and the closure inductively generated by a single mapping. If a space $\mathscr{P}$ is inductively generated by the empty family, then $\mathscr{P}$ is a discrete space and hence $\mathscr{P}$ is inductively generated by any constant mapping into $\mathscr{P}$. For spaces inductively generated by a non-void family the reduction is described in the theorem which follows.

33 A.8. Theorem. *Let $\{f_a \mid a \in A\}$ be a non-void inductive family of mappings for closure spaces with a common range carrier $\mathscr{P}$ and let f be the reduced sum of the family $\{f_a\}$ i.e. $f = \{\langle a, x\rangle \to f_a x\} : \Sigma\{\mathbf{D}^* f_a\} \to \mathscr{P}$. Then $\{f_a\}$ is an inductive generating family if and only if the mapping f is an inductive generating mapping.*

Proof. Let i_a denote the canonical embedding of $\mathbf{D}^* f_a$ into the sum space $\Sigma\{\mathbf{D}^* f_a\}$, i.e. $i_a = \mathrm{inj}_a : \mathbf{D}^* f_a \to \Sigma\{\mathbf{D}^* f_a\}$. Clearly $f_a = f \circ i_a$ for each a in A. Since $\{i_a\}$ is an inductive generating family (e.g. by 33 A.3 (e)), the statement follows from 33 A.6.

33 A.9. Inductive factorization theorem. *Let $\{f_a \mid a \in A\}$ be an inductive family of mappings for closure spaces with a common range carrier $\mathscr{P}$ which is a space. There exists a unique inductive generating family $\{g_a\}$ for closure spaces with a common range carrier $\mathscr{Q}$ such that $|\mathscr{P}| = |\mathscr{Q}|$ and $f_a = h \circ g_a$ for each a where h is the identity mapping of $\mathscr{Q}$ onto $\mathscr{P}$. The mapping h is continuous if and only if all the mappings f_a are continuous.*

Proof. Write $\mathscr{P} = \langle P, u\rangle$ and let us consider the closure v inductively generated by the family $\{f_a : \mathbf{D}^* f_a \to P\}$. If $g_a = f_a : \mathbf{D}^* f_a \to \langle P, v\rangle$ and $h = \mathrm{J} : \langle P, v\rangle \to \langle P, u\rangle$, then $\{g_a\}$ is an inductive generating family for closure spaces and $f_a = h \circ g_a$ for each a. By 33 A.5 the mapping h is continuous if and only if all the mappings f_a are continuous.

Remark. Since $f_a = h \circ g_a$ for each index a, we can write

$$\{f_a\} = h \circ [\{g_a\}] \,(= \{h \circ g_a\})$$

and this formula is sometimes termed the canonical inductive factorization of the inductive family $\{f_a\}$ (compare with 32 A.11).

33 A.10. Remark. Let $\{f_a\}$ be an inductive family of mappings for closure spaces with common range carrier $\mathscr{P}$ and let

$$(*) \qquad \{f_a\} = h_i \circ [\{g_a^i\}], \quad i = 1, 2,$$

where h_i are bijective mappings and $\{g_a^i\}$, $i = 1, 2$, are inductive generating families of mappings for closure spaces. If $k = h_1^{-1} \circ h_2$, then $k^{-1} = h_2^{-1} \circ h_1$, $g_a^1 = k \circ g_a^2$ and $g_a^2 = k^{-1} \circ g_a^1$ for each a; hence k as well as k^{-1} is continuous by 33 A.5, and consequently is a homeomorphism. Thus the factorization $(*)$ with $\{g_a^i\}$ an inductive generating family is unique up to a homeomorphism k. It is to be noted that if we are given $\{g_a^i\}$, then the mapping h_i need not be completely determined unless $\{\mathbf{E}g_a^i\}$ is a cover of the common range carrier of all g_a^i.

In 33 A.4 we described the closure inductively generated by a family of mappings $\{f_a\}$ by a simple formula depending on closures of domain spaces of mappings f_a. We shall often need descriptions of neighborhoods, open sets and closed sets in a space $\mathscr{P}$ inductively generated by a family of mappings $\{f_a\}$ in terms of the corresponding notions for the domain spaces of the mappings f_a. The required results are listed in the following proposition, the proof of which depends on 33 A.4 and the descriptions of suprema of families of closure operations.

33 A.11. Theorem. *Let $\mathscr{P}$ be a closure space inductively generated by a family of mappings $\{f_a\}$. A subset U of $\mathscr{P}$ is a neighborhood of a point x of $\mathscr{P}$ if and only if $x \in U$ and $f_a^{-1}[U]$ is a neighborhood of $f_a^{-1}[x]$ in the domain $\mathbf{D}^*f_a$ of f_a for each a. A subset U of $\mathscr{P}$ is open if and only if the set $f_a^{-1}[U]$ is open for each a, and finally, a subset X of $\mathscr{P}$ is closed if and only if the set $f_a^{-1}[X]$ is closed for each a.*

Proof. According to 33 A.4 the closure structure of $\mathscr{P}$ is the least upper bound of the family $\{u_a\}$ where u_a is the closure inductively generated by the mapping f_a considered as a mapping of $\mathbf{D}^*f_a$ into the underlying set of $\mathscr{P}$. By 31 A.4 a set $X \subset |\mathscr{P}|$ is a neighborhood of x in $\mathscr{P}$, is open or is closed if and only if it has the corresponding property relative to the closure u_a for each a. Hence it remains to prove that if $f : \langle Q, v\rangle \to \langle P, u\rangle$ is an inductive generating mapping, then $X \subset P$ is a neighborhood of x in $\langle P, u\rangle$ or X is open in $\langle P, u\rangle$ or X is closed in $\langle P, u\rangle$ if and only if $x \in X$ and $f^{-1}[X]$ is a neighborhood of $f^{-1}[x]$ or $f^{-1}[X]$ is open in $\langle Q, v\rangle$ or $f^{-1}[X]$ is closed in $\langle Q, v\rangle$, respectively. It will suffice to prove the first statement because the second one is an immediate consequence of the first (a set is open if and only if it is a neighborhood of all its points) and the third follows from the second (a set is closed if and only if its complement is open). To prove the first recall that, according to 33 A.4, we have $uY = Y \cup f[vf^{-1}[Y]]$ for each $Y \subset P$. By definition, $X \subset P$ is a neighborhood of x in $\langle P, u\rangle$ if and only if $x \in P - u(P - X)$, that is, if and only if $x \in P - ((P - X) \cup f[vf^{-1}[P - X]])$, i.e. if and only if $x \in X$ and $f^{-1}[x] \subset Q - vf^{-1}[P - X]$. However, the last inclusion means that $f^{-1}[X]$ is a neighborhood of $f^{-1}[x]$ in $\langle Q, v\rangle$ (because of the trivial equality $f^{-1}[P - X] = Q - f^{-1}[X]$), which accomplishes the proof.

Usually a space inductively generated by a family of mappings $\{f_a\}$ inherits very few of the properties of the domain spaces of the mappings f_a. This explains the fact that the inductive construction of spaces occurs so frequently. Indeed, often very complicated spaces with many extraordinary properties can be constructed by a suitable choice of an inductive generating family of mappings, usually with very simple and reasonable domain spaces, and conversely, an examination of a complicated space can be simplified by a suitable inductive generating family of mappings.

B. INDUCTIVE CONSTRUCTION FOR TOPOLOGICAL SPACES

We know that a closure space projectively generated by a family of mappings into topological spaces is a topological space. It turns out that a closure space inductively generated by a family of mappings of topological spaces need not be topological. For example, if $u = \sup \{u_a\}$ in $\mathbf{C}(P)$, then u is inductively generated by the family of mappings $\{\mathrm{J} : \langle P, u_a \rangle \to P\}$ and if all the u_a are topological, then u need not be topological. It may be in place to give the simplest example. If a space $\langle P, u \rangle$ is not topological then necessarily the cardinal of P is at least 3 because there exists a subset X of P such that $X \neq \emptyset$, $uX - X \neq \emptyset$ and $uuX - uX \neq \emptyset$. Consider the set $P = (1, 2, 3)$ and define a closure u for P as follows: $u(1) = (1, 2)$, $u(2) = (2, 3)$ and $u(3) = (3)$. Obviously the closure u is not inductively generated by any mapping whose domain is a topological three-point space. Nevertheless, u is inductively generated by a mapping f whose domain is a topological four-point space. Such a mapping f can be constructed as follows: let $Q = (x_1, x_2, x_3, x_4)$ be a four-point set, v the closure for Q such that $v(x_1) = (x_1, x_2)$, $v(x_2) = x_2$, $v(x_3) = (x_3, x_4)$ and $v(x_4) = (x_4)$; and $fx_1 = 1$, $fx_4 = 3$ and $fx_2 = fx_3 = 2$. Clearly $\langle Q, v \rangle$ is a topological space and the above closure u is inductively generated by the mapping $f : \langle Q, v \rangle \to P$.

Now we shall show that every closure space is inductively generated by a family of mappings whose domain carriers are topological.

33 B.1. Example. Let $\langle P, u \rangle$ be a closure space. For each $a \in P \times \exp P$, $a = \langle x, X \rangle$, let Q_a be the set $X \cup (x)$, v_a the closure operation for Q_a such that $y \in v_a Y - Y$ if and only if $Y \subset Q_a$, $y = x \in uY - Y$ and finally, let f_a be the identity mapping of $\langle Q_a, v_a \rangle$ into $\langle P, u \rangle$. It follows from formula (**) of 33 A.4 that a family $\{f_a \mid a \in A\}$ where $A \subset P \times \exp P$ inductively generates $\langle P, u \rangle$ whenever the set A has the following property:

(*) if $y \in uY - Y$ then there exists a $\langle y, X \rangle \in A$ such that $y \in u(X \cap Y)$.

In particular, the space $\langle P, u \rangle$ is inductively generated by the family $\{f_a \mid a \in \in P \times (P)\}$; in this case each mapping f_a is bijective and $u = \sup \{v_a \mid a \in P \times (P)\}$. Each space $\langle Q_a, v_a \rangle$ has at most one cluster point, namely x if $a = \langle x, X \rangle$, and evidently every space with at most one cluster point is topological. Therefore each space $\langle Q_a, v_a \rangle$ is topological. Thus we have proved that

(a) *Every space is inductively generated by a family of injective mappings whose domain carriers are topological spaces (each with at most one cluster point).*

If the space $\langle P, u \rangle$ is quasi-discrete, then the set of all $\langle x, X \rangle$ such that X is a one-point set and $x \in uX - X$ possesses the property (*), and therefore

(b) *Every quasi-discrete space is inductively generated by a family of one-to-one mappings whose domain carriers are two-point feebly semi-separated spaces (and hence topological).*

If the space $\langle P, u \rangle$ is semi-separated, then each space $\langle Q_a, v_a \rangle$ is semi-separated (because v_a is finer than a relativization of a semi-separated closure, namely u). Evidently every semi-separated closure space possessing at most one cluster point is paracompact (29 ex. 7). It follows that

(c) *Every semi-separated closure space is inductively generated by a family of bijective mappings the domain carriers of which are paracompact separated spaces.*

By an appropriate modification of the construction of 33 B.1 we shall prove the following interesting result.

33 B.2. Theorem. *Every closure space is inductively generated by a family of mappings the domain carriers of which are paracompact separated spaces, each possessing at most one cluster point.*

Proof. Let $\langle P, u\rangle$ be a closure space. For each $a \in (P \times \exp P)$, $a = \langle x, X\rangle$, let R_a be the set $X \cup (x)$, and v_a be the closure for R_a such that $y \in v_a Y - Y$ if and only if $Y \subset R_a$, $y = x$ and $x \in u(Y - F)$ for each finite subset F of Y, and finally, let f_a be the identity mapping of $\langle R_a, v_a\rangle$ into $\langle P, u\rangle$. It is evident that each $\langle R_a, v_a\rangle$ is a semi-separated space with at most one cluster point, and hence each $\langle R_a, v_a\rangle$ is a paracompact separated space. Let v be the closure inductively generated by the family $\{f_a \mid a \in (P \times \exp P)\}$. Clearly v is finer than u and, in addition, $x \in vX - X$ if and only if $x \in u(X - F) - X$ for each finite subset F of X. As a consequence, if u is semi-separated then $u = v$ and we obtain a new proof of statement (c) of 33 B.1, and if $\langle P, u\rangle$ is not semi-separated then $v \neq u$ because $x \in u(y)$ for some $y \in$ $\in P - (x)$ but $x \notin \emptyset = u((y) - (y)) \supset v(y) - (y)$. Thus, if $\langle P, u\rangle$ is not semi-separated, then we must add some further mappings. Let B be the set of all $(\langle x, y\rangle)$, $\langle x, y\rangle \in P \times P$, such that $x \neq y$ and $x \in u(y)$. Notice that $B \cap (P \times \exp P) = \emptyset$ because the elements of B are one-point sets but the elements of $P \times \exp P$ are pairs, and hence not classes. For each $a \in B$ let $\langle R_a, v_a\rangle$ be any semi-separated space with exactly one cluster point, say r_a (e.g. we can take the subspace of reals consisting of all n^{-1}, $n = 1, 2, \ldots$, and the point 0 as r_a), and let us consider the mapping f_a of $\langle R_a, v_a\rangle$ into $\langle P, u\rangle$ which carries r_a into x and $R_a - (r_a)$ into y, where $a = (\langle x, y\rangle)$. Now it follows from 33 A.4 (**) that the closure space $\langle P, u\rangle$ is inductively generated by the family $\{f_a \mid a \in B \cup (P \times \exp P)\}$.

Remark. Notice that in 33 B.1 (c) the space $\langle P, u\rangle$ is assumed to be semi-separated and the mappings f_a are bijective, and hence injective, but in 33 B.2 the space $\langle P, u\rangle$ is not assumed to be semi-separated so that the mappings f_a need not be injective; in fact, if $\langle P, u\rangle$ is not semi-separated and $\langle P, u\rangle$ is inductively generated by a family of mappings f_a whose domain carriers are semi-separated, then at least one f_a is not injective.

Topological spaces are of principal importance and a closure inductively generated by a family of mappings of topological spaces need not be a topological closure. Therefore we shall study the finest topological closure making continuous all mappings of a given inductive family of mappings. For convenience we shall consider any class of spaces K and we shall try to carry over the theory presented in 33 A. In conclusion direct proofs will be given for the case where K is the class of all topological spaces. It is to be noted that theorems for general projective-stable K may be applied to various classes of closure spaces.

33 B.3. Definition. Let K be a class of closure spaces. A *closure u for a set P* is said to be *K-inductively generated by a family of mappings* $\{f_a\}$ if $\{f_a\}$ is an inductive family of mappings for closure spaces with common range carrier P or $\langle P, u\rangle$, and u is the finest closure such that $\langle P, u\rangle \in K$ and all mappings $f_a : \mathbf{D}^* f_a \to \langle P, u\rangle$ are continuous; *the family* $\{f_a\}$ is said to be a *K-inductive generating family for* $\langle P, u\rangle$. These definitions are applied to a collection of mappings $\mathscr{F}$ as to the family $\{f \mid f \in \mathscr{F}\}$, and to a single mapping f as to the collection (f). If K is the class of all topological spaces then we shall say "*topologically inductively generated*" and "*topological inductive generating family*" instead of "K-inductively generated" and "K-inductive generating family", respectively.

From the definition we obtain immediately

33 B.4. Theorem. *Let K be a class of spaces and let L be the class consisting of the closure structures of spaces of K. Let $\{f_a\}$ be an inductive family of mappings for closure spaces with the common range carrier $\langle P, u\rangle$. Then u is the closure K-inductively generated by the family $\{f_a\}$ if and only if u is the upper modification in L of the closure v inductively generated by the family $\{f_a : \mathbf{D}^* f_a \to P\}$.*

33 B.5. Corollary. *If the closure K-inductively generated by a family $\{f_a\}$ exists for each inductive family $\{f_a\}$ for closure spaces, then every closure u has its upper modification in L, and consequently L is order-complete and completely meet-stable in $\mathbf{C}$. Conversely, if every closure has an upper modification in L, then the closure K-inductively generated by an inductive family $\{f_a\}$ for closure spaces exists for each $\{f_a\}$.*

33 B.6. *Suppose that K is a class of spaces and L is the class consisting of the closure structures of spaces of K. Let u be a closure for a set P and v a closure from $L \cap \mathbf{C}(P)$ such that the following condition is fulfilled:*

(*) *A mapping $f : \langle P, u\rangle \to \mathscr{Q}$, $\mathscr{Q} \in K$, is continuous if and only if the mapping $g = f : \langle P, v\rangle \to \mathscr{Q}$ is continuous.*

Then v is the upper modification of u in L. Conversely, if $K = \operatorname{proj} K$, i.e. if K is projective-stable, then the upper modification v of u in L is the unique closure satisfying condition (*).

Proof. I. Suppose that a closure $v \in L$ fulfils (*). Since $\mathrm{J}: \langle P, v\rangle \to \langle P, v\rangle$ is continuous and $\langle P, v\rangle \in K$, by condition (*) the mapping $\mathrm{J} : \langle P, u\rangle \to \langle P, v\rangle$ is continuous, i.e. v is coarser than u. If $w \in L$ is a closure coarser than u, then the mapping $\mathrm{J} : \langle P, u\rangle \to \langle P, w\rangle$ is continuous, and hence by condition (*), the mapping $\mathrm{J} : \langle P, v\rangle \to \langle P, w\rangle$ is continuous, i.e. w is coarser then v. Thus v is actually the upper modification of u in L.

II. Conversely, suppose that K is projective-stable and $u \in \mathbf{C}(P)$. By 32 B.9 there exists the upper modification v of u in L. We shall prove that v fulfils (*). If g is continuous, then f is continuous because v is coarser than u. Conversely, if f is continuous and if w is the closure projectively generated by the mapping $f : P \to \mathcal{Q}$, then $w \in L$ because $\mathcal{Q} \in K$ and K is projective-stable, and clearly w is coarser than u. But v is the finest closure of L coarser than u and hence v is finer than w. Since $f : \langle P, w \rangle \to \mathcal{Q}$ is continuous, $g = f : \langle P, v \rangle \to \mathcal{Q}$ is also continuous.

Remark. Notice that 33 B.6 is a generalization of Theorem 16 B.4 on topological modification, and of Theorem 24 B.15 on uniformizable modification.

33 B.7. *Let K be a projective-stable class of closure spaces and let L be the class of closure structures of spaces of K. In order that an inductive family of mappings $\{f_a\}$ with the common range carrier $\langle P, u \rangle$ be a K-inductive generating family it is necessary and sufficient that*

a mapping f of $\langle P, u \rangle$ into a space $\mathcal{Q}$ from K is continuous if and only if all the mappings $f \circ f_a$ are continuous.

Proof. I. Let us consider the closure v for P inductively generated by the family $\{f_a : \mathbf{D}^* f_a \to P\}$, and let u be K-inductively generated by $\{f_a\}$. By 33 B.4 the closure u is the upper modification of v in L and hence, by 33 B.6, if $\mathcal{Q} \in K$, then a mapping f of $\langle P, u \rangle$ into $\mathcal{Q}$ is continuous if and only if the mapping $f : \langle P, v \rangle \to \mathcal{Q}$ is continuous; but v is inductively generated by $\{f_a : \mathbf{D}^* f_a \to P\}$ and consequently $f : \langle P, v \rangle \to \mathcal{Q}$ is continuous if and only if all the mappings $(f : \langle P, v \rangle \to \mathcal{Q}) \circ (f_a : \mathbf{D}^* f_a \to \langle P, v \rangle)$ are continuous. Since $(f : \langle P, v \rangle \to \mathcal{Q}) \circ (f_a : \mathbf{D}^* f_a \to \langle P, v \rangle) = f \circ f_a$, we find that f is continuous if and only if all the mappings $f \circ f_a$ are continuous. – II. We have proved that the condition is necessary. But evidently at most one $u \in L$ fulfils the condition and hence the condition is sufficient.

Remark. Notice that Theorem 33 A.5 is obtained for $K = \mathbf{C}$.

Now we shall prove that 33 B.7 implies the theorem on associativity. It is to be noted that, for the case $\mathbf{C} = K$ we obtain Theorem 33 A.6 which was proved independently of theorem 33 A.5.

33 B.8. Theorem on associativity. *Suppose that K is a projective-stable class of closure spaces and $\{f_a \mid a \in A\}$ is an inductive family of mappings for closure spaces with the common range carrier $\langle P, u \rangle$. For each a in A let $\mathbf{D}^* f_a$ be K-inductively generated by a family $\{g_{ab} \mid b \in B_a\}$. Then $\{f_a\}$ is a K-inductive generating family if and only if the family $\{f_a \circ g_{ab} \mid a \in A,\ b \in B_a\}$ is a K-inductive generating family.*

Proof. Notice that both inductive families have the same common range carrier, namely $\langle P, u \rangle$. By 33 B.7 the statements that $\{f_a\}$ or $\{f_a \circ g_{ab}\}$ are K-inductive generating families are equivalent to the statements that, if $\mathcal{Q} \in K$ and f is a mapping of $\langle P, u \rangle$ into $\mathcal{Q}$, then f is continuous if and only if all $f \circ f_a$ are continuous or all $f \circ (f_a \circ g_{ab})$ are continuous, respectively. But each $\{g_{ab} \mid b \in B_a\}$ is a K-inductive generating family and therefore, again by 33 B.7, all $(f \circ f_a) \circ g_{ab}$ are continuous if and only if all the $f \circ f_a$ are continuous; since $(f \circ f_a) \circ g_{ab} = f \circ (f_a \circ g_{ab})$, the proof is complete.

33 B.9. *K*-inductive factorization theorem. *Let K be a projective-stable class of closure spaces. If $\{f_a\}$ is an inductive family of mappings for closure spaces with the common range carrier $\langle P, u\rangle$ belonging to K, then there exists a K-inductive generating family $\{g_a\}$ and an identity mapping h such that $f_a = h \circ g_a$ for each a. All the mappings f_a are continuous if and only if the mapping h is continuous.*

Proof. Let v be the closure K-inductively generated by the family $\{f_a : \mathbf{D}^*f_a \to P\}$, $h = \mathsf{J} : \langle P, v\rangle \to \langle P, u\rangle$, and $g_a = f_a : \mathbf{D}^*f_a \to \langle P, v\rangle$.

Remark. Theorem 33 A.7 on partial commutativity for inductive generation for closure spaces is not true for K-inductive generation. This will be shown for the case $K = \tau\mathbf{C}$ in 33 B.15.

33 B.10. Theorem. *Let K be a projective-stable class of closure spaces and let κ be the single-valued relation on $\mathbf{C}$ which assigns to each closure space $\mathscr{P}$ its upper modification in K, i.e., $\kappa\langle P, u\rangle = \langle P, \kappa u\rangle$ where κu is the upper modification of u in the class of closure structures of spaces of K. Let $\{f_a \mid a \in A\}$ be a non-void inductive family of mappings for closure spaces with the common range carrier $\langle P, u\rangle$. Each of the following two conditions is necessary and sufficient for $\{f_a\}$ to be a K-inductive generating family:*

(a) *the reduced sum f of $\{f_a\}$ is a K-inductive generating family;*

(b) *the mapping $g = f : \kappa\mathbf{D}^*f \to \langle P, u\rangle$ is a K-inductive generating mapping.*

Proof. It is almost self-evident that the two conditions are equivalent. Let i_a stand for the mapping $\mathrm{inj}_a : \mathbf{D}^*f_a \to \kappa\mathbf{D}^*f$. Since $f_a = g \circ i_a$ for each a, to prove that (b) is necessary and sufficient, it is enough to show (by 33 B.8) that $\{i_a\}$ is a K-inductive generating family; this follows from 33 B.4 and the fact that $\{\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f\}$ is an inductive generating family.

Remark. It is natural to define the *K-sum* of a family $\{\mathscr{P}_a\}$ of closure spaces as the upper modification of the sum space $\Sigma\{\mathscr{P}_a\}$ in K. It is to be noted that the K-sum of $\{\mathscr{P}_a\}$ may actually be distinct from $\Sigma\{\mathscr{P}_a\}$, e.g. for K take the class of all accrete spaces.

33 B.11. Theorem. *Every non-void inductive family topologically inductively generates exactly one closure operation. If a closure u for a set P is topologically inductively generated by a family of mappings $\{f_a\}$ and for each a the mapping f_a topologically inductively generates u_a, then u is the least upper bound of $\{u_a\}$ in the ordered set $\tau\mathbf{C}(P)$ of all topological closures for P. If a closure u is topologically inductively generated by a mapping f then a set $X \subset \mathbf{E}^*f$ is open (closed) if and only if $f^{-1}[X]$ is open (closed) in $\mathbf{D}^*f$. If a closure u for a set P is topologically inductively generated by a family $\{f_a\}$ then $X \subset P$ is open (closed) in $\langle P, u\rangle$ if and only if $f_a^{-1}[X]$ is open (closed) in $\mathbf{D}^*f_a$ for each a.*

Proof. I. Uniqueness is clear and existence can be proved in the same way as in 33 A.4.

II. The formula $u = \sup\{u_a\}$ (in $\tau\mathbf{C}(P)$) can be proved in the same way as in 33 A.4.

III. Let f be a mapping of a space $\mathscr{Q}$ into a set P. Consider the collection $\mathscr{U}$ of all $U \subset P$ such that $f^{-1}[U]$ is open in $\mathscr{Q}$. It is easy to verify that $\mathscr{U}$ is the collection of all open sets for some topological closure u for P (use 15 A.6). Now, as in the proof of 33 A.4, one can show that u is topologically inductively generated by f. The statement concerning closed sets follows from the one concerning open sets.

IV. The description of open and closed sets of a space topologically inductively generated by a family $\{f_a\}$ is an immediate consequence of II, III and the description 31 B.6 of open sets relative to the least upper bound in $\tau\mathbf{C}(P)$ of a family of topological closures.

33 B.12. Theorem. *Let $\mathscr{P}$ be a topological space and for each $a \in A$, let f_a be a mapping of a space into $\mathscr{P}$. Then $\mathscr{P}$ is topologically inductively generated by the family $\{f_a\}$ if and only if the following condition is fulfilled:*

If f is a mapping of $\mathscr{P}$ into a topological space $\mathscr{R}$, then f is continuous if and only if $f \circ f_a$ is continuous for each a in A.

Proof. I. First let $\{f_a\}$ be a topological inductive generating family. If f is continuous, then all $f \circ f_a$ are continuous as compositions of continuous mappings. Conversely suppose that all compositions $f \circ f_a$ are continuous. Since $\mathscr{R}$ is topological, to prove f is continuous it is enough to show that $Y = f^{-1}[X]$ is open in P for each open subset X of $\mathscr{R}$. Since $\mathscr{P}$ is topologically inductively generated by $\{f_a\}$, to prove Y is open in $\mathscr{P}$ it is enough to show that $f_a^{-1}[Y]$ is open in $\mathbf{D}^* f_a$ for each a. Since $f \circ f_a$ is continuous, the set $(f \circ f_a)^{-1}[X] = f_a^{-1}[f^{-1}[X]]$ must be open for each a. The continuity of f follows. – II. The proof of sufficiency of the condition follows the proof of 33 A.5.

33 B.13. Theorem. *Let us suppose that $\mathscr{P}$ is a topological space and $\{f_a \mid a \in A\}$ is a family, each f_a being a mapping of a topological space $\mathscr{Q}_a$ into $\mathscr{P}$. For each a in A let $\mathscr{Q}_a$ be topologically inductively generated by a family of mappings $\{g_{ab} \mid b \in B_a\}$. Then $\mathscr{P}$ is topologically inductively generated by $\{f_a\}$ if and only if it is topologically inductively generated by the family $\{f_a \circ g_{ab} \mid a \in A, b \in B_a\}$.*

Proof. The proof proceeds as that of 33 A.6; instead of the description (**) of inductively generated closures we must use the description of open sets from 33 B.11.

33 B.14. Factorization theorem for topological inductive generating families. *Suppose that $\{f_a \mid a \in A\}$ is a family, each f_a being a continuous mapping of a space $\mathscr{Q}_a$ into a space $\mathscr{P}$. There exists a topological inductive generating family $\{h_a \mid a \in A\}$ and a one-to-one continuous mapping g such that $f_a = g \circ h_a$ for each a in A.*

By 33 A.7 the operation of taking the inductively generated closure commutes, in a certain sense, with the operation of taking a subspace. The analogue for topologically inductively generated closures is not true as we shall now show.

33 B.15. If a space $\mathscr{P}$ is topologically inductively generated by a mapping f (from a topological space, or even a normal space) and if $\mathscr{R}$ is a subspace of $\mathscr{P}$, then $\mathscr{R}$ need not be topologically inductively generated by the mapping g where g is a mapping of the subspace $f^{-1}[|\mathscr{R}|]$ of $\mathbf{D}^*f$ into $\mathscr{R}$ which coincides with f on $\mathbf{D}^*g$. Moreover, if a mapping $\langle f, \langle Q, v\rangle, P\rangle$ inductively generates a closure u which is not topological, then there exists a subset R of P such that the restriction g of f to a mapping of the subspace $f^{-1}[R]$ of $\langle Q, v\rangle$ into R topologically inductively generates a closure which is not the relativization to R of u. In fact, if $\langle P, u\rangle$ is not topological, then (by 17 A.7) there exists a subset R of P such that the topological modification τw of the relativization w of u to R is not a relativization of τu to R. On the other hand the closure w is inductively generated by g and τw and τu are topologically inductively generated by g and f, respectively.

C. QUOTIENT SPACES

An embedding is a projective generating injective mapping. In a certain sense, which will not be made precise, the "dual" concept is an inductive generating surjective mapping. A space $\mathscr{Q}$ is a homeomorph of a subspace of a given space $\mathscr{P}$ if and only if $\mathscr{Q}$ is the domain carrier of a projective generating injective mapping f whose range carrier is $\mathscr{P}$. The dual concept of "a homeomorph of a subspace of $\mathscr{P}$" is "a quotient of space $\mathscr{P}$"; $\mathscr{Q}$ is a quotient of $\mathscr{P}$ if $\mathscr{Q}$ is the range carrier of an inductive generating surjective mapping with domain carrier $\mathscr{P}$.

The first part of the subsection concerns quotients and related concepts. The second part is devoted to an examination of the inductive progeny of a given class (and corresponds to 32 B).

33 C.1. Definition. If f is a mapping from a closure space $\mathscr{P}$ (i.e. $\mathscr{P} = \mathbf{D}^*f$), then the *quotient of* $\mathscr{P}$ *under* f (the *topological quotient of* $\mathscr{P}$ *under* f, respectively), denoted by $\mathscr{P}/f$ ($\mathscr{P}/_T f$, respectively), is defined to be the set $\mathbf{E}f$ endowed with the closure inductively (topologically inductively, respectively) generated by the mapping $f: \mathscr{P} \to \mathbf{E}f$. A *quotient* (*topological quotient*) *mapping* is a mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$ such that the space $\mathscr{P}/f$ ($\mathscr{P}/_T f$, respectively) is a subspace of $\mathscr{Q}$.

Thus a surjective mapping is a quotient or a topological quotient mapping if and only if it is, respectively, an inductive or topological inductive generating mapping. Evidently each inductive generating mapping is a quotient mapping and each topological inductive generating mapping is a topological quotient mapping. The converse is not true. Indeed, if f is an inductive generating mapping then $|\mathbf{E}^*f| - \mathbf{E}f$ is an open discrete subset of $\mathbf{E}^*f$ (by Corollary (c) of 33 A.4) but this need not be true if f is only a quotient mapping. Moreover, if f is a quotient (topological quotient) mapping of $\mathscr{P}$ into $\mathscr{Q}$ and if $\mathscr{Q}_1$ is any space such that $\mathbf{E}f \subset \mathscr{Q}_1$ and the subspaces $f[|\mathscr{P}|]$ of $\mathscr{Q}$

and $f[|\mathscr{P}|]$ of $\mathscr{Q}_1$ are canonically homeomorphic, then the mapping $f : \mathscr{P} \to \mathscr{Q}_1$ is also a quotient (topological quotient) mapping.

It has already been noted that an embedding is a projective generating mapping but not an inductive generating mapping. It is clear that an embedding is always a quotient mapping and if the domain carrier is topological, then it is also a topological quotient mapping.

For convenience we review some propositions about quotient sets under equivalences. Let ϱ be an equivalence relation on a $P \neq \emptyset$, that is, ϱ is a symmetric reflexive and transitive relation for P such that $\varrho[P] = P$. The quotient of P under ϱ, denoted by P/ϱ, is the set of all equivalence classes, i.e. the sets of the form $\varrho[x]$, $x \in P$. Thus P/ϱ is a disjoint non-void cover of P and each of its elements is also non-void; stated in other words, P/ϱ is a decomposition of P. The mapping $\{x \to \varrho[x]\}$ of P onto P/ϱ will be called the canonical mapping of P onto P/ϱ and will usually be denoted by π. If $\mathscr{D}$ is any decomposition of P then there exists exactly one equivalence ϱ on P such that $\mathscr{D} = P/\varrho$; it is $\varrho = \bigcup\{D \times D \mid D \in \mathscr{D}\}$. Now we are prepared to introduce the concept of the quotient of a space under an equivalence or a decomposition.

33 C.2. Definition. Suppose that $\langle P, u\rangle$ is a closure space, ϱ is an equivalence relation on P and π is the canonical mapping of P onto P/ϱ. The *quotient* of $\langle P, u\rangle$ *under* ϱ (the topological quotient of $\langle P, u\rangle$ under ϱ), denoted by $\langle P, u\rangle/\varrho$ ($\langle P, u\rangle/_T\varrho$, respectively) is the set P/ϱ endowed with the closure inductively (topologically inductively) generated by the mapping $\pi : \langle P, u\rangle \to P/\varrho$; stated in other words,

$$\langle P, u\rangle/\varrho = \langle P, u\rangle/\pi$$
$$\langle P, u\rangle/_T\varrho = \langle P, u\rangle/_T\pi \,.$$

It follows from the definition that the quotient of a space P under an equivalence ϱ can be always considered as the quotient of P under the canonical mapping π of P onto P/ϱ. Conversely, it is easy to verify that every quotient P/f can be obtained by a canonical homeomorphism from $P/\{fx = fy\}$ onto P/f. More precisely,

33 C.3. Theorem. *Let f be a quotient mapping of a closure space $\mathscr{P}$ onto another one $\mathscr{Q}$ (thus $\mathscr{Q} = \mathscr{P}/f$) and let π be the canonical mapping of $\mathscr{P}$ onto the quotient space $\mathscr{P}/\{fx = fy\}$ (thus $\mathscr{P}/\{fx = fy\} = \mathscr{P}/\pi$). There exists a homeomorphism $\tilde{f}$ such that $f = \tilde{f} \circ \pi$.*

Proof. Clearly there is exactly one mapping $\tilde{f}$ such that $f = \tilde{f} \circ \pi$, that this mapping is one-to-one and $\pi = \tilde{f}^{-1} \circ f$. Both mappings f and π being quotient mappings onto, they are inductive generating mappings. In consequence, the mappings $\tilde{f} \circ \pi\,(= f)$ and $\tilde{f}^{-1} \circ f\,(= \pi)$ being continuous, the mappings $\tilde{f}$ and $\tilde{f}^{-1}$ are continuous by 33 A.5.

33 C.4. Theorem. *Let f be a topological quotient mapping of a closure space $\mathscr{P}$ onto another one $\mathscr{Q}$ (thus $\mathscr{Q} = \mathscr{P}/_T f$, in particular, $\mathscr{Q}$ is topological) and let π be the canonical mapping of $\mathscr{P}$ onto the topological quotient space $\mathscr{P}/_T\{fx = fy\}$*

(thus $\mathscr{P}/_T\{fx = fy\} = \mathscr{P}/_T \pi$). Then there exists a homeomorphism $\tilde{f}$ such that $f = \tilde{f} \circ \pi$.

The proof proceeds as that of 33 C.3; instead of 33 A.5 we apply 33 B.12.

Remark. Let X be a subset of a space P and let $\varrho = (X \times X) \cup \bigcup\{(\langle x, x\rangle) \mid x \in \in P - X\}$. The spaces P/ϱ and $P/_T \varrho$ are often said to be obtained by identifying or topologically identifying the points of the set X.

Often we shall need earlier results adapted for (topological) quotients and (topological) quotient mappings. For easier references we summarize these in two propositions which follow.

33 C.5. A) *Let $\mathscr{P}$ be a closure space, ϱ be an equivalence on $\mathscr{P}$ and π be the canonical mapping of $\mathscr{P}$ onto the quotient space $\mathscr{P}/\varrho$. Then*

(a) *π is an inductive generating mapping for $\mathscr{P}/\varrho$.*

(b) *A mapping f of $\mathscr{P}/\varrho$ into a space is continuous if and only if the composition $f \circ \pi$ is continuous.*

(c) *A mapping f of $\mathscr{P}/\varrho$ into a space is a quotient mapping if and only if $f \circ \pi$ is a quotient mapping.*

(d) *If $\mathscr{Q}$ is a subspace of $\mathscr{P}$ such that $|\mathscr{Q}| = \pi^{-1}[Y]$ for some Y and $\sigma = \varrho \cap \cap (|\mathscr{Q}| \times |\mathscr{Q}|)$, then $\mathscr{Q}/\sigma$ is a subspace of $\mathscr{P}/\varrho$.*

(e) *If σ is an equivalence on $\mathscr{P}$ such that $\varrho \subset \sigma$, then the canonical mapping of $\mathscr{P}/\varrho$ onto $\mathscr{P}/\sigma$ is a quotient mapping, that is $\mathscr{P}/\sigma = (\mathscr{P}/\varrho)/f$.*

(f) *If $X = \pi^{-1}[Y]$, then $\pi[\overline{X}] = \overline{Y}$.*

(g) *$Y \subset |\mathscr{P}/\varrho|$ is open (closed) in $\mathscr{P}/\varrho$ if and only if the set $\pi^{-1}[Y]$ is open (closed) in $\mathscr{P}$.*

B) *Let f, g and h be mappings such that $h = g \circ f$ and f is an inductive generating mapping. If one of the mappings g and h is a quotient mapping or an inductive generating mapping, then the other also has the corresponding property.*

C) *The composition of two quotient mappings need not be a quotient mapping.*

33 C.6. A) *Let $\mathscr{P}$ be a closure space, ϱ be an equivalence on $\mathscr{P}$ and π be the canonical mapping of $\mathscr{P}$ onto the topological quotient $\mathscr{P}/_T\varrho$. Then*

(a) *π is a topological inductive generating mapping for $\mathscr{P}$.*

(b) *A mapping f of $\mathscr{P}/_T\varrho$ into a topological space is continuous if and only if the composition $f \circ \pi$ is continuous.*

(c) *A mapping f of $\mathscr{P}/_T\varrho$ into a space $\mathscr{R}$ is a topological quotient mapping if and only if $f \circ \pi$ is a topological quotient mapping.*

(d) *Partial commutativity with formation of subspaces does not hold (see* 33 B.15*).*

(e) *If σ is an equivalence on $\mathscr{P}$ such that $\varrho \subset \sigma$, then the canonical mapping f of $\mathscr{P}/_T\varrho$ onto $\mathscr{P}/_T\sigma$ is a topological quotient mapping, that is, $\mathscr{P}/_T\sigma = (\mathscr{P}/_T\varrho)/_Tf$.*

(f) $\mathscr{P}/_T\varrho = \tau(\mathscr{P}/\varrho)$.

(g) $Y \subset |\mathscr{P}/_T\varrho|$ *is open (closed) if and only if* $\pi^{-1}[Y]$ *is open (closed) in* $\mathscr{P}$.

B) *Let f, g and h be mappings such that $h = g \circ f$ and f is a topological inductive generating mapping. If one of the mappings g and h is a topological quotient mapping or a topological inductive generating mapping, then the other mapping also has the corresponding property.*

C) *The composition of two topological quotient mappings need not be a topological quotient mapping.*

The concluding part is devoted to various examples. We begin with the spaces of components and quasi-components of a space; for earlier results needed, see 20 B and 21 B.

33 C.7. If $\mathscr{P}$ is a closure space and $\mathscr{C}$ is the collection of all components of $\mathscr{P}$, then the union σ of $\{C \times C \mid C \in \mathscr{C}\}$ is an equivalence on $\mathscr{P}$ (20 B.4). The quotient spaces $\mathscr{P}/\sigma$ and $\mathscr{P}/_T\sigma$ are called the component space of $\mathscr{P}$ and the topological component space of $\mathscr{P}$. Similarly the union σ_1 of all $C \times C$, C being a quasi-component of $\mathscr{P}$, is an equivalence (20 B.9). The quotient spaces $\mathscr{P}/\sigma_1$ and $\mathscr{P}/_T\sigma_1$ will be called the quasi-component space and the topological quasi-component space of $\mathscr{P}$. As always, $\mathscr{P}/_T\sigma = \tau(\mathscr{P}/\sigma)$ and $\mathscr{P}/_T\sigma_1 = \tau(\mathscr{P}/\sigma_1)$.

(a) Since $\sigma \subset \sigma_1$ (20 B.9), there exists a mapping f of $\mathscr{P}/\sigma$ onto $\mathscr{P}/\sigma_1$ and f_T of $\mathscr{P}/_T\sigma$ onto $\mathscr{P}/_T\sigma_1$ such that $\pi_1 = f \circ \pi$ and $\pi_{1T} = f_T \circ \pi_T$, where π, π_T, π_1, π_{1T} are canonical mappings of $\mathscr{P}$ onto $\mathscr{P}/\sigma$, $\mathscr{P}/_T\sigma$, $\mathscr{P}/\sigma_1$, $\mathscr{P}/_T\sigma_1$ respectively. Since π_1 and π are quotient mappings, the mapping f is a quotient mapping by 33 C.5. Since π_{1T} and π_T are topological quotient mappings, the mapping f_T is a topological quotient mapping (33 C.6); stated in other words, $\mathscr{P}/\sigma_1 = (\mathscr{P}/\sigma)/f$ and $\mathscr{P}/_T\sigma_1 = (\mathscr{P}/_T\sigma)/_T f_T$.

(b) A space $\mathscr{P}$ is feebly locally connected if and only if one, and then all, of the spaces $\mathscr{P}/\sigma$, $\mathscr{P}/\sigma_1$, $\mathscr{P}/_T\sigma$, $\mathscr{P}/_T\sigma_1$, are discrete.

(c) The associated mappings $\{X \to \pi[X]\}$ and $\{X \to \pi_1[X]\}$ carry quasi-components into quasi-components. In particular, quasi-components of $\mathscr{P}/\sigma_1$, and hence of $\mathscr{P}/_T\sigma_1$, are one-point sets if $|\mathscr{P}| \neq \emptyset$.

(d) The components of $\mathscr{P}/\sigma$ are one-point sets if $|\mathscr{P}| \neq \emptyset$: if $C \subset |\mathscr{P}/\sigma|$ is connected and $\mathscr{Q}$ is the subspace $\pi^{-1}[C]$ of $\mathscr{P}$, then the quotient $\mathscr{Q}/(\pi \mid \mathscr{Q})$ is a subspace of $\mathscr{P}/\sigma$. But $\mathscr{Q}/(\pi \mid \mathscr{Q})$ is the component space of $\mathscr{Q}$. Hence, to prove that C is a one-point set it is enough to notice that the component space of a space is connected if and only if the space is connected.

It should be remarked that the range carrier of an inductive generating mapping inherits very few of the properties of the domain carrier. There are two very significant special cases of inductive generating mappings which preserve more properties; they will be considered in the next section. Here we shall investigate properties of classes of spaces invariant under inductive constructions (compare with 32 B).

33 C.8. Definition. If K is a class of spaces and L is the class of closure structures of spaces of K, then $\operatorname{ind} K$ denotes the class of all spaces inductively generated by families of mappings with domain carriers in K, and $\operatorname{ind} L$ denotes the class of closure structures of spaces of $\operatorname{ind} K$. A class K is said to be *inductive-stable* if $\operatorname{ind} K = K$. The classes $\operatorname{ind} K$ and $\operatorname{ind} L$ are called the *inductive progeny of* K and L, respectively.

33 C.9. *If K is any class of spaces then the class* $\operatorname{ind} K$ *contains all discrete spaces. Next,* $K \subset \operatorname{ind} K$ *and* $\operatorname{ind}\operatorname{ind} K = \operatorname{ind} K$, *i.e.* $\operatorname{ind} K$ *is inductive-stable and contains* K. *Finally, if* $K \subset K'$ *then* $\operatorname{ind} K \subset \operatorname{ind} K'$.

Proof. The empty family inductively generates every discrete space and hence $\operatorname{ind} K$ contains all discrete spaces. A homeomorphism is an inductive generating mapping and therefore $K \subset \operatorname{ind} K$. The formula $\operatorname{ind}\operatorname{ind} K = \operatorname{ind} K$ follows from the associativity theorem 33 A.6. The last statement is evident.

33 C.10. *Let K be a class of spaces. A space $\mathscr{P} = \langle P, u\rangle$ belongs to the class* $\operatorname{ind} K$ *if and only if $\mathscr{P}$ is the quotient space of a space $\mathscr{R}$ under a mapping f, where $\mathscr{R}$ is the sum of a family $\{\mathscr{R}_a \mid a \in A\}$ such that all $\mathscr{R}_a$, excepting at most one, say $\mathscr{R}_\alpha$, belong to K and $\mathscr{R}_\alpha = \langle P, v\rangle$ where v is the discrete closure for P.*

Proof. Every such space belongs to $\operatorname{ind}\operatorname{ind} K = \operatorname{ind} K$. Conversely, suppose $\langle P, u\rangle \in \operatorname{ind} K$ and take an inductive generating family $\{f_b \mid b \in B\}$ for $\langle P, u\rangle$ such that $\mathbf{D}^* f_b \in K$ for each b in B. Choose an element α, $\alpha \notin B$, put $A = B \cup (\alpha)$, and let f_α be the identity mapping of $\langle P, v\rangle$ onto $\langle P, u\rangle$, where v is the discrete closure for P. It is evident that $\{f_a \mid a \in A\}$ is an inductive generating family for $\langle P, u\rangle$ and the reduced sum f of $\{f_a \mid a \in A\}$, i.e. $f : \{\langle a, x\rangle \to f_a x\} : \Sigma\{\mathbf{D}^* f_a\} \to \langle P, u\rangle$, is a surjective inductive generating mapping (33 A.8). Thus $\langle P, u\rangle = \Sigma\{\mathbf{D}^* f_a\}/f$.

33 C.11. *Let K be a class of closure spaces and let L be the class of closure structures of spaces of K. In order that K be inductive-stable (i.e.* $\operatorname{ind} K = K$*) it is necessary and sufficient that*

(a) *if f is an inductive generating mapping and $\mathbf{D}^* f \in K$, then $\mathbf{E}^* f \in K$; and*

(b) *every closure has its lower modification in L.*

Remark. Recall that, by lemma 31 B.2, condition (b) is equivalent to the following statement: L contains all discrete closures and L is completely join-stable in $\mathbf{C}$. In the proof we shall use this equivalence.

Proof. If $\operatorname{ind} K = K$, then clearly (a) is fulfilled; also, L contains all discrete closures (33 C.9) and L is completely join-stable in $\mathbf{C}$ because $\sup\{u_a\}$ in $\mathbf{C}$, where $\{u_a\}$ is a family in some $\mathbf{C}(P)$, is inductively generated by the family $\{\mathsf{J} : \langle P, u_a\rangle \to P\}$, and hence, by the foregoing remark, (b) is fulfilled. Conversely, assuming (a) and (b) let us consider a space $\langle P, u\rangle$ inductively generated by a family $\{f_a\}$ with domain carriers in K. If $\{f_a\}$ is empty, then u is discrete and hence, by (b), belongs to L. If $\{f_a\}$ is non-void, then consider the family $\{u_a\}$ where each u_a is inductively generated by the mapping $f_a : \mathbf{D}^* f_a \to P$. By (a) $u_a \in L$ for each a and hence $u = \sup\{u_a\}$ (by 33 A.4) belongs to L (by (b)). Thus $\langle P, u\rangle \in K$.

33 C.12. Examples. (a) It follows from 31 B.5 that neither the class of all topological spaces nor the class of all uniformizable spaces is inductive-stable.

(b) The class of all discrete spaces is inductive-stable.

(c) The class of all quasi-discrete spaces is inductive-stable (show that each quotient of a quasi-discrete space is quasi-discrete and the sum of a family of quasi-discrete spaces is a quasi-discrete space).

(d) An important inductive-stable class will be considered in Section 35 (**S**-spaces, i.e. spaces which can be described by means of the convergence of sequences).

(e) Let L consist of all accrete and all discrete closures. Clearly every closure has a lower as well as an upper modification in L, and quotients and subspaces of spaces from K belong to K. On the other hand, K is neither inductive-stable nor projective-stable, and in fact K is not closed under products or sums.

In conclusion to point out the duality between the concepts considered, we shall state a description of the projective and inductive progeny of a given class of spaces; the proof follows from earlier results.

33 C.13. *Let K be a class of spaces.* A) *Let K_1 be the class of all discrete spaces. The inductive progeny of K consists of quotients of sums of spaces of $K \cup K_1$. In particular, K is inductive-stable if and only if K contains all discrete spaces, K is closed under sums, and quotients of spaces of K belong to K.*

B) *Let K_2 be the class of all accrete spaces. The projective progeny of K consists of all subspaces of products of spaces of $K \cup K_2$. In particular, K is projective-stable if and only if K contains all accrete spaces, K is closed under products (i.e. K is completely productive), and subspaces of spaces of K belong to K (i.e. K is hereditary).*

D. EXAMPLES

We shall introduce the following concepts: the inductive product of a family of closure spaces, a closure space inductively generated by a collection of its subspaces, a strictly inductively generating family. Then we shall define the meaning of commonly employed expressions as e.g. a space obtained by identifying the points of prescribed sets, a space obtained by pasting together pairs of prescribed points, a space obtained by sewing together spaces of a given family of spaces along a given family of homemorphisms or subspaces.

The subsection ends with an exposition of some constructions which can be used to obtain an example of a regular separated topological space which is not uniformizable (33 D.6) and an example of an infinite regular separated topological space such that each continuous function on it is constant (33 D.7) – of course, such a space is not uniformizable.

33 D.1. Inductive products. In 17 D.1 the inductive product of two spaces $\mathscr{P}$ and $\mathscr{Q}$, denoted by ind $(\mathscr{P} \times \mathscr{Q})$ or $\mathscr{P} \times_{\text{ind}} \mathscr{Q}$, was defined by specifying neighborhoods of points (the "crosses"). In terms of this section, Theorem 17 D.3 can be restated as follows: The space ind $(\mathscr{P} \times \mathscr{Q})$ is inductively generated by the family of all canonical embeddings, i.e. the mappings $\{x \to \langle x, y\rangle\} : \mathscr{P} \to |\mathscr{P}| \times |\mathscr{Q}|$, $y \in |\mathscr{Q}|$, and $\{y \to \langle x, y\rangle\} : \mathscr{Q} \to |\mathscr{P}| \times |\mathscr{Q}|$, $x \in |\mathscr{P}|$. Now we shall introduce the concept of the inductive product of a family of closure spaces.

Let $\{P_a \mid a \in A\}$ be a family of sets and let P be the product of $\{P_a\}$. For each x in P and α in A there is a one-to-one map $f_{x,\alpha}$ of the set P_α into P, called the canonical embedding of P_α into P corresponding to x, which assigns to each $z \in P_\alpha$ that point of P whose α-th coordinate is z and the other coordinates coincide with corresponding coordinates of x. Thus $f_{x,\alpha} = f_{y,\alpha}$ whenever $x_a = y_a$ for each $a \neq \alpha$.

(a) Let $\{u_a \mid a \in A\}$ be a family, each u_a being a closure for P_a. There is defined the product closure $\Pi\{u_a\}$ for P, which is, as it has already been shown, projectively generated by the family of all projections $\mathrm{pr}_a : P \to \langle P_a, u_a\rangle$. This closure will be sometimes called the projective product of $\{u_a\}$. Now we shall define the *inductive product* of $\{u_a\}$ and the *topological inductive product* of $\{u_a\}$ to be the closure generated inductively or topologically inductively, respectively, by the family of all canonical embeddings $f_{x,a} : \langle P_a, u_a\rangle \to P$, $x \in P$, $a \in A$. The space $\langle P, u\rangle$, where u is the inductive or the topological inductive product, will be denoted accordingly by ind $\Pi\{\langle P_a, u_a\rangle\}$ or τind $\Pi\{\langle P_a, u_a\rangle\}$. (Thus the letter τ can be taken as the topological modification.) Now, since $\{f_{x,a}\}$ is an inductive generating family for ind $\Pi\{P_a\}$ and $\{f_{x,a}\}$ is also a topological inductive generating family for τind $\Pi\{P_a\}$, we obtain the following result:

(b) If $\{\mathscr{P}_a\}$ is a family of spaces then a mapping g of ind $\Pi\{\mathscr{P}_a\}$ into a space $\mathscr{R}$ is continuous if and only if each composite $g \circ f_{x,a}$ is continuous, and a mapping g of τind $\Pi\{\mathscr{P}_a\}$ into a topological space is continuous if and only if all composites $g \circ f_{x,a}$ are continuous. Roughly speaking, a mapping from ind $\Pi\{\mathscr{P}_a\}$ is continuous if and only if it is continuous in each coordinate separately, and similarly for mappings from τind $\Pi\{\mathscr{P}_a\}$.

(c) Mappings from products which are not continuous (relative to the projective product closures) but which are continuous relative to the inductive product closures frequently occur in mathematics. The fundamental problem is the following: "to express" a given "inductively" continuous mapping in terms of "projectively" continuous mappings. For example, it may be shown that if f is a continuous mapping of the inductive product ind $(\mathscr{P} \times \mathscr{Q})$ of spaces $\mathscr{P}$ and $\mathscr{Q}$ into a pseudometrizable space $\mathscr{R}$, then there exists a sequence $\{f_n\}$ of continuous mappings of the projective product $\mathscr{P} \times \mathscr{Q}$ into $\mathscr{R}$ such that for each x the sequence $\{f_n x\}$ converges to fx, stated in other words, f is of the first Baire class on $\mathscr{P} \times \mathscr{Q}$.

(d) We leave to the reader the task of defining the canonical embeddings into $P_1 \times P_2 \times \ldots \times P_n$ and the inductive product ind $(\mathscr{P}_1 \times \ldots \times \mathscr{P}_n)$. It is easily seen that

$$\text{ind}\,(\mathscr{P}_1 \times \ldots \times \mathscr{P}_n) = (\ldots((\mathscr{P}_1 \times_{\text{ind}} \mathscr{P}_2) \times_{\text{ind}} \mathscr{P}_3) \ldots) \times_{\text{ind}} \mathscr{P}_n .$$

33 D.2. Spaces inductively generated by a given family of subspaces. Let us consider a closure space $\langle P, u \rangle$ and a collection $\mathscr{X}$ of subspaces of $\langle P, u \rangle$. For each $X \in \mathscr{X}$ let f_X be the canonical mapping of X into P. The family $\{f_X\}$ inductively generates a closure v for P which is finer than u. If $v = u$, then the space $\langle P, u \rangle$ is said to be *inductively generated by the collection* $\mathscr{X}$ of its subspaces. If $\langle P, u \rangle$ is topological, then also the closure topologically inductively generated by $\{f_X\}$ is finer than u and if it coincides with u, then $\langle P, u \rangle$ is said to be *topologically inductively generated by the collection* $\mathscr{X}$ of its subspaces.

(a) In order that a closure space $\mathscr{P}$ be topologically inductively generated by a collection $\mathscr{X}$ of subspaces of $\mathscr{P}$ it is necessary and sufficient that a set $U \subset \mathscr{P}$ be open if and only if the set $U \cap X$ is open in X for each $X \in \mathscr{X}$.

(b) In order that a closure space $\mathscr{P}$ be inductively generated by a collection $\mathscr{X}$ of its subspaces it is necessary that, for each $y \in \mathscr{P}$ and $Y \subset |\mathscr{P}|$, $y \in \overline{Y}$ if and only if $y \in Y$ or $y \in X \cap \overline{X \cap Y}$ for some X in $\mathscr{X}$. (Notice that $X \cap \overline{X \cap Y}$ is the closure of $X \cap Y$ in X.)

(c) If $\mathscr{X}$ is an interior cover of a space $\mathscr{P}$, then $\mathscr{P}$ is inductively generated by $\mathscr{X}$, and if $\mathscr{P}$ is topological, then $\mathscr{P}$ is also topologically inductively generated by $\mathscr{X}$. — Obvious.

(d) Of course, the condition from (c) is not necessary. For example, every metrizable space is inductively generated by the collection of all countable subspaces with exactly one accumulation point. Indeed, if $x \in \overline{Y} - Y$, then there exists a sequence $\{x_n\}$ in Y which converges to x. Clearly $X = (x) \cup \mathbf{E}\{x_n\}$ is a countable subspace with exactly one accumulation point, namely x. Such spaces will be investigated in Section 35 devoted to convergence, in particular the **L**-spaces and **S**-spaces (that is, spaces such that their closure structure can be described in terms of convergent sequences).

(e) A space is feebly locally connected if and only if it is inductively generated by the collection of all open connected subspaces. A topological space is locally connected if and only if it is inductively (topologically inductively) generated by the collection of all locally connected open subspaces. (Trivial.) The "if" part of the latter statement of (e) can be strenghtened as follows:

(f) If a space $\mathscr{P}$ is inductively generated by a collection of its locally connected subspaces, then $\mathscr{P}$ is locally connected.

For the proof of (f) and a further discussion of local connectedness, see the exercises.

In the last two examples 33 D.1 and 33 D.2 all inductive generating families $\{f_a\}$ were formed by embeddings.

33 D.3. Definition. An *inductive (topological inductive) generating family* $\{f_a\}$ is said to be *strict* if all f_a are embeddings. The meaning of the expressions of the type "strict inductive generating family for a set or a space" is obvious and therefore the definitions are omitted.

Let $\{f_a\}$ be a strict inductive (topological inductive) generating family for a space $\mathscr{P}$. For each a and b in A let $\mathscr{R}_{ab}$ be the subspace of $\mathscr{P}$ whose underlying set is $\mathbf{E}f_a \cap \cap \mathbf{E}f_b$, and $\mathscr{R}_a$ be the subspace of $\mathscr{P}$ whose underlying set is $\mathbf{E}f_a$. Thus the mappings $f_a : \mathbf{D}^*f_a \to \mathscr{R}_a$ are homeomorphisms, and $\mathscr{R}_{ab} = \mathscr{R}_{ba}$. Let $\mathscr{R}'_{ab}$ denote the subspace of $\mathbf{D}^*f_a$ with underlying set $f_a^{-1}[|\mathscr{R}_{ab}|]$. Clearly $f_a : \mathscr{R}'_{ab} \to \mathscr{R}_{ab}$ is a homeomorphism. Hence the mapping $f_b^{-1} \circ f_a : \mathscr{R}'_{ab} \to \mathscr{R}'_{ba}$, which will be denoted by $l(f_a, f_b)$, is also a homeomorphism. Clearly $l(f_a, f_b) = l^{-1}(f_b, f_a)$.

If K is any class then $\{f_a\}$ might be termed a K-structure if each $l(f_a, f_b)$ belongs to K; if, in addition, $\{\mathbf{E}f_a\}$ is an interior cover of $\mathscr{P}$, then $\{f_a\}$ might be termed a strong K-structure. For example, if $n \in \mathbf{N}$ and K is the set of all analytic homeomorphism of open subsets of $\mathscr{C}^n$ onto open subsets of $\mathscr{C}^n$, i.e. such homeomorphisms which are simultaneously analytic, then a pair $\langle \mathscr{P}, \{f_a\} \rangle$, where $\mathscr{P}$ is a separated connected closure space and $\{f_a\}$ is a strong K-structure for $\mathscr{P}$, is called an n-dimensional analytic manifold. Similarly real k-differentiable n-dimensional manifolds are defined. Of course a K-structure need not be a strong K-structure.

The next two examples concern the operations of pasting and sewing together of spaces which occur frequently e.g. in function theory.

33 D.4. Suppose that $\mathscr{P}$ is a closure space. If ϱ is an equivalence relation on $\mathscr{P}$ then there is defined the quotient space $\mathscr{P}/\varrho$. We may say that $\mathscr{P}/\varrho$ is obtained from P by *pasting together all points of each equivalence class*. Now let us suppose that we are given a relation σ for $\mathscr{P}$, that is, $\sigma \subset |\mathscr{P}| \times |\mathscr{P}|$; we want to find the smallest equivalence on $\mathscr{P}$ containing σ, in other words, there is prescribed which points must be pasted to which points, and we want to describe all pairs $\langle x, y \rangle$ of points which must be necessarily pasted together. The required equivalence ϱ is obviously the intersection of all equivalences containing σ. There is a direct construction of σ. First recall that (see 1 C) a relation α for P is an equivalence on P if and only if $\alpha \supset \Delta_P$ (reflexivity and $\alpha[P] = P$), $\alpha = \alpha^{-1}$ (symmetry) and $\alpha \circ \alpha = \alpha$ (transitivity). Now the construction goes as follows: put $\sigma_0 = \Delta_P \cup (\sigma \cup \sigma^{-1})$ and by induction $\sigma_{n+1} = \sigma_n \circ \sigma_n$, $n \in \mathbf{N}$; it is easily seen that $\varrho = \bigcup\{\sigma_n\}$. Indeed, by induction we have $\sigma \subset \sigma_n \subset \varrho$ for each n and hence $\sigma \subset \sigma' = \bigcup\{\sigma_n\} \subset \varrho$; on the other hand σ' is an equivalence because $\sigma' \supset \Delta$, σ' is symmetric as the union of symmetric relations, and finally clearly $\sigma' \circ \sigma' = \sigma'$.

Now let $\{\mathscr{P}_a \mid a \in A\}$ be a family of spaces and let $\mathscr{P}$ be the sum of $\{\mathscr{P}_a \mid a \in A\}$. If ϱ is an equivalence on $\mathscr{P}$, then the quotient space $\mathscr{P}/\varrho$ is sometimes said to be obtained from the family $\{\mathscr{P}_a \mid a \in A\}$ by *pasting together all pairs of points* $\langle x, y \rangle \in \varrho$. Sometimes there is given a relation σ in $\mathscr{P}$ which prescribes which points must be pasted together with which points; and if we want to construct the resulting quotient space we must find the smallest equivalence on $\mathscr{P}$ containing σ. This can be done as above.

33 D.5. There is a special case of pasting which will be called sewing. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be closure spaces and let f be a homeomorphism of a subspace $\mathscr{Q}_1$ of $\mathscr{P}_1$ onto a subspace $\mathscr{Q}_2$ of $\mathscr{P}_2$. Let $\mathscr{P}$ be the sum of $\mathscr{P}_1$ and $\mathscr{P}_2$, g the mapping of the subspace

$\mathrm{inj}_1[\mathcal{Q}_1]$ of $\mathcal{P}$ onto the subspace $\mathrm{inj}_2[\mathcal{Q}_2]$ of $\mathcal{P}$ "induced" by f, and let ϱ be the smallest equivalence on $\mathcal{P}$ containing the graph of g. Evidently $\varrho = g \cup g^{-1} \cup \Delta_P$. The quotient space $\mathcal{P}/\varrho$ is said to be obtained from spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ by *sewing together along the homeomorphism* f. If the homeomorphism f mentioned above is uniquely determined by subspaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and the context then we say that the space $\mathcal{P}$ is obtained from spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ by sewing together along subspaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

The reader will find no difficulties in adapting the construction to obtain the concept of a space obtained by *sewing together a given family of spaces along a given family of homeomorphisms.*

33 D.6. An example of a regular topological space which is not uniformizable. Let Q and P be infinite sets and let the cardinal of P be greater than the cardinal of Q. Choose a ξ in P and an η in Q. Let u be the closure for P such that ξ is the only cluster point of $\mathcal{P} = \langle P, u\rangle$ and the complements of neighborhoods of ξ are finite, and let v be the closure for Q such that the point η is the only cluster point of $\mathcal{Q} = \langle Q, v\rangle$ and the complements of neighborhoods of the point η are finite. We know that the product space $\langle P, u\rangle \times \langle Q, v\rangle$ is normal (29 B.8) and hence the subspace $\mathcal{R} = \langle R, w\rangle$ of $\langle P, u\rangle \times \langle Q, v\rangle$, where $R = P \times Q - (\langle \xi, \eta\rangle)$, is uniformizable. Let $X = (P - (\xi)) \times (\eta)$, $Y = (\xi) \times (Q - (\eta))$. Clearly X and Y are closed in $\mathcal{R}$ and we have shown in 29 B.8 that X and Y are not separated in $\mathcal{R}$ and hence $\mathcal{R}$ is not normal. We shall need the following property of continuous functions on $\mathcal{R}$, $\mathcal{P}$ and $\mathcal{Q}$.

(a) Let f be a continuous function on $\langle P, u\rangle$. For each positive real r there exists a neighborhood U of ξ such that $|fx - f\xi| < r$ for each x in U. The complement of U is finite and thus the inequality is true for each x in P except for a finite number of x's. If A is a countable set of positive reals such that $0 = \inf A$, then $fx = f\xi$ if and only if $|fx - f\xi| < r$ for each r in A, and hence $fx = f\xi$ for all x except for a countable number. Of course a similar result holds for continuous functions on $\langle Q, v\rangle$.

(b) Let f be a continuous function on $\mathcal{R}$. There exists a real number c such that

(1) $f\langle x, \eta\rangle = c$ for each $x \in P$ except for a countable number of x' s;

(2) If $\{y_n\}$ is a one-to-one sequence in $Q - (\eta)$, then the sequence $\{f\langle \xi, y_n\rangle\}$ converges to c. As a consequence, the number c is determined by values of f on any countably infinite subset of Y.

(3) There exists a subset P' of P containing ξ and a subset Q' of Q containing η such that the cardinal of $P - P'$ is at most card Q, the cardinal of $Q - Q'$ is countable and $fz = c$ for each $z \in P' \times Q' - (\langle \xi, \eta\rangle)$ and hence f is constant on $P' \times Q' - (\langle \xi, \eta\rangle)$.

(c) Let us consider the sum $\mathcal{S}$ of the family $\{\mathcal{R} \mid n \in \mathsf{N}\}$. Thus $|\mathcal{S}| = \mathsf{N} \times |\mathcal{R}|$ and the closure structure of $\mathcal{S}$ is inductively generated by the family $\{\mathrm{inj}_n : \mathcal{R} \to \mathcal{S} \mid n \in \mathsf{N}\}$. Let ϱ be the smallest equivalence on $\mathcal{S}$ such that:

(1) $\langle n, x\rangle \varrho \langle n + 1, x\rangle$ for each $x \in X$ and each even n

(2) $\langle n, y\rangle \varrho \langle n + 1, y\rangle$ for each $y \in Y$ and each odd n.

It is evident that two distinct points are equivalent only in cases (1) and (2) (remember that $X \cap Y = \emptyset$). Let $\mathscr{T}$ be the quotient of $\mathscr{S}$ under ϱ, i.e. $\mathscr{T} = \mathscr{S}/\varrho$. The subspace $(n) \times \mathscr{R}$ of $\mathscr{S}$ will be termed the n-th sheet and denoted by $\mathscr{R}_n$, the set $(n) \times X$ the main row of the n-th sheet, and the set $(n) \times Y$ the main column of the n-th sheet. By (1) each even sheet and the following odd sheet are sewed together along the main rows (i.e. by the homeomorphism $\{\langle n, x\rangle \to \langle n + 1, x\rangle \mid x \in X\}$) and, by (2), each odd sheet and the following even sheet are sewed together along the main columns (i.e. by the homeomorphism $\{\langle n, y\rangle \to \langle n + 1, y\rangle \mid y \in Y\}$). Let h denote the canonical mapping of $\mathscr{S}$ onto $\mathscr{T}$ and h_n the mapping $h \circ \mathrm{inj}_n : \mathscr{R} \to \mathscr{T}$. First we shall show that

(3) $\{h_n\}$ is a strict inductive generating family for $\mathscr{T}$ and the family $\{\mathbf{E}h_n\}$ is a closed locally finite and star-finite cover of $\mathscr{T}$ (i.e. each member intersects only a finite number of members).

Since h is an inductive generating mapping and $\{\mathrm{inj}_n : \mathscr{R} \to \mathscr{S}\}$ is an inductive generating family, by 33 A.6 $\{h_n\}$ is an inductive generating family. Clearly each h_n is injective. Since $\mathscr{R}$ is a topological space, to prove that h_n is an embedding, it is sufficient to prove that $h_n[F]$ is closed for each closed $F \subset \mathscr{R}$; and this follows from the fact that $M_k = h_k^{-1}[h_n[F]]$ is closed for each k (clearly $M_n = F$, M_k is $\emptyset$ if $|k - n| > 1$ and M_k is $F \cap X$ or $F \cap Y$ in the remaining cases) because $\{h_k\}$ is an inductive generating family. We have also proved that $\mathbf{E}h_n$ is closed for each n. If $|k - n| > 1$ then $\mathbf{E}h_n \cap \mathbf{E}h_k = \emptyset$ and therefore $\{\mathbf{E}h_n\}$ is clearly star-finite. If z is any point of $\mathscr{T}$ then the star U of z in $\{\mathbf{E}h_n\}$ is a neighborhood of z because $h_n^{-1}[z] \neq \emptyset$ implies $h_n^{-1}[U] = |\mathscr{R}|$. As a consequence, $\{\mathbf{E}h_n\}$ is locally finite.

(4) $\mathscr{T}$ is a semi-separated totally disconnected space (sets simultaneously open and closed form a base for $\mathscr{T}$). As a consequence, $\mathscr{T}$ is a separated uniformizable space.

Clearly $\mathscr{R}$ is a totally disconnected space, and using this fact the reader find without difficulties that $\mathscr{T}$ is totally disconnected. Finally, the space is semi-separated because $h_n^{-1}[z]$ is a one-point set for each n and $\mathscr{R}$ is semi-separated.

The space $\mathscr{T}$ is separated and uniformizable but for each continuous function f on $\mathscr{T}$ we have $\bigcap\{f[h_n[R]]\} \neq \emptyset$. We shall prove somewhat more.

(5) If f is a continuous function on $\mathscr{T}$, then there exist a real c and a countable subset X' of X such that $f \circ h_n x = c$ for each $x \in X - X'$ (remember that the cardinal of $X - X'$ is infinite).

The composite $f_n = f \circ h_n$ is a continuous function on $\mathscr{R}$ and therefore, by (b) (1), there exists a real c_n and a countable subset X_n of X such that $f_n z = c_n$ for each $z \in X - X_n$. Put $X' = \bigcup\{X_n\}$; of course, the cardinal of X' is countable and $f_n z = c_n$ for each $z \in X - X'$. We shall prove that $c_n = c_k$ for each n and k. It is sufficient to show $c_n = c_{n+1}$ for each n. If n is even and $z \in X$, then clearly $h_n z = h_{n+1} z$ and therefore $c_n = c_{n+1}$. Assuming that n is odd, choose a one-to-one sequence $\{y_n\}$ in Y. By (b) (2), the sequence $\{f_n y_k \mid k\}$ converges to c_n and the sequence $\{f_{n+1} y_k \mid k\}$ converges to c_{n+1}. Since $h_n y_k = h_{n+1} y_k$ we obtain $f_n y_k = f_{n+1} y_k$ and hence $c_n = c_{n+1}$. It should be noted that if Q is uncountable then (5) follows immediately from (b) (3).

Now we are prepared to exhibit the required example.

(d) Let T_0 be a set consisting of all points of $|\mathcal{T}|$ and one further point, say ζ. Let $\mathcal{T}_0$ be the space whose underlying set is T_0, such that $\mathcal{T}$ is an open subspace of $\mathcal{T}_0$ and the complements in T_0 of sets of the form $\mathbf{E}h_n$ form a local sub-base at ζ. We shall prove that $\mathcal{T}_0$ is a separated regular topological space but $\mathcal{T}_0$ is not uniformizable. First let us notice that the sets

$$U_n = (\zeta) \cup \bigcup\{\mathbf{E}h_k \mid k \geqq n\}$$

form a local base at ζ. If f is a continuous function on $\mathcal{T}_0$ which is zero on $|\mathcal{T}_0| - U_n$, $n > 0$, then $0 \in f[\mathbf{E}h_k]$ for each k (by (c)(5)) and therefore $f\zeta = 0$. Consequently, $\mathcal{T}_0$ is not uniformizable. Clearly each set U_n is closed and $\bigcap\{U_n\} = (\zeta)$. Using the facts that $\mathcal{T}$ is an open subspace of $\mathcal{T}_0$ and $\mathcal{T}$ is a separated regular space the reader will find without difficulties that $\mathcal{T}$ is a regular separated space. Evidently $\mathcal{T}_0$ is a topological space.

33 D.7. A regular separated topological space without non-constant continuous functions.

(a) Let $\mathcal{L}$ be a closure space and let L_1 and L_2 be two equipollent subsets of $\mathcal{L}$ such that L_1 is dense in $\mathcal{L}$, $L_1 \cap L_2 = \emptyset$ and each continuous function on $\mathcal{L}$ is constant on L_2. Let φ be a bijective relation for L_1 and L_2 and let σ be the smallest equivalence on $\mathcal{L}$ containing φ. If f is a continuous function on the quotient space $\mathcal{L}/\sigma$, then f is constant because f is constant on the dense set $g[L_2]$ where g is the canonical mapping of $\mathcal{L}$ into $\mathcal{L}/\sigma$. If $\mathcal{L}$ is regular, separated and topological, then the quotient $\mathcal{L}/\sigma$ need not be regular or topological but $\mathcal{L}/\sigma$ is always separated. If $\mathcal{L}_1$ is the topological regular modification of $\mathcal{L}/\sigma$, then every continuous function on $\mathcal{L}_1$ is constant but $\mathcal{L}_1$ need not be separated. However, as we shall see later, there exist spaces $\mathcal{L}/\sigma$ such that $\mathcal{L}_1$ is separated.

We want to construct a space $\mathcal{L}$ and an equivalence σ such that $\mathcal{L}/\sigma$ possesses some additional properties which will allow us to prove that $\mathcal{L}_1$ is separated.

(b) Suppose that we are given a topological regular separated space $\mathcal{X}$, a closed subset Z of $\mathcal{X}$, a proper filter $\mathcal{A}$ of sets on Z, a point ζ of $\mathcal{X}$ and a dense subset M of $\mathcal{X}$ such that

(1) $M \cap Z = \emptyset$ and ζ non $\in M$; and

(2) $\zeta \notin Z$ but for each continuous function f on $\mathcal{X}$ there exists an A in $\mathcal{A}$ such that $(f\zeta) = f[A]$, i.e. $f\zeta$ is the only value of f on A.

Remark. Such a space $\mathcal{X}$ exists and, in addition, $\mathcal{X}$ may possess many additional properties. As an example, let $\mathcal{X}$ be the space $\mathcal{T}_0$ from 33 D.6, Z be the set $h_0[X]$, $\mathcal{A}$ be the filter having for a base the collection of all subsets of Z whose complements in Z are of cardinal less than the cardinal of Z, ζ be the point ζ of $\mathcal{T}_0$ and M be the set of all $h_n\langle x, y\rangle$, $\langle x, y\rangle \in (P - (\xi)) \times (Q - (\eta))$.

Let us consider the sum space $\mathcal{X}_1 = \Sigma\{\mathcal{X} \mid m \in M\}$. Thus $M \times |\mathcal{X}|$ is the underlying set of $\mathcal{X}_1$ and $\mathcal{X}_1$ is inductively generated by the family of mappings $\{\text{inj}_m :$

$: \mathscr{K} \to \mathscr{K}_1\}$. Let τ be the smallest equivalence on $\mathscr{K}_1$ containing all the pairs $\langle\langle m, z\rangle, \langle n, z\rangle\rangle$, $z \in Z$, $m \in M$, $n \in M$, and consider the quotient $\mathscr{L}$ of $\mathscr{K}_1$ under τ, i.e. $\mathscr{L} = \mathscr{K}_1/\tau$, and the canonical mapping g of $\mathscr{K}_1$ onto $\mathscr{L}$. Roughly speaking, $\mathscr{L}$ is obtained from $\mathscr{K}$ by sewing together card M copies of $\mathscr{K}$ along Z; more precisely, $\mathscr{L}$ is obtained from $\mathscr{K}_1$ by sewing together all the subspaces $(m) \times Z$, $m \in M$. The space $\mathscr{L}$ has the following two properties:

(3) The mapping g is one-to-one on $M \times M$ and $L_1 = g[M \times M]$ is dense in $\mathscr{L}$.

(4) The mapping g is one-to-one on $M \times (\zeta)$ and each continuous function f on $\mathscr{L}$ is constant on $L_2 = g[M \times (\zeta)]$.

Proof. Clearly g is one-to-one on $M \times M$. The set M is dense in $\mathscr{K}$ and therefore the set $M \times M$ is dense in $\mathscr{K}_1$. Since g is continuous, L_1 is dense in $\mathscr{L}$. Evidently g is one-to-one on $M \times (\zeta)$. Let f be a continuous function on $\mathscr{L}$. Let us consider the continuous function $h = f \circ g$ on $\mathscr{K}_1$. We shall prove that h is constant on $M \times (\zeta)$ (which implies that f is constant on L_2). Let $h_m = h \circ (\text{inj}_m: \mathscr{K} \to \mathscr{K}_1)$. We must show that $h_{m_1}\zeta = h_{m_2}\zeta$ for each m_1 and m_2 in M. By (2) we can choose $A_i \in \mathscr{A}$ such that $h_{m_i}[A_i] = (h_{m_i}\zeta)$. If $A = A_1 \cap A_2$, then $(h_{m_i}\zeta) = h_{m_i}[A]$ and clearly $h_{m_1}[A] = h_{m_2}[A]$, which implies that $(h_{m_1}\zeta) = (h_{m_2}\zeta)$ and hence $h_{m_1}\zeta = h_{m_2}\zeta$.

(c) If $\mathscr{K}$ is the space $\mathscr{T}_0$ of 33 D.6 and if ζ, Z, $\mathscr{A}$ and M are defined as stated in Remark subsequent to (b) (2), $\mathscr{L}$ is the space defined in (b) and $\mathscr{L}_1$ is the space defined in (a), then $\mathscr{L}_1$ is a separated regular topological space such that each continuous function on $\mathscr{L}_1$ is constant.

We have proved that every continuous function on $\mathscr{L}_1$ is constant. The fact that $\mathscr{L}_1$ is separated is left to the reader as an exercise on topological modification. We want to point out that the set L_1 is isolated; this can be used to give a very simple proof. It is to be noted that (c) is due to J. Novák (who made the assumption that the cardinal of both spaces $\mathscr{P}$ and $\mathscr{Q}$ of 33 D.6 are uncountable).

(d) If $\mathscr{L}_1$ is an infinite countable regular topological space, then $\mathscr{L}_1$ is paracompact (because each cover has a σ-locally finite refinement) and hence $\mathscr{L}_1$ is uniformizable. Therefore a space which is regular and topological but not uniformizable, is necessarily uncountable. Since the space $\mathscr{R}$ of 33 D.6 can be taken with cardinal $\aleph_1$, the cardinal of the space $\mathscr{L}_1$ of (c) may also be taken as $\aleph_1$.

34. HYPERSPACES AND CONTINUITY OF CORRESPONDENCES

This section is devoted to an examination of the "continuity" of correspondences for closure spaces. It is natural to introduce three kinds of continuity, namely upper semi-continuous, lower semi-continuous and continuous correspondences. For mappings all three kinds of continuity coincide with the usual one (in the sense of Definition **16 A.1**). The definitions will be based on hyperspaces of a given space which are introduced and studied in subsection A. Subsection B is concerned with defining and developing the properties of correspondences. In subsection C an important result of B enables us to prove, e.g., that the quotient of a topological group under a homomorphism is a topological group. In the last subsection some special kinds of correspondences are considered, mainly inversely upper or lower semi-continuous quotient mappings.

A preliminary comment seems to be necessary.

Let f be a mapping of a closure space $\mathscr{P} = \langle P, u \rangle$ into another one $\mathscr{Q} = \langle Q, v \rangle$. We know that the following two conditions are necessary and sufficient for the mapping f to be continuous:

(1) If V is a neighborhood of y in $\mathscr{Q}$, then $f^{-1}[V]$ is a neighborhood of the set $f^{-1}[y]$ in $\mathscr{P}$.

(2) If $y \notin vY$, then $f^{-1}[y] \cap uf^{-1}[Y] = \emptyset$. Evidently condition (i) implies the following condition (i'), i = 1, 2:

(1') If V is open in $\mathscr{Q}$, then $f^{-1}[V]$ is open in $\mathscr{P}$.

(2') If Y is closed in $\mathscr{Q}$, then $f^{-1}[Y]$ is closed in $\mathscr{P}$.

If $\mathscr{Q}$ is topological, then (i') implies (i), i = 1, 2. Now let f be a domain-full correspondence of $\mathscr{P}$ into $\mathscr{Q}$. Thus $f = \langle \operatorname{gr} f, \mathscr{P}, \mathscr{Q} \rangle$ where $\operatorname{gr} f$ is a relation such that $\mathbf{D} \operatorname{gr} f = |\mathscr{P}|$ and $\mathbf{E} \operatorname{gr} f \subset |\mathscr{Q}|$. It is easy to see that (1) implies (1') and (2) implies (2'). On the other hand condition (1) is not equivalent to condition (2), and, in addition, (1) does not imply (2), and (2) does not imply (1). Evidently it is sufficient to show that in the class of all topological spaces (1') does not imply (2') and (2') does not imply (1'). If π is the projection of a product space $\mathscr{Q}$ onto one of its coordinate spaces, say $\mathscr{P}$, then π carries open sets into open sets (by **17 C.7**) and hence the correspondence $\pi^{-1} : \mathscr{P} \to \mathscr{Q}$ fulfils condition (1'). On the other hand, π need not carry closed sets into closed sets (27 ex. 5) and hence the correspondence $\pi^{-1} : \mathscr{P} \to \mathscr{Q}$ need not fulfil condition (2'). An example of a (continuous) mapping g (for topological spaces) which carries closed sets into closed sets but which does not carry open sets into open sets was given in 27 ex. 4; the correspondence g^{-1} fulfils (2′) but not (1′).

A domain-full correspondence satisfying condition (1) (condition 2) is said to be *lower (upper) semi-continuous* and a correspondence satisfying both conditions is said to be *continuous*. It is to be noted that the theory developed will usually be applied to a correspondence f^{-1} where f is a continuous mapping. Therefore it is convenient to introduce the following terminology: a correspondence f is said to be *inversely lower semi-continuous* if the inverse correspondence f^{-1} is lower semi-continuous, and similarly we define *inversely upper semi-continuous* correspondences and *inversely continuous* correspondences. Thus, e.g., a mapping f of a closure space $\mathscr{Q}$ into a closure space $\mathscr{P}$ is inversely lower semi-continuous if (and only if) $x \in \operatorname{int} U$ implies $f[x] \subset \operatorname{int} f[U]$, and if $\mathscr{Q}$ is topological, then this condition can be replaced by the requirement that f carry open sets into open sets. A mapping which carries open (closed) sets into open (closed) sets is termed open (closed). Thus a mapping f of a topological space into a closure space is inversely lower (upper) semi-continuous if and only if f is open (closed).

It turns out that a simultaneously continuous and inversely upper or lower semi-continuous mapping is a quotient mapping and quotient mappings of this kind have many important properties. We mention two: The product of inversely lower semi-continuous quotient mappings is a quotient mapping, and this is the result which is needed for the proof of the fact that quotients of a topological group are topological groups. A domain-restriction of a quotient mapping need not be a quotient mapping; on the other hand a domain-restriction of an inversely upper (lower) semi-continuous quotient mapping to a closed (open) subspace is a quotient mapping. Next, a quotient of a space $\mathscr{P}$ inherits very few of the properties of $\mathscr{P}$. Inverse lower or upper semi-continuity of the canonical mapping onto a quotient often enter as essential additional assumptions.

The examination of the continuity of correspondences can be reduced to an examination of continuous mappings as follows. Given a space $\mathscr{P}$, we define certain spaces $\mathbf{H}_+(\mathscr{P})$, $\mathbf{H}_-(\mathscr{P})$ and $\mathbf{H}(\mathscr{P})$ whose underlying set is the collection of all non-void subsets of $|\mathscr{P}|$, and then we define: a correspondence ranging in $\mathscr{P}$ is upper semi-continuous, lower semi-continuous or continuous if the mapping $\{x \to f[(x)]\}$ of the subspace $\mathbf{D}f$ of $\mathbf{D}^*f$ into the space $\mathbf{H}_+(\mathscr{P})$, $\mathbf{H}_-(\mathscr{P})$ or $\mathbf{H}(\mathscr{P})$ respectively, is continuous.

It is to be noted that the terms lower and upper semi-continuous historically originate from semi-continuous functions introduced in 18 D (see ex. 15). Keep in mind that an upper semi-continuous function in the sense of 18 D need not be continuous, but an upper semi-continuous function in the sense introduced here is continuous. We hope that this ambiguity will not lead to any confusion. In this section functions semi-continuous in the sense of 18 D will be not considered.

A. HYPERSPACES

In 12 A the star of a set in a cover was introduced. Here we shall need the so-called *combinatorial star of a set Y in $\mathcal{X}$* which is defined to be the set of all $X \in \mathcal{X}$ intersecting Y and which will be denoted by star $(Y, \mathcal{X})$. We shall simply say a star because the star in the sense of 12 A will not be needed. For convenience let us agree to denote by $\exp' X$ the set of all non-void subsets of X, that is, $\exp' X = \exp X - (\emptyset)$.

34 A.1. Definition. Let $\langle P, u \rangle$ be a closure space. The *hyperspace of upper semi-continuity* $\mathbf{H}_+(P, u)$ of $\langle P, u \rangle$ is defined as follows: the underlying set of $\mathbf{H}_+(P, u)$ is the set $\exp' P$ and

(1) $\mathbf{E}\{\exp' U \mid U \text{ is a neighborhood of } X \text{ in } \langle P, u \rangle\}$

is a local base at X in $\mathbf{H}_+(P, u)$ for each X. The *hyperspace* $\mathbf{H}_-(P, u)$ *of lower semi-continuity* is defined as follows: the underlying set of $\mathbf{H}_-(P, u)$ is $\exp' P$ and

(2) $\mathbf{E}\{\text{star}(U, \exp' P) \mid X \cap \text{int } U \neq \emptyset\}$

is a local sub-base at X in $\mathbf{H}_-(P, u)$. Finally, *the hyperspace* $\mathbf{H}(P, u)$ *of continuity* is defined to be the set $\exp' P$ endowed with the infimum of closure structures of spaces $\mathbf{H}_+(P, u)$ and $\mathbf{H}_-(P, u)$.

If $\mathcal{P} = \langle P, u \rangle$ then we shall write $\mathbf{H}(\mathcal{P})$ instead of $\mathbf{H}(P, u)$ and similarly for $\mathbf{H}_+(\mathcal{P})$ and $\mathbf{H}_-(\mathcal{P})$.

Of course it must be shown that (1) and (2) is a local base and a local sub-base at X for some closure operations for $\exp' P$; that is, according to 14 B.10, 11 it must be shown that each element of (1) and also each element of (2) contains X, and in addition, the intersection of two elements of (1) contains an element. The former statement is obvious and the latter follows from the obvious equality

$$\exp'(U_1 \cap U_2) = (\exp' U_1) \cap (\exp' U_2)$$

and the fact that the intersection of two neighborhoods is a neighborhood.

From the definition we obtain at once

34 A.2. *Let $\langle P, u \rangle$ be a closure space and f be the relation $\mathbf{E}\{\langle x, (x) \rangle \mid x \in P\}$. Then the mappings $f: \langle P, u \rangle \to \mathbf{H}_+(P, u)$, $f: \langle P, u \rangle \to \mathbf{H}_-(P, u)$ and $f: \langle P, u \rangle \to \mathbf{H}(P, u)$ are embeddings (which will be called the canonical embeddings of $\langle P, u \rangle$ into $\mathbf{H}_+(P, u)$, $\mathbf{H}_-(P, u)$ and $\mathbf{H}(P, u)$ respectively).*

Now let $X \in \mathbf{H}_-(P, u)$ and let star $(U_i, \exp' P)$, $i \leqq n$, be canonical neighborhoods of X. Then $\bigcap\{\text{star}(U_i, \exp P)\}$ is a neighborhood of X and clearly

(3) $\bigcap\{\text{star}(U_i, \exp P)\} = \mathbf{E}\{Y \mid Y \in P,\ U_i \cap Y \neq \emptyset \text{ for each } i\}$.

Since (2) was a local sub-base at X in $\mathbf{H}_-(P, u)$, the collection of all sets of the form (3) form a local base at X in $\mathbf{H}_-(P, u)$.

Now we shall describe a certain type of local bases in $\mathbf{H}(P, u)$. According to 31 A, if w is the infimum of two closures w_1 and w_2, say for a set Q, and $\mathcal{U}_1$ and $\mathcal{U}_2$ are local bases at a point x in $\langle Q, w_1\rangle$ and $\langle Q, w_2\rangle$ respectively, then $\mathcal{U}_1 \cup \mathcal{U}_2$ is a local sub-base at x in $\langle Q, w\rangle$. It follows that the sets of the form

(4) $\mathbf{E}\{Y \mid Y \subset U,\ Y \cap U_i \neq \emptyset \quad \text{for } i = 0, \ldots, n\}$

form a local sub-base at X in $\mathbf{H}(P, u)$, where U is a neighborhood of X in P, $n \in \mathrm{N}$ and U_i are sets such that $X \cap \operatorname{int} U_i \neq \emptyset$ for each $i \leqq n$. But clearly the sets of the form (4) form a filter base, and consequently, the collection of all sets of the form (4) is a local base at X in $\mathbf{H}(P, u)$. The set (4) coincides with the set

(5) $\mathbf{E}\{Y \mid Y \subset \bigcup \mathcal{U},\ Y \cap V \neq \emptyset \quad \text{for each } V \text{in } \mathcal{U}\}$

where $\mathcal{U}$ consists of U and all sets $U_i \cap U$, $i \leqq n$. In addition, $\mathcal{U}$ is finite and interiorly covers X. Conversely, if $\mathcal{U}$ is any finite collection which interiorly covers X in $\langle P, u\rangle$ and if $U = \bigcup \mathcal{U}$ and $\{U_i\}$ is any finite sequence whose range is $\mathcal{U}$, then the set (5) is equal to the set (4). It follows, exactly the sets of the form (4) can be of the form (5).

34 A.3. Theorem. *Let $\langle P, u\rangle$ be a closure space and let $X \in \exp' P$. The collection of all sets of the form* (5) *is a local base at X in $\mathbf{H}(P, u)$ where $\mathcal{U}$ varies over all finite subcollections of* $\exp' P$ *such that* $\{\operatorname{int} U \mid U \in \mathcal{U}\}$ *covers X and* $\operatorname{int} U \cap X \neq \emptyset$ *for each U in $\mathcal{U}$.*

34 A.4. Theorem. *If $\langle P, u\rangle$ is a topological space, then all three hyperspaces $\mathbf{H}_+(P, u)$, $\mathbf{H}_-(P, u)$ and $\mathbf{H}(P, u)$ are also topological spaces and the collection of all* $\exp' U$, *U open in $\langle P, u\rangle$, is an open base for $\mathbf{H}_+(P, u)$; the collection of all* star $(U, \exp' P)$, *U open in $\langle P, u\rangle$, is an open sub-base for $\mathbf{H}_-(P, u)$; and finally, the collection of all sets of the form*

(6) $\mathbf{E}\{Y \mid Y \subset \bigcup \mathcal{U},\ U \in \mathcal{U} \Rightarrow U \cap Y \neq \emptyset\}$

where $\mathcal{U}$ is a finite collection of open subsets of $\langle P, u\rangle$, is an open base for $\mathbf{H}(P, u)$.

First we shall prove

34 A.5. *Let $\langle P, u\rangle$ be a closure space. If U is open in $\langle P, u\rangle$ then the set* $\exp' U$ *is open in $\mathbf{H}_+(P, u)$ and the set* star $(U, \exp' P)$ *is open in $\mathbf{H}_-(P, u)$. If $\mathcal{U}$ is a finite collection of open sets then the set* (6) *is open in $\mathbf{H}(P, u)$.*

Proof of 34 A.5. Let U be an open subset of $\langle P, u\rangle$. If $\emptyset \neq X \subset U$ then $X \cap \operatorname{int} U = X \cap U = X \neq \emptyset$ and hence, by definition, $\exp' U$ is a neighborhood of X in $\mathbf{H}_+(P, u)$. The set X was chosen arbitrarily, and hence $\exp' U$ is a neighborhood of each of its elements which means that $\exp' U$ is open. Similarly, if $X \cap U \neq \emptyset$ then $X \cap \operatorname{int} U \neq \emptyset$ ($U = \operatorname{int} U$), and hence star $(U, \exp' P)$ is a neighborhood of X in $\mathbf{H}_-(P, u)$. Since X was arbitrarily chosen in star $(U, \exp' P)$, star $(U, \exp' P)$ is open in $\mathbf{H}_-(P, u)$. Using 34 A.3 it is easily seen that the set (6) is a neighborhood of each of its points in $\mathbf{H}(P, u)$.

Proof of 34 A.4. Let $\langle P, u\rangle$ be a topological space; thus $\operatorname{int} U$ is open for each $U \subset P$. We shall only prove the statement for $\mathbf{H}_+(P, u)$; the statements for $\mathbf{H}_-(P, u)$ and $\mathbf{H}(P, u)$ can be proved in a similar way. By 34 A.5 the sets $\exp' U$, U open in $\langle P, u\rangle$, are open in $\mathbf{H}_+(P, u)$. It remains to show that every neighborhood $\mathcal{U}$ of any $X \in \mathbf{H}_+(P, u)$ contains an $\exp' U$ containing X with U open in $\langle P, u\rangle$. By definition there exists a neighborhood V of the set X in $\langle P, u\rangle$ such that $\exp' V \subset \mathscr{U}$. Put $U = \operatorname{int} V$. The set U is an open neighborhood of X and clearly $\exp' U \subset \exp' V$.

34 A.6. *If $\mathscr{Q}$ is a subspace of a closure space $\mathscr{P}$ then $\mathbf{H}_+(\mathscr{Q})$, $\mathbf{H}_-(\mathscr{Q})$ and $\mathbf{H}(\mathscr{Q})$ are subspaces of $\mathbf{H}_+(\mathscr{P})$, $\mathbf{H}_-(\mathscr{P})$ and $\mathbf{H}(\mathscr{P})$ respectively.*

The straightforward verification is left to the reader.

34 A.7. Theorem. *Let $\mathscr{Q}$ be a subspace of a closure space $\mathscr{P}$, $\mathscr{R}$ be the collection of all sets $X \subset |\mathscr{P}|$ intersecting $|\mathscr{Q}|$ and let f be the single-valued relation on $\mathscr{R}$ which assigns to each X the set $X \cap |\mathscr{Q}|$. If $\mathscr{Q}$ is closed in $\mathscr{P}$ then the mapping f of the subspace $\mathscr{R}$ of $\mathbf{H}_+(\mathscr{P})$ into $\mathbf{H}_+(\mathscr{Q})$ is continuous. If $\mathscr{Q}$ is open in $\mathscr{P}$, then the mapping f of the subspace $\mathscr{R}$ of $\mathbf{H}_-(\mathscr{P})$ into $\mathbf{H}_-(\mathscr{Q})$ is continuous.*

Corollary. *Under the assumptions of 34 A.7, if $\mathscr{Q}$ is simultaneously open and closed in $\mathscr{P}$, then the mapping of the subspace $\mathscr{R}$ of $\mathbf{H}(\mathscr{P})$ into $\mathbf{H}(\mathscr{Q})$ is continuous.*

Proof of 34 A.7. I. Let $\mathscr{Q}$ be closed, X be any point of $\mathscr{R}$, and $\mathscr{U}$ be any neighborhood of fX in $\mathbf{H}_+(\mathscr{Q})$. By definition, there exists a neighborhood U of the set fX in $\mathscr{Q}$ such that $\exp' U \subset \mathscr{U}$. Put $V = (|\mathscr{P}| - |\mathscr{Q}|) \cup U$. Since $\mathscr{Q}$ is closed, V is necessarily a neighborhood of the set X in $\mathscr{P}$. By definition of $\mathbf{H}_+(\mathscr{P})$, $\exp' V$ is a neighborhood of the point X in $\mathbf{H}_+(P)$. In consequence $\mathscr{W} = \mathscr{R} \cap (\exp' V)$ is a neighborhood of the point X in the subspace $\mathscr{R}$ of $\mathbf{H}_+(P)$. Clearly $f[\mathscr{W}] \subset \exp' U \subset \mathscr{U}$. Thus we have proved: if $X \in \mathscr{R}$ and $\mathscr{U}$ is a neighborhood of fX, then there exists a neighborhood $\mathscr{W}$ of X such that $f[\mathscr{W}] \subset \mathscr{U}$. By definition, the mapping f is continuous.

II. Now let $\mathscr{Q}$ be open, X be a point of $\mathscr{R}$ and $\mathscr{U}$ be a neighborhood of fX in $\mathbf{H}_-(\mathscr{Q})$. By definition of $\mathbf{H}_-(\mathscr{Q})$ there exist subsets U_i of $\mathscr{Q}$ such that $(\operatorname{int}_{\mathscr{Q}} U_i) \cap fX \neq \emptyset$ and $\bigcap_i \operatorname{star}(U_i, \exp' |\mathscr{Q}|) \subset \mathscr{U}$. Since $\mathscr{Q}$ is open, we have $\operatorname{int}_{\mathscr{Q}} U_i = \operatorname{int}_{\mathscr{P}} U_i$; consequently $(\operatorname{int}_{\mathscr{P}} U_i) \cap X \neq \emptyset$, which implies that $\operatorname{star}(U_i, \exp' |\mathscr{P}|)$ is a neighborhood of X in $\mathbf{H}_-(\mathscr{P})$. Hence $\mathscr{W}_i = \mathscr{R} \cap \operatorname{star}(U_i, \exp' |\mathscr{P}|)$ ($= \operatorname{star}(U_i, \exp' |\mathscr{P}|)$, but we need not this) is a neighborhood of X in the subspace $\mathscr{R}$ of $\mathbf{H}_-(\mathscr{P})$. Clearly

$$\bigcap_i f[\mathscr{W}_i] \subset \bigcap_i \operatorname{star}(U_i, \exp' |\mathscr{Q}|) \subset \mathscr{U},$$

which establishes the continuity.

Remark. Let $\mathscr{Q}$ be a closed (open) subspace of a space $\mathscr{P}$. The collection of all $X \subset |\mathscr{P}|$, $X \cap |\mathscr{Q}| \neq \emptyset$, is an open (closed) subset of $\mathbf{H}_+(\mathscr{P})$ ($\mathbf{H}_-(\mathscr{P})$, respectively).

B. CONTINUITY OF CORRESPONDENCES

A *correspondence for closure spaces* $\mathscr{P}$ and $\mathscr{Q}$ is a correspondence f such that $\mathbf{D}^*f = \mathscr{P}$ and $\mathbf{E}^*f = \mathscr{Q}$, i.e., $f = \langle \operatorname{gr} f, \mathscr{P}, \mathscr{Q}\rangle$ where $\operatorname{gr} f$ is a relation for $|\mathscr{P}|$ and $|\mathscr{Q}|$ (i.e. $\mathbf{D} \operatorname{gr} f \subset |\mathscr{P}|$, $\mathbf{E} \operatorname{gr} f \subset |\mathscr{Q}|$) which is called the graph of f. Thus every mapping for closure spaces is a correspondence for closure spaces. Correspondences and related concepts were introduced in subsection 7 B. For convenience we recall the terminology and conventions needed. As in the case of mappings we often write f instead of $\operatorname{gr} f$, e.g.

$f[X]$, $\mathbf{E}f$, and occasionally, gr f instead of f. A correspondence f is *domain-full* if $|\mathbf{D}^*f| = \mathbf{D}f$, *range-full* if $|\mathbf{E}^*f| = \mathbf{E}f$, and *full* if it is range-full and domain-full. A correspondence f for $\mathscr{P}$ and $\mathscr{Q}$ is said to be *for* $\mathscr{P}$ *ranging on* $\mathscr{Q}$ if $|\mathbf{E}^*f| = \mathbf{E}f$, *on* $\mathscr{P}$ *ranging in* $\mathscr{Q}$ if $|\mathbf{D}^*f| = \mathbf{D}f$, and *on* $\mathscr{P}$ *ranging on* $\mathscr{Q}$ or *on* $\mathscr{P}$ *onto* $\mathscr{Q}$ if it is both on $\mathscr{P}$ ranging into $\mathscr{Q}$ and for $\mathscr{P}$ ranging on $\mathscr{Q}$.

34 B.1. Definition. A correspondence f for closure spaces $\mathscr{P}$ and $\mathscr{Q}$ is said to be *upper semi-continuous, lower semi-continuous* or *continuous* if the mapping $\{x \to f[x]\}$ of the subspace $\mathbf{D}f$ of $\mathscr{P}$ into $\mathbf{H}_+(\mathscr{Q})$, $\mathbf{H}_-(\mathscr{Q})$ or $\mathbf{H}(\mathscr{Q})$, respectively, is continuous (in the usual sense, that is, in the sense of Definition 16 A.1). A correspondence f for closure spaces is said to be *inversely upper semi-continuous, inversely lower semi-continuous* or *inversely continuous* if the inverse correspondence is, respectively, upper semi-continuous, lower semi-continuous or continuous. Finally, a correspondence f will be termed *bilaterally upper semi-continuous, bilaterally lower semi-continuous* or *bilaterally continuous* if both f and f^{-1} are upper semi-continuous, lower semi-continuous or continuous, respectively. Instead of bilaterally continuous we shall usually say *bicontinuous.*

First we shall prove that the continuity of a correspondence f depends on subspaces $\mathbf{D}f$ of $\mathbf{D}^*f$ and $\mathbf{E}f$ of $\mathbf{E}^*f$ only.

34 B.2. Theorem. *Let f be a correspondence for closure spaces $\mathscr{P}$ and $\mathscr{Q}$, let $\mathscr{P}_1$ be a subspace of $\mathscr{P}$ such that $|\mathscr{P}_1| = \mathbf{D}f$, and $\mathscr{Q}_1$ be a subspace of $\mathscr{Q}$ such that $|\mathscr{Q}_1| = \mathbf{E}f$. Then f is upper semi-continuous, lower semi-continuous or continuous if and only if the correspondence $f: \mathscr{P}_1 \to \mathscr{Q}_1$ has the corresponding property.* Roughly speaking, the upper or lower semi-continuity or the continuity of a correspondence f only depends on the relativization of the closure structure of $\mathbf{D}^*f$ to $\mathbf{D}f$ and the relativization of the closure structure of $\mathbf{E}^*f$ to $\mathbf{E}f$.

Proof. It is self-evident that f is upper or lower semi-continuous if and only if the correspondence $f: \mathscr{P}_1 \to \mathscr{Q}$ has the corresponding property. Since $\mathscr{Q}_1$ is a subspace of $\mathscr{Q}$, by 34 A.6 $\mathbf{H}_+(\mathscr{Q}_1)$, $\mathbf{H}_-(\mathscr{Q}_1)$ and $\mathbf{H}(\mathscr{Q}_1)$ is a subspace of $\mathbf{H}_+(\mathscr{Q})$, $\mathbf{H}_-(\mathscr{Q})$ and $\mathbf{H}(\mathscr{Q})$ respectively, and therefore, e.g., the mapping $\{x \to f[x]\} : \mathscr{P}_1 \to \mathbf{H}_+(\mathscr{Q}_1)$ is continuous if and only if the mapping $\{x \to f[x]\} : \mathscr{P}_1 \to \mathbf{H}_+(\mathscr{Q})$ is continuous.

According to the preceding theorem we may restrict our attention to full correspondences.

Every struct-mapping is a struct-correspondence, in particular, a mapping for closure spaces is a correspondence for closure spaces. Thus for mappings we have two definitions of continuity, the one introduced in 16 A.1 and the second one introduced in 34 B.1. It is stated in the proposition to follow that the two definitions are equivalent, and moreover, upper semi-continuity as well as lower semi-continuity is equivalent to continuity.

34 B.3. *The following properties of a mapping f for closure spaces are equivalent: f is continuous in the sense of Definition* 16 A.1, *f is continuous in the sense of Definition* 34 B.1, *f is upper semi-continuous, f is lower semi-continuous.*

Proof. Let f be a mapping of a closure space $\mathscr{P}$ into a closure space $\mathscr{Q}$. By 34 A.2 the canonical mappings, say g_+, g_- and g, of $\mathscr{Q}$ into $\mathbf{H}_+(\mathscr{Q})$, $\mathbf{H}_-(\mathscr{Q})$ and $\mathbf{H}(\mathscr{Q})$ respectively, are embeddings. By Definition 34 B.1 the mapping f is upper semi-continuous (lower semi-continuous, continuous) if and only if the mapping $g_+ \circ f$ ($g_- \circ f$, $g \circ f$) is continuous in the usual sense, i.e. in the sense of Definition 16 A.1; since g_+ (g_-, g, respectively) is an embedding, $g_+ \circ f$ ($g_- \circ f$, $g \circ f$, respectively) is continuous in the usual sense if and only if the mapping f is continuous in the usual sense.

Remark. A direct proof is probably simpler.

Now we shall give direct descriptions of lower and upper semi-continuity.

34 B.4. Theorem. *Suppose that f is a correspondence for closure spaces $\mathscr{P}$ and $\mathscr{Q}$. Each of the following two conditions is necessary and sufficient for the correspondence f to be lower semi-continuous:*

(a) *If V is a neighborhood of y in $\mathscr{Q}$, then $f^{-1}[V]$ is a neighborhood of the set $f^{-1}[y]$ in the subspace $\mathbf{D}f$ of $\mathscr{P}$.*

(b) *If $x \in \mathbf{D}f$ and V is a neighborhood of a point of $f[x]$ in $\mathscr{Q}$ (i.e. if $f[x] \cap \cap \operatorname{int} V \neq \emptyset$), then there exists a neighborhood U of x in $\mathscr{P}$ such that $V \cap f[z] \neq \emptyset$ for each $z \in U$.*

Each of the following two conditions is necessary and sufficient for the correspondence f to be upper semi-continuous:

(c) *If $x \in \mathbf{D}f$ and V is a neighborhood of the set $f[x]$ in $\mathscr{Q}$, then the set $\mathbf{E}\{z \mid f[z] \subset \subset V\}$ is a neighborhood of x in the subspace $\mathbf{D}f$ of $\mathscr{P}$.*

(d) *If $x \in \mathbf{D}f$ and V is a neighborhood of the set $f[x]$ in $\mathscr{Q}$, then there exists a neighborhood U of x in $\mathscr{P}$ such that $f[z] \subset V$ for each z in U.*

Proof. Evidently (a) is equivalent to (b), and (c) is equivalent to (d). We shall prove that condition (a) (condition (c)) is necessary and sufficient for f to be lower (upper) semi-continuous. Clearly we may and shall assume that f is domain-full. Consider the mappings

$$g = \{x \to f[x]\} : \mathscr{P} \to \mathbf{H}_-(\mathscr{Q}), \quad h = \{x \to f[x]\} : \mathscr{P} \to \mathbf{H}_+(\mathscr{Q}).$$

By definition, the correspondence f is lower or upper semi-continuous if and only if the mapping g or h, respectively, is continuous. Evidently one has $f^{-1}[V] = g^{-1}[\operatorname{star}(V, \exp' |\mathscr{Q}|)]$ and $\mathbf{E}\{z | fz \subset V\} = h^{-1}[\exp' V]$ for each $V \subset |\mathscr{Q}|$.

Now, the equivalences immediately follow from Definition 34 A.1 of the closure structures of $\mathbf{H}_-(\mathscr{Q})$ and $\mathbf{H}_+(\mathscr{Q})$ (given $\emptyset \neq Y \subset |\mathscr{Q}|$, the sets $\operatorname{star}(V, \exp' |\mathscr{Q}|)$, $Y \cap \operatorname{int} V \neq \emptyset$, form a local sub-base at Y in $\mathbf{H}_-(\mathscr{Q})$ and the sets $\exp' V$, $Y \subset \operatorname{int} V$, form a local base at Y in $\mathbf{H}_+(\mathscr{Q})$).

It follows immediately from the preceding theorem that if a full correspondence f is lower (upper) semi-continuous then the inverse correspondence f^{-1} carries open (closed) sets into open (closed) sets.

34 B.5. Definition. A *correspondence* f for closure spaces is said to be *open* (*closed*) if the image of each open (closed) subset of the subspace $\mathbf{D}f$ of $\mathbf{D}^*f$ is open (closed) in the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$. A correspondence f is said to be *inversely open* (*closed*) if the inverse correspondence f^{-1} is open (closed).

Now the corollary of 34 B.4 mentioned above can be stated as follows:

34 B.6. *Every lower semi-continuous correspondence is inversely open and every upper semi-continuous correspondence is inversely closed.*

It is easily seen that an inversely open (closed) correspondence need not be lower (upper) continuous. For example, if $\mathscr{P}$ is the topological modification of a space $\mathscr{Q}$, then the mapping $\mathsf{J} : \mathscr{P} \to \mathscr{Q}$ is both inversely open and inversely closed, but it is continuous if and only if $\mathscr{P} = \mathscr{Q}$, i.e. if $\mathscr{Q}$ is topological. On other hand, for topological spaces the equivalence is true as stated in the following theorem.

34 B.7. Theorem. *In order that a correspondence f ranging in a topological space be lower (upper) semi-continuous it is necessary and sufficient that f be inversely open (closed).*

Proof. Recall that, in a topological space, the interior of a set is open and apply 34 B.4. An alternate proof follows from the fact that hyperspaces of a topological space are topological spaces.

Remark. It should be noted that an open (closed) correspondence is often defined to be a correspondence f which carries open (closed) subsets of $\mathbf{D}^*f$ into open (closed) subsets of $\mathbf{E}^*f$. In functional analysis the term closed correspondence (usually closed mapping) often means that the graph is closed in the product of the domain carrier with the range carrier.

We leave to the reader the task of the formulation of results for inversely lower and upper semi-continuous correspondences which are obtained by applying 34 B.4, 34 B.6 and 34 B.7 to the inverse of a correspondence. We restrict ourselves to mappings.

34 B.8. *In order that a mapping f of a closure space $\mathscr{P}$ into another closure space $\mathscr{Q}$ be inversely lower semi-continuous it is necessary and sufficient that $f[U]$ be a neighborhood of fx in the subspace $\mathbf{E}f$ of $\mathscr{Q}$ for each neighborhood U of any point x of $\mathscr{P}$. In order that a mapping f of a closure space $\mathscr{P}$ into another closure space $\mathscr{Q}$ be inversely upper semi-continuous it is necessary and sufficient that for each y in $\mathbf{E}f$ and each neighborhood U of $f^{-1}[y]$ in $\mathscr{P}$ there exist a neighborhood V of y in $\mathscr{Q}$ such that $f^{-1}[z] \subset U$ for each $z \in V$ (i.e. $f^{-1}[V] \subset U$). Any inversely lower (upper) semi-continuous mapping is open (closed), and an open (closed) mapping of a topological space in a closure space is inversely lower (upper) semi-continuous.*

34 B.9. Examples. (a) *The projections of the product of a family of closure spaces onto coordinate spaces are inversely lower semi-continuous.* This is obvious (and was stated in 17 C.7). By 17 ex. 5, the projections need not be closed and thus certainly need not be inversely upper semi-continuous.

(b) *Every projective generating mapping f is inversely continuous.* In fact, if U is a neighborhood of a point x of $\mathbf{D}^*f$, then there exists a neighborhood V of fx such that $f^{-1}[V] \subset U$; since $f[U] \supset \mathbf{E}f \cap f[f^{-1}[V]] = \mathbf{E}f \cap V, f[U]$ is a neighborhood of fx in $\mathbf{E}f$, which establishes the inverse lower semi-continuity of f (by 34 B.8); now U is also an arbitrary neighborhood of $f^{-1}[fx]$, and $y \in V$ implies $f^{-1}[y] \subset U$, which establishes the inverse upper semi-continuity.

(c) *Every continuous inversely upper semi-continuous mapping as well as every continuous inversely lower semi-continuous mapping is a quotient mapping.* Indeed, let f be a continuous and inversely lower or upper semi-continuous mapping. Without loss of generality we may and shall assume that f is surjective. (Under this assumption the mapping f is a quotient mapping if and only if f is an inductive generating mapping.) We shall show that f is an inductive generating mapping, that is,

$$y \in \overline{Y} (\text{in } \mathbf{E}^*f) \Leftrightarrow f^{-1}[y] \cap \overline{f^{-1}[Y]} \neq \emptyset \,(\text{in } \mathbf{D}^*f)\,.$$

The implication $\Leftarrow$ means the continuity of f. The implication $\Rightarrow$ is proved as follows. Assuming $f^{-1}[y] \cap \overline{f^{-1}[Y]} = \emptyset$ we must show $y \notin \overline{Y}$. By our assumption the set $U = |\mathbf{D}^*f| - f^{-1}[Y]$ is a neighborhood of $f^{-1}[y]$. Now, if f is inversely lower semi-continuous, then $f[U] = |\mathbf{E}^*f| - Y$ is a neighborhood of y in $\mathbf{E}^*f$ which does not intersect Y, and hence $y \notin \overline{Y}$; and if f is inversely upper semi-continuous then there exists a neighborhood V of y in $\mathbf{E}^*f$ such that $f^{-1}[V] \subset U$ and hence $V \cap Y = \emptyset$ which implies that $y \notin \overline{Y}$.

(d) Not every quotient mapping is inversely lower or upper semi-continuous, and in fact, inversely lower or upper semi-continuous quotient mappings form a very small but important class of quotient mappings. Subsection C will be devoted to an examination of this kind of quotient mappings and also to its generalization to families of mappings. Here we want to show that a quotient mapping need not be inversely lower or upper semi-continuous. Since the quotient of a topological space need not be a topological space it is sufficient to prove the following proposition.

(e) *If f is a surjective continuous and inversely lower or upper semi-continuous mapping, then $\mathbf{E}^*f$ is topological provided that $\mathbf{D}^*f$ is topological.*

Suppose that f is continuous and surjective, and $\mathbf{D}^*f$ is topological. Assuming that f is inversely upper semi-continuous, and hence closed, we shall prove that the closure $\overline{Y}$ of any subset Y of $\mathbf{E}^*f$ is closed. By the continuity f we have $f[\overline{f^{-1}[Y]}] \subset \subset \overline{Y}$; since $\mathbf{D}^*f$ is topological, the set $\overline{f^{-1}[Y]}$ is closed in $\mathbf{D}^*f$, and consequently, f being closed and surjective, $f[\overline{f^{-1}[Y]}]$ is closed in $\mathbf{E}^*f$, which implies that $\overline{Y}$ is closed in $\mathbf{E}^*f$. Assuming that f is inversely lower semi-continuous (and hence open) we shall prove that $\operatorname{int} V$ is open in $\mathbf{E}^*f$ for each $V \subset \mathbf{E}f$. According to the continuity of f we have $\operatorname{int} f^{-1}[V] \supset f^{-1}[\operatorname{int} V]$. The set $U = \operatorname{int} f^{-1}[V]$ is open in $\mathbf{D}^*f$ and hence, f being an open surjective mapping, $f[U]$ is open in $\mathbf{E}^*f$. Clearly $\operatorname{int} V \subset \subset f[U] \subset V$, which shows that $\operatorname{int} V = f[U]$ and concludes the proof.

The remainder of the subsection is devoted to an examination of composites, restrictions and products of lower and upper semi-continuous and continuous correspondences.

34 B.10. *If g is a domain-restriction of a correspondence f and f is upper semi-continuous, lower semi-continuous or continuous, then g has the corresponding property.*

Proof. Let f be a correspondence for closure spaces $\mathscr{P}$ and $\mathscr{Q}$, and let g be the domain-restriction of f to a subspace $\mathscr{P}_1$ of $\mathscr{P}$, i.e. $\mathbf{D}^*g = \mathscr{P}_1$, $\mathbf{E}^*g = \mathscr{Q}$, $\mathbf{D}g = |\mathscr{P}_1| \cap \mathbf{D}f$ and $\operatorname{gr} g = \operatorname{gr} f \cap (\mathbf{D}g \times \mathbf{E}f)$. The statements follow from definition 34 B.1 and from the fact that the restriction of a continuous mapping is a continuous mapping. E.g., the mapping $\{x \to g[x]\} : \mathbf{D}g \to \mathbf{H}_+(\mathscr{Q})$ is a domain-restriction of the mapping $\{x \to f[x]\} : \mathbf{D}f \to \mathbf{H}_+(\mathscr{Q})$ where $\mathbf{D}g$ is considered as a subspace of $\mathscr{P}_1$ and $\mathbf{D}f$ is considered as a subspace of $\mathscr{P}$ (remember that $\mathscr{P}_1$ is a subspace of $\mathscr{P}$); it follows that if the latter mapping is continuous, i.e. f is upper semi-continuous, then the former mapping is continuous, i.e. g is upper semi-continuous.

A similar result for range-restrictions is not true. This will be seen from the proof of the following important theorem.

34 B.11. Theorem. *Let g be the range-restriction of a correspondence f for closure spaces. If f is upper semi-continuous and $\mathbf{E}g$ is closed in $\mathbf{E}f$, then g is also upper semi-continuous. If f is lower semi-continuous and $\mathbf{E}g$ is open in $\mathbf{E}f$, then g is also lower semi-continuous.*

Proof. Let g be the range-restriction of f, i.e. $\mathbf{E}^*g$ is a subspace of $\mathbf{E}^*f$, $\mathbf{D}^*g = \mathbf{D}^*f$ and $\operatorname{gr} g = \operatorname{gr} f \cap (\mathbf{D}f \times \mathbf{E}g)$. Thus $\mathbf{D}g$ is the set of all $x \in \mathbf{D}f$ such that $f[x] \cap \mathbf{E}g \neq \emptyset$ and $g[x] = \mathbf{E}g \cap f[x]$. We shall only prove the statement concerning upper semi-continuity, leaving the similar proof for lower continuity to the reader. Suppose that f is upper semi-continuous. According to 34 B.2 we may assume that f is range-full, and it is sufficient to prove that the correspondence $g : \mathbf{D}^*g \to \mathbf{E}^*f$ is upper semi-continuous, i.e. to verify that the mapping $g_1 = \{x \to g[x]\} : \mathbf{D}g \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous. Consider the subspace $\mathscr{X}$ of $\mathbf{H}_+(\mathbf{E}^*f)$ consisting of all $f[x]$, $x \in \mathbf{D}g$, i.e. of all $f[x]$ such that $f[x] \cap \mathbf{E}g \neq \emptyset$, and the mapping $h_1 = \{x \to f[x]\} : \mathbf{D}g \to \mathscr{X}$, $h_2 = \{X \to X \cap \mathbf{E}g\} : \mathscr{X} \to \mathbf{H}_+(\mathbf{E}^*f)$. Clearly $g_1 = h_2 \circ h_1$. The mapping h_1 is continuous because $\{x \to f[x]\} : \mathbf{D}g \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous by 34 B.10, and $\mathscr{X}$ is a subspace of $\mathbf{H}_+(\mathbf{E}^*f)$ containing the actual range of g. The mapping h_2 is continuous by 34 A.7.

Remark. A proof without hyperspaces is probably more clear. If $\mathbf{D}^*f$ is a topological space then it is sufficient to prove that f is inversely closed or open and this is almost evident.

Corollary. *Under the assumptions of 34 B.11, if $\mathbf{E}g$ is simultaneously open and closed in $\mathbf{E}f$ and if f is continuous, then g is also continuous.*

We leave to the reader the task of formulation of the results for inverse upper and lower semi-continuity and inverse continuity which are obtained by applying the results of 34 B.10 and 34 B.11 and Corollary of 34 B.11 to the inverse of a correspondence. Nevertheless, we shall state and prove these results for mappings.

34 B.12. Theorem. *Let f be a mapping of a closure space $\mathscr{P}$ into a space $\mathscr{Q}$ and let g be the domain-restriction of f to a subspace $\mathscr{P}_1$ of $\mathscr{P}$. If $|\mathscr{P}_1| = f^{-1}[f[|\mathscr{P}_1|]]$ (i.e. $|\mathscr{P}_1| = f^{-1}[\mathbf{E}g]$) and f is inversely upper semi-continuous, inversely lower semi-continuous or inversely continuous, then g has the corresponding property. If $|\mathscr{P}_1|$ is closed (open) in $\mathscr{P}$ and f is inversely upper (lower) semi-continuous, then g is also inversely upper (lower) semi-continuous. If $|\mathscr{P}_1|$ is simultaneously open and closed in $\mathscr{P}$ and f is inversely continuous, then g is also inversely continuous.*

Proof. The first group of statements follows from 34 B.10, the second one from 34 B.11, and the last statement follows from Corollary of 34 B.11 or from the second group of statements.

34 B.13. *If f is a mapping simultaneously continuous and inversely upper (lower) semi-continuous, then the domain-restriction of f to any closed (open) subspace of $\mathbf{D}^*f$ is a simultaneously continuous and inversely upper (lower) semi-continuous mapping, and in particular, by* 34 B. 9 (c), *a quotient mapping.*

Proof. Any restriction of a continuous mapping is a continuous mapping, and therefore the results follow from 34 B.12.

If f and g are correspondences such that $\mathbf{E}^*f = \mathbf{D}^*g$, then the composite $g \circ f$ is defined to be the correspondence $\operatorname{gr} g \circ \operatorname{gr} f : \mathbf{D}^*f \to \mathbf{E}^*g$ (7 C.1). It is not true that the composite of two upper (lower) semi-continuous correspondences is an upper (lower) semi-continuous correspondence. The composite of two continuous correspondences may fail to be a continuous correspondence. The theorem which follows gives the best positive results.

34 B.14. Theorem. *The composite $g \circ f$ of two upper (lower) semi-continuous correspondences is upper (lower) semi-continuous provided that the set $\mathbf{E}f \cap \mathbf{D}g$ is closed (open) in the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$, in particular, if $\mathbf{E}f \subset \mathbf{D}g$, e.g. if g is domain-full.*

Proof. Let $h = g \circ f$. I. Suppose that both f and g are lower semi-continuous correspondences. By 34 B.4 it suffices to show that if $z \in h[x]$ and W is a neighborhood of z in $\mathbf{E}^*h$ $(= \mathbf{E}^*g)$, then $U = h^{-1}[W]$ is a neighborhood of x in $\mathbf{D}^*h$ $(= \mathbf{D}^*f)$. Choose a y such that $y \in f[x]$ and $z \in g[y]$. Since g is lower semi-continuous, the set $V = g^{-1}[W]$ is a neighborhood of y in $\mathbf{D}g$, and hence $V \cap \mathbf{E}f$ is a neighborhood of y in $\mathbf{E}f \cap \mathbf{D}g$. Now, if $\mathbf{E}f \cap \mathbf{D}g$ is open in $\mathbf{E}f$, then $V \cap \mathbf{E}f$ is a neighborhood of y in $\mathbf{E}f$, and hence $f^{-1}[V \cap \mathbf{E}f]$ is a neighborhood of x because f is lower semi-continuous. Since $f^{-1}[g^{-1}[W] \cap \mathbf{E}f] = h^{-1}[W]$, the proof is complete. — II. Suppose that both f and g are upper semi-continuous and $\mathbf{D}g \cap \mathbf{E}f$ is closed in $\mathbf{E}f$. By

34 B.4 it suffices to show that if $x \in \mathbf{D}h$ and W is a neighborhood of $h[x]$ in $\mathbf{E}^*h$, then there exists a neighborhood U of x such that $h[U] \subset W$. Put $X = f[x] \cap \mathbf{D}g$. Clearly $g[X] = h[x]$. Since g is upper semi-continuous, by 34 B.4 there exists a neighborhood V of X in $\mathbf{D}g$ such that $g[V] \subset W$. Put $V_1 = (\mathbf{E}f - \mathbf{D}g) \cup V$. Since $\mathbf{E}f - \mathbf{D}g$ is open in $\mathbf{E}f$ (its complement $\mathbf{E}f \cap \mathbf{D}g$ is closed in $\mathbf{E}f$ by our assumption) and V is a neighborhood of $X = f[x] \cap \mathbf{D}g$, V_1 is a neighborhood of $f[x]$ in $\mathbf{E}f$. By 34 B.4 there exists a neighborhood U of x in $\mathbf{D}f$ such that $f[U] \subset V_1$. Since $g[V] = g[V_1]$ we obtain $h[U] = g[f[U]] \subset g[V_1] = g[V] \subset W$. The proof is complete.

Remark. Notice that theorem 34 B.11 is an immediate consequence of the preceding theorem. If h is a range-restriction of f, let us consider the correspondence $g = \mathrm{J}_{\mathbf{E}h} : \mathbf{E}^*f \to \mathbf{E}^*f$ which is bicontinuous. Clearly $h = g \circ f$; apply 34 B.14.

The preceding theorem was proved without using hyperspaces. Notice that f is upper semi-continuous if and only if the mapping $h_1 = \{x \to f[x]\} : \mathbf{D}f \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous, and g is upper semi-continuous if and only if the mapping $h_2 = \{y \to g[y]\} : \mathbf{D}g \to \mathbf{H}_+(\mathbf{E}^*g)$ is continuous. Clearly the composite $h_2 \circ h_1$ does not exist even if both f and g are full (with a trivial exception) because the elements of $\mathbf{E}^*h_1$ are subsets of $\mathbf{D}^*h_2$ (if $|\mathbf{E}^*f| = \mathbf{D}g$). Nevertheless, a proof based on hyperspaces is very simple but requires the following "symmetric" characterizations of semi-continuous and continuous correspondences by means of hyperspaces.

34 B.15. Theorem. *Let f be a domain-full correspondence. Then f is upper semi-continuous if and only if the mapping*

$$h_1 = \{X \to f[X]\} : \mathbf{H}_+(\mathbf{D}^*f) \to \mathbf{H}_+(\mathbf{E}^*f)$$

is continuous; f is lower semi-continuous if and only if the mapping

$$h_2 = \{X \to f[X]\} : \mathbf{H}_-(\mathbf{D}^*f) \to \mathbf{H}_-(\mathbf{E}^*f)$$

is continuous; and finally, f is continuous if and only if the mapping

$$h_3 = \{X \to f[X]\} : \mathbf{H}(\mathbf{D}^*f) \to \mathbf{H}(\mathbf{E}^*f)$$

is continuous.

Proof. Evidently the last statement is an immediate consequence of the first two statements. We shall prove the first statement only. If h_1 is continuous then $\{x \to f[x]\} : \mathbf{D}^*f \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous as the composite of two continuous mappings, namely the canonical embedding of $\mathbf{D}^*f$ into $\mathbf{H}_+(\mathbf{D}^*f)$ followed by h_1. Conversely, assuming that $\{x \to f[x]\} : \mathbf{D}^*f \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous, we shall show that h_1 is continuous. Let X be any element of $\mathbf{H}_+(\mathbf{D}^*f)$ and $Y = h_1X$, i.e. $Y = f[X]$. Let $\mathcal{U}$ be a neighborhood of Y in $\mathbf{H}_+(\mathbf{E}^*f)$. We must find a neighborhood $\mathcal{V}$ of X in $\mathbf{H}_+(\mathbf{D}^*f)$ such that $h_1[\mathcal{V}] \subset \mathcal{U}$, i.e. $Z \in \mathcal{V} \Rightarrow f[Z] \in \mathcal{U}$. Choose a canonical neighborhood $\exp' U \subset \mathcal{U}$ of Y in $\mathbf{H}_+(\mathbf{E}^*f)$; thus U is a neighborhood of Y. Since $\{x \to f[x]\} : \mathbf{D}^*f \to \mathbf{H}_+(\mathbf{E}^*f)$ is continuous, we can choose a neighborhood V of X such that $f[x] \in \exp' U$ for each $x \in V$, i.e. $f[x] \subset U$ for each $x \in V$, and hence $f[V] \subset U$. The collection $\mathcal{V} = \exp' V$ is a neighborhood of X in $\mathbf{H}_+(\mathbf{D}^*f)$, and evidently $h_1[\mathcal{V}] \subset \exp' U \subset \mathcal{U}$ which completes the proof.

34 B.16. Definition. The *product of a family* $\{f_a\}$ *of correspondences for closure spaces* is defined to be the correspondence f such that

$$\mathbf{D}^*f = \Pi\{\mathbf{D}^*f_a\}, \quad \mathbf{E}^*f = \Pi\{\mathbf{E}^*f_a\}$$

and that the graph of f is the relational product of the family $\{\operatorname{gr} f_a\}$; thus

$$\Pi\{f_a\} = \langle \Pi_{\mathrm{rel}}\{\operatorname{gr} f_a\}, \Pi\{\mathbf{D}^*f_a\}, \Pi\{\mathbf{E}^*f_a\}\rangle .$$

The definition of the product $f \times g$ is evident.

Remark. It is obvious that

$$(\Pi\{f_a\})^{-1} = \Pi\{f_a^{-1}\},$$

i.e. the inverse of a product is the product of inverses.

It turns out that the product of upper semi-continuous correspondences need not be an upper semi-continuous correspondence. For example let f be the mapping of the space Q of rational numbers into the one-point space (0). Clearly f^{-1} is an upper semi-continuous and inversely continuous full correspondence. On the other hand the product correspondence $g = f^{-1} \times (\mathsf{J} : \mathsf{Q} \to \mathsf{Q})$ is not upper semi-continuous because its inverse $g^{-1} = f \times (\mathsf{J} : \mathsf{Q} \to \mathsf{Q})$ followed by the homeomorphism $\{\langle 0, x\rangle \to x\} : (0) \times \mathsf{Q} \to \mathsf{Q}$ is the projection of $\mathsf{Q} \times \mathsf{Q}$ onto Q which is not closed (by 27 ex. 5) and therefore is not inversely upper semi-continuous. On the other hand,

34 B.17. *The product of a family of lower semi-continuous correspondences is a lower semi-continuous correspondence. The product of inversely lower semi-continuous correspondences is an inversely lower semi-continuous correspondence.*

Proof. By virtue of the remark following 34 B.16 the two statements are equivalent. We shall prove the first one. Let f be the product of a non-void family $\{f_a \mid a \in A\}$ of lower semi-continuous correspondences. Without loss of generality we may and shall assume that all the f_a are full, and hence that f is full. Let $x \in \mathbf{D}f$, $y \in f[x]$, and let V be a neighborhood of y in $\mathbf{E}^*f$. We must show that $f^{-1}[V]$ is a neighborhood of x in $\mathbf{D}^*f$. Choose a canonical neighborhood W of y such that $W \subset V$; thus

$$W = \mathbf{E}\{z \mid z \in \mathbf{E}f,\ a \in A' \Rightarrow \operatorname{pr}_a z \in W_a\},$$

where A' is a finite subset of A and W_a is a neighborhood of $\operatorname{pr}_a y$ in $\mathbf{E}^*f_a$ for each a. By our assumption $f_a^{-1}[W_a]$ is a neighborhood of $\operatorname{pr}_a x$ in $\mathbf{D}^*f_a$ for each a in A'. Since each f_a is full we have $f_a^{-1}[|\mathbf{E}^*f_a|] = |\mathbf{D}^*f_a|$ for each a. Consequently

$$U = \mathbf{E}\{t \mid t \in \mathbf{D}f,\ a \in A' \Rightarrow \operatorname{pr}_a t \in f_a^{-1}[W_a]\}$$

is a neighborhood of x in $\mathbf{D}^*f$. But clearly $U = f^{-1}[W] \subset f^{-1}[V]$. The proof is complete.

Remark. A *correspondence* f is said to be *proper* if f is a lower semi-continuous correspondence such that, for each simultaneously lower semi-continuous and inversely upper semi-continuous correspondence g, the product correspondence $f \times g$ is inversely upper semi-continuous. It turns out that the product of any family of proper correspondences is a proper correspondence. It should be noted that proper mappings are often called perfect mappings. We do not intend to present the theory of proper correspondences though this theory is very important and interesting. We only note that a full proper correspondence carries over many properties of the domain carrier to the range carrier and conversely.

C. QUOTIENTS OF TOPOLOGIZED ALGEBRAIC STRUCTS

A surjective inductive generating mapping f need not be inversely lower semi-continuous. On the other hand if the closure structures in question are compatible for some group structures such that f is a homomorphism, then f is inversely lower semi-continuous and, in fact, the following somewhat more general result is true.

34 C.1. Theorem. *Let f be a quotient homomorphism of a topologized group $\mathscr{G} = \langle G, \sigma, u \rangle$ onto a topologized group $\mathscr{H} = \langle H, \mu, v \rangle$, i.e. the mapping f: : $\langle G, u \rangle \to \langle H, v \rangle$ is a quotient mapping and $f : \langle H, \sigma \rangle \to \langle H, \mu \rangle$ is a homomorphism. If the inversion of $\mathscr{G}$ is continuous then the inversion of $\mathscr{H}$ is also continuous. If $\mathscr{G}$ is inductively continuous, then $\mathscr{H}$ is inductively continuous and f is inversely lower semi-continuous, in particular, f is open. If $\mathscr{G}$ is continuous, then $\mathscr{H}$ is continuous. If $\mathscr{G}$ is a topological group, then $\mathscr{H}$ is a topological group.*

Proof. I. Since f is a homomorphism, we have

$(*)\ f \circ \sigma = \mu \circ (f \times f)$,

and if g is the inversion of $\mathscr{G}$ and h is the inversion of $\mathscr{H}$, then

$(**)\ f \circ g = h \circ f.$

If g is continuous, then $f \circ g$ is continuous, by $(**)$ $h \circ f$ is continuous, and f being a surjective quotient mapping, h is continuous. Thus, if the inversion of $\mathscr{G}$ is continuous, then the inversion of $\mathscr{H}$ is continuous.

II. Suppose that $\langle \sigma, u \rangle$ is inductively continuous, i.e. let the mapping $\sigma' = \sigma$: : ind $(\langle G, u \rangle \times \langle G, u \rangle) \to \langle G, u \rangle$ be continuous, $\mu' = \mu$: ind $(\langle H, v \rangle \times \langle H, v \rangle) \to$ $\to \langle H, v \rangle$ and $f' = f : \langle G, u \rangle \to \langle H, v \rangle$. The equality $(*)$ implies $f' \circ \sigma' = \mu' \circ$ ind $(f' \times f')$ where ind $(f' \times f')$ is the inductive product of f' and f'. Since f' is a quotient mapping, ind $(f' \times f')$ is a quotient mapping and therefore, to prove that μ' is continuous, it is sufficient to show that $\mu' \circ$ ind $(f' \times f')$ is continuous and this follows from the above equality because σ' and f' are continuous. Thus $\langle \mu, v \rangle$ is inductively continuous.

III. If $f \times f$ is a quotient mapping then the equality $(*)$ implies that $\langle \mu, v \rangle$ is continuous whenever $\langle \sigma, u \rangle$ is continuous. The product of quotient mappings need not be a quotient mapping but the product of two continuous inversely lower semi-continuous mappings is a continuous inversely lower semi-continuous and hence a quotient mapping, and this is our case.

IV. Assuming that $\langle \sigma, u \rangle$ is inductively continuous we shall prove that f is inversely lower semi-continuous. Let U be a neighborhood of a point x of $\langle G, u \rangle$; we must show that $f[U]$ is a neighborhood of fx, and f being a quotient mapping, it is sufficient to prove that $f^{-1}[f[U]]$ is a neighborhood of $f^{-1}[fx]$. Clearly $f^{-1}[fx] = x\sigma[K]$ and $f^{-1}[f[U]] = [U]\,\sigma[K]$ where K is the inverse image under f of the neutral element of $\mathscr{H}$. If y is any element of K then $[U]\,\sigma y$ is a neighborhood of $x\sigma y$ because U is a neighborhood of x and the mapping $\{z \to z\sigma y\} : \langle G, u \rangle \to \langle G, u \rangle$ is a homeomorphism which carries U into $[U]\,\sigma y$ and x into $x\sigma y$. Since $[U]\,\sigma y \subset [U]\,\sigma[K]$, $[U]\,\sigma[K]$ is a neighborhood of each element of $x\sigma[K]$.

V. If $\mathscr{G}$ is a topological group, then the inversion of $\mathscr{H}$ is continuous by I and $\mathscr{H}$ is continuous by III, and therefore $\mathscr{H}$ is a topological group.

Remark. Remember that a surjective quotient mapping is an inductive generating mapping and therefore we may apply the theorems of Section 33, in particular, of subsection 33 C.

Assume that $\langle G, \sigma \rangle$ is a group; now if H is any subgroup of $\langle G, \sigma \rangle$ such that $x\sigma[H] = [H]\,\sigma x$ for each x, then the relation $\varrho = \mathbf{E}\{\langle x, y \rangle \mid x\sigma y^{-1} \in H\}$ is stable under σ and hence there exists a unique group structure μ on the quotient set G/ϱ such that the mapping $\{x \to x\sigma[H]\} : \langle G, \sigma \rangle \to \langle G/\varrho, \mu \rangle$ is a homomorphism. Such a subgroup H (which is an ideal in the sense of 8 D.4) is often said to be an invariant subgroup of $\langle G, \sigma \rangle$ and the group $\langle G/\varrho, \mu \rangle$ is said to be the quotient of $\langle G, \sigma \rangle$ under H and is denoted by $\langle G, \sigma \rangle / H$.

34 C.2. *Let $\mathscr{G} = \langle G, \sigma, u \rangle$ be a topological group and let H be an invariant subgroup of $\langle G, \sigma \rangle$. Let v be the closure inductively generated by the canonical mapping $\{x \to [H]\,\sigma x\}$ of G onto $\langle G, \sigma \rangle / H$. By* 34 C.1 *the topologized group $\langle \langle G, \sigma \rangle / H; v \rangle$ is a topological group which will be denoted by $\mathscr{G}/H$ and will be called the quotient of $\mathscr{G}$ under H. The mapping $\{x \to [H]\,\sigma x\}$ of $\mathscr{G}$ onto $\mathscr{G}/H$ will be called the canonical mapping of $\mathscr{G}$ onto $\mathscr{G}/H$. By* 34 C.1 *the canonical mapping of $\mathscr{G}$ onto $\mathscr{G}/H$ is inversely lower semi-continuous, i.e. open ($\mathscr{G}$ is a topological space).*

A topological group is separated if and only if it is semi-separated, and therefore the following proposition holds.

34 C.3. *A quotient $\mathscr{G}/H$ of a topological group $\mathscr{G}$ is separated if and only if H is closed in $\mathscr{G}$.*

34 C.4. *Let $\mathscr{G}$ be a topological group and let H be the closure of the neutral element e of $\mathscr{G}$. Then H is an invariant subgroup of $\mathscr{G}$ and the quotient group $\mathscr{G}/H$ is separated.*

Proof. Let us consider an inner automorphism f of $\mathscr{G}$. Since $fe = e$ and f is continuous we have that $f\overline{(e)} \subset \overline{(fe)} = \overline{(e)}$, which shows that H is stable under each automorphism of $\mathscr{G}$. Since (e) is a subgroup, the closure H of (e) is a subgroup also. Thus H is an invariant subgroup of $\mathscr{G}$. The quotient group $\mathscr{G}/H$ is separated by 34 C.3.

34 C.5. *The quotient group $\mathscr{G}/H$ is discrete if and only if H is open in $\mathscr{G}$.*

34 C.6. *Let f be the canonical mapping of a topological group G (written multiplicatively) onto its quotient group G/H. If K is any subgroup of G, then $f[K]$ is a subgroup of G/H, $[K] . [H]$ a subgroup of G, H is an invariant subgroup of $[K] . [H]$ and the topological groups $[K] . [H]/H$ and $f[K]$ are isomorphic.*

Proof. Evidently $f[K] = f[[K] . [H]]$ and therefore without loss of generality we may and shall assume $K = [K] . [H]$, i.e. $K \supset H$. We have $K = f^{-1}[f[K]]$ and therefore, by 33 A.7, the restriction g of the inductive generating mapping (see the remarks following 34 C.1) f to a mapping of K onto $f[K]$ is an inductive generating mapping and hence a quotient mapping. Now if h is the canonical mapping of K onto K/H then there exists a unique mapping k such that $h \circ k = g$; in addition k is bijective. Since h and g are inductive generating mappings, k is a homeomorphism (33 C.5).

34 C.7. Theorem. *Let f be a quotient homomorphism of a topological ring $\mathscr{R}_1 = \langle R_1, \sigma_1, \mu_1, u_1 \rangle$ onto a topologized ring $\mathscr{R}_2 = \langle R_2, \sigma_2, \mu_2, u_2 \rangle$. Then $\mathscr{R}$ is a topological ring.*

Proof. By 34 C.1 the group $\langle R_2, \sigma_2, u_2 \rangle$ is topological and f is inversely lower semi-continuous. Since f is a homomorphism we have $\operatorname{gr} f \circ \mu_1 = \mu_2 \circ (\operatorname{gr} f \times \operatorname{gr} f)$ and consequently $f' \circ \mu_1' = \mu_2' \circ (f' \times f')$, where $f' = f : \langle R_1, u_1 \rangle \to \langle R_2, u_2 \rangle$, and $\mu_i' = \mu_i : \langle R_i, u_i \rangle \times \langle R_i, u_i \rangle \to \langle R_i, u_i \rangle$, $i = 1, 2$. Since μ_1' is continuous, $f' \circ \mu_1'$ is also continuous and therefore, by the above equality, $\mu_2' \circ (f' \times f')$ is continuous. Since f' is continuous and inversely lower semi-continuous, the product $f' \times f'$ has the same property and hence $f' \times f'$ is a quotient mapping. Since $f' \times f'$ is surjective, $f' \times f'$ is an inductive generating mapping and the continuity of $\mu_2' \circ (f' \times f')$ implies that μ_2' is continuous. Thus the multiplication of $\mathscr{R}_2$ is continuous and hence $\mathscr{R}_2$ is a topological ring.

Remark. We leave to the reader the simple task of defining the concept of a quotient of a topological ring.

34 C.8. Theorem. *Let f be a quotient homomorphism of a topological module (algebra) $\mathscr{L}_1$ over a topological ring $\mathscr{R}$ onto a topologized module (algebra) $\mathscr{L}_2$ over $\mathscr{R}$. Then $\mathscr{L}_2$ is a topological module (algebra) over $\mathscr{R}$.*

Proof. According to 34 C.7 it remains to prove that the external multiplication of $\mathscr{L}_2$ is continuous. Denoting by ϱ_i' the topologized external multiplication of $\mathscr{L}_i$ we have $f \circ \varrho_1' = \varrho_2' \circ ((\mathrm{J} : \mathscr{R} \to \mathscr{R}) \times f)$. Since f and $\mathrm{J} : \mathscr{R} \to \mathscr{R}$ are simultaneously continuous and inversely lower semi-continuous, this product has the same property and the continuity of ϱ_2' follows by the same argument as the continuity of μ_2' in the proof of 34 C.7.

Remark. We leave to the reader as a simple task the definition of the concept of a quotient of a topological module or algebra.

D. EXAMPLES AND REMARKS

We know that each of the following conditions is necessary and sufficient for a mapping f of a closure space $\langle P, u\rangle$ into a closure space $\langle Q, v\rangle$ to be continuous:

$$(*) \qquad X \subset P \Rightarrow f[uX] \subset vf[X]$$

$$(**) \qquad Y \subset Q \Rightarrow uf^{-1}[Y] \subset f^{-1}[vY].$$

If f is a correspondence then the conditions $(*)$ and $(**)$ are not equivalent because $(*)$ is equivalent to the lower semi-continuity of f and $(**)$ is equivalent to the upper semi-continuity of f as stated in the following theorem.

34 D.1. Theorem. *Let f be a correspondence on a closure space $\langle P, u\rangle$ ranging in a closure space $\langle Q, v\rangle$. Then f is lower semi-continuous (upper semi-continuous) if and only if condition $(*)$ (condition $(**)$) is fulfilled.*

Proof. I. Assuming $(*)$, if $x \in u(P - f^{-1}[V])$, then $f[x] \subset vf[P - f^{-1}[V]]$ and clearly $V \cap f[P - f^{-1}[V]] = \emptyset$. Consequently, if V is a neighborhood of a point y of $f[x]$, then $f^{-1}[V]$ is a neighborhood of x, which shows that f is lower semi-continuous (by 34 B.4). Conversely, if f is lower semi-continuous, then $f^{-1}[V]$ is a neighborhood of each x such that V is a neighborhood of a point of $f[x]$ (by 34 B.4). If $y \notin vf[X]$, then $V = Q - f[X]$ is a neighborhood of y, and hence $U = f^{-1}[Q - - f[X]]$ is a neighborhood of each point of $f^{-1}[y]$. Since clearly $U \cap X = \emptyset$, no point of $f^{-1}[y]$ belongs to uX, and hence $y \notin f[uX]$. — II. Assuming $(**)$ we shall prove that, for each $x \in P$ and each neighborhood V of $f[x]$ in $\langle Q, v\rangle$, there exists a neighborhood U of x such that $f[U] \subset V$. By 34 B.4 f will be upper semi-continuous. Put $U = P - f^{-1}[Q - V]$. Clearly $f[U] \cap (Q - V) = \emptyset$ and hence $f[U] \subset V$. Since $f[x] \cap v(Q - V) = \emptyset$, $(**)$ gives $x \notin uf^{-1}[Q - V]$ which means that U is a neighborhood of x. Finally, suppose that f is upper semi-continuous and $x \notin f^{-1}[vY]$. We must show that $x \notin uf^{-1}[Y]$. Clearly the set $V = Q - Y$ is a neighborhood of each point of $f[x]$. By 34 B.4 we can choose a neighborhood U of x such that $f[U] \subset \subset V$. Clearly $U \cap f^{-1}[Y] = \emptyset$ which shows that $x \notin uf^{-1}[Y]$.

34 D.2. Theorem. *Let f be a correspondence on a closure space $\langle P, u\rangle$ ranging on a closure space $\langle Q, v\rangle$. Then f is simultaneously lower semi-continuous and inversely upper semi-continuous if and only if*

$$X \subset P \quad \textit{implies} \quad f[uX] = vf[X],$$

and f is simultaneously upper semi-continuous and inversely lower semi-continuous if and only if

$$Y \subset Q \quad \textit{implies} \quad f^{-1}[vY] = uf^{-1}[Y].$$

Proof. We shall prove only the first statement. Let $X \subset P$. If f is lower semi-continuous, then $f[uX] \subset vf[X]$ (by 34 D.1). If f is inversely upper semi-continuous, i.e. f^{-1} is upper semi-continuous, then applying the foregoing theorem to f^{-1} we obtain $f[uX] \supset vf[X]$ (of course $(f^{-1})^{-1} = f$).

Remark. A mapping f of a space $\langle P, u\rangle$ onto a space $\langle Q, v\rangle$ is a quotient mapping if and only if $vf[X] = f[uX]$ for each set X such that $f^{-1}[f[X]] = X$. Now it is evident from 34 D.2 that every simultaneously continuous and inversely upper or lower semi-continuous mapping is a quotient mapping.

Now we restrict our attention to mappings. We have shown (34 B.9 (c) or the preceding remark) that every continuous inversely upper or lower semi-continuous mapping is a quotient mapping. On the other hand we have shown (34 B.9 (d)) that a quotient mapping need not be inversely upper or lower semi-continuous. Inversely upper or lower semi-continuous quotient mappings form a very important class of quotient mappings because many properties of domain carriers are carried over to range carriers and conversely. First we shall introduce the current terminology for quotient spaces under an equivalence and we shall give an interesting characterization of the concepts introduced.

34 D.3. Definition. An *equivalence* ϱ on a closure space $\mathscr{P}$ is said to be *upper semi-continuous, lower semi-continuous* or *continuous* if the canonical mapping of $\mathscr{P}$ onto $\mathscr{P}/\varrho$ is, respectively, inversely upper semi-continuous, inversely lower semi-continuous or continuous. A *decomposition* $\mathscr{D}$ of a non-void space $\mathscr{P}$ (i.e. a disjoint cover of $\mathscr{P}$ which is a collection consisting of non-void sets) is said to be *upper semi-continuous, lower semi-continuous* or *continuous* if the equivalence $\bigcup\{D \times D \mid D \in \mathscr{D}\}$ possesses the corresponding property.

34 D.4. Theorem. *An equivalence ϱ on $\mathscr{P}$ is upper semi-continuous, lower semi-continuous or continuous if and only if the quotient $\mathscr{P}/\varrho$ is a subspace of $\mathbf{H}_+(\mathscr{P})$, $\mathbf{H}_-(\mathscr{P})$ or $\mathbf{H}(\mathscr{P})$ respectively.*

It is more convenient to prove the following somewhat more general statement.

34 D.5. *A continuous mapping f of a closure space $\mathscr{P}$ onto a closure space $\mathscr{Q}$ is inversely upper semi-continuous, inversely lower semi-continuous or inversely continuous if and only if the mapping $\{y \to f^{-1}[y]\} : \mathscr{Q} \to \mathbf{H}_+(\mathscr{P})$, $\{y \to f^{-1}[y]\} : \mathscr{Q} \to \mathbf{H}_-(\mathscr{P})$ or $\{y \to f^{-1}[y]\} : \mathscr{Q} \to \mathbf{H}(\mathscr{P})$ respectively, is an embedding.*

Proof. Denote by σ the relation $\{y \to f^{-1}[(y)] \mid y \in |\mathscr{Q}|\}$. By definition, regardless of the continuity of f, the mapping $\sigma : \mathscr{Q} \to \mathbf{H}_+(\mathscr{P})$ is continuous if and only if the mapping f is inversely upper semi-continuous, and similarly for $\sigma : \mathscr{Q} \to \mathbf{H}_-(\mathscr{Q})$ and $\sigma : \mathscr{Q} \to \mathbf{H}(\mathscr{Q})$. Consequently "if" is evident in all three statements. "Only if" will be proved for inverse upper semi-continuity only. Suppose that the mapping $g = \sigma : \mathscr{Q} \to \mathbf{H}_+(\mathscr{P})$ is continuous. Let j be the canonical embedding of $\mathscr{Q}$ into $\mathbf{H}_+(\mathscr{Q})$ and let $\mathscr{Q}'$ be the subspace of $\mathbf{H}_+(\mathscr{Q})$ whose underlying set is $\mathbf{E}j$. Let g' be the mapping of $\mathscr{Q}'$ into $\mathbf{H}_+(\mathscr{P})$ such that $gy = g'jy$ for each y in $|\mathscr{Q}|$. Since j is an embedding, g' is continuous, and to prove that g is an embedding it is sufficient to show that g' is an embedding. Since the mapping f is continuous, f is upper semi-continuous and therefore, by 34 B.15, the mapping

$$h = \{X \to f[X]\} : \mathbf{H}_+(\mathscr{P}) \to \mathbf{H}_+(\mathscr{Q})$$

is continuous. If $\eta \in |\mathcal{Q}'|$, i.e. $\eta = (y)$, $y \in |\mathcal{Q}|$, then $hg'\eta = \eta$. Thus g' is an embedding.

Remark. If the mapping $\{y \to f^{-1}[y]\} : \mathcal{Q} \to \mathbf{H}_+(\mathcal{P})$ in an embedding then f need not be continuous.

34 D.6. By 34 B.9 (e) the quotient of a topological space $\mathcal{P}$ under an upper or lower semi-continuous equivalence ϱ is a topological space. A new proof can be obtained from 34 D.4. Indeed, by 34 A.4 the hyperspaces of a topological space are topological, and hence $\mathcal{P}/\varrho$ is a topological space as the subspace of a topological space.

The domain-restriction of a quotient mapping f to a subspace $\mathcal{R}$ of $\mathbf{D}^*f$ need not be a quotient mapping even if $\mathcal{R}$ is both closed and open in $\mathbf{D}^*f$. E.g., if a closure u for a set P is the supremum of a family $\{u_a\}$ and $u_a \neq u$ for each a, then $\langle P, u\rangle$ is inductively generated by the family of mappings $\{J : \langle P, u_a\rangle \to \langle P, u\rangle\}$ and hence the reduced sum f of this family is an inductive generating mapping for $\langle P, u\rangle$ and the domain carrier of f is the sum space $\mathcal{Q} = \Sigma\{\langle P, u_a\rangle\}$. Clearly the restriction f_a of f to each subspace $(a) \times P$ of $\mathcal{Q}$ is an injective mapping which is not an embedding and hence f_a is not a quotient mapping. Clearly $(a) \times P$ is both open and closed in $\mathcal{Q}$.

On the other hand, if f is an inversely upper (lower) semi-continuous quotient mapping and $\mathcal{R}$ is a closed (open) subspace of $\mathbf{D}^*f$, then $f|\mathcal{R}$ is both continuous and inversely upper (lower) semi-continuous mapping (34 B.13) and therefore a quotient mapping.

Let f be a mapping of a space $\langle P, u\rangle$ onto a space $\langle Q, v\rangle$, Y_1 and Y_2 be subsets of Q and let $X_i = f^{-1}[Y_i]$, $i = 1, 2$. If f is continuous and Y_1 and Y_2 are separated or semi-separated in $\langle Q, v\rangle$, then X_1 and X_2 have the same property in $\langle P, u\rangle$. If f is a quotient mapping and X_1 and X_2 are semi-separated in $\langle P, u\rangle$, then Y_1 and Y_2 are semi-separated in $\langle Q, v\rangle$; however, a similar result for separated sets is not true.

34 D.7. *Let f be an inversely upper semi-continuous mapping of a space $\mathcal{P}$ onto a space $\mathcal{Q}$, Y_1 and Y_2 be subsets of $\mathcal{Q}$ and $X_i = f^{-1}[Y_i]$, $i = 1, 2$. If X_1 and X_2 are separated in $\mathcal{P}$, then Y_1 and Y_2 are separated in $\mathcal{Q}$.*

Proof. Let U_i be a neighborhood of X_i in $\mathcal{P}$, $i = 1, 2$, such that $U_1 \cap U_2 = \emptyset$. Let V_i be the set of all $y \in |\mathcal{Q}|$ such that $f^{-1}[y] \subset U_i$. Since $U_1 \cap U_2 = \emptyset$ we have $V_1 \cap V_2 = \emptyset$. Since $f^{-1}[f[X_i]] = X_i$ we have $f[X_i] = Y_i \subset V_i$. Finally, since f is inversely upper semi-continuous, V_i is a neighborhood of Y_i in $\mathcal{Q}$ (by 34 B.8).

From this fact we shall derive the following results.

34 D.8. *Let ϱ be an upper semi-continuous equivalence on a space $\mathcal{P}$. Then*

(a) *$\mathcal{P}/\varrho$ is separated if and only if each two distinct fibres are separated in $\mathcal{P}$.*

(b) *If any neighborhood of any fibre $\varrho[x]$ contains the closure of a neighborhood of $\varrho[x]$ and each fibre is closed, then $\mathcal{P}/\varrho$ is regular and separated.*

(c) *If each fibre is closed and $\mathcal{P}$ is normal (hereditarily normal) then $\mathcal{P}/\varrho$ is a separated normal (hereditarily normal) space.*

Proof. Let f be the canonical mapping of $\mathcal{P}$ onto $\mathcal{P}/\varrho$.

I. The "only if" part of (a) follows from continuity of f and "if" follows from 34 D.7.

II. The assumptions of (b) imply that any two distinct fibres are separated; by (a) $\mathscr{P}/\varrho$ is separated. Next, if $\varrho[x]$ does not belong to the closure of a set $\mathscr{X}$ in $\mathscr{P}/\varrho$ then $|\mathscr{P}| - f^{-1}[\mathscr{X}]$ is a neighborhood of $\varrho[x]$ in $\mathscr{P}$ and hence the sets $f^{-1}[\mathscr{X}]$ and $\varrho[x]$ are separated (by our assumption). By 34 D.7 the sets $\mathscr{X}$ and $(\varrho[x])$ are separated in $\mathscr{P}/\varrho$, which shows that $\mathscr{P}/\varrho$ is regular.

III. Let $\mathscr{P}$ be normal and let the fibres be closed. By 34 D.6 $\mathscr{P}/\varrho$ is topological, and by (b) $\mathscr{P}/\varrho$ is a separated regular space. If Y_1 and Y_2 are disjoint closed subsets of $\mathscr{P}/\varrho$, then $X_1 = f^{-1}[Y_1]$ and $X_2 = f^{-1}[Y_2]$ are disjoint closed subsets of $\mathscr{P}$ (because f is continuous) and hence, $\mathscr{P}$ being normal, X_1 and X_2 are separated. Since f is inversely upper semi-continuous, the sets Y_1 and Y_2 are separated (by 34 D.7) which shows that $\mathscr{P}/\varrho$ is normal.If $\mathscr{P}$ is hereditarily normal, then $\mathscr{P}/\varrho$ is normal (this was just proved) and hence by 30 A.4, it remains to show that any two semi-separated subsets Y_1 and Y_2 of $\mathscr{P}/\varrho$ are separated in $\mathscr{P}/\varrho$. The sets $X_1 = f^{-1}[Y_1]$ and $X_2 = f^{-1}[Y_2]$ are semi-separated in $\mathscr{P}$ because f is continuous. Again by 30 A.4 the sets X_1 and X_2 are separated, and finally, by 34 D.7 the sets Y_1 and Y_2 are separated.

34 D.9. *Let ϱ be a lower semi-continuous equivalence on a closure space $\mathscr{P}$. Then:*

(a) *The local character of $\mathscr{P}/\varrho$ is less than or equal to the local character of $\mathscr{P}$.*

(b) *If $\mathscr{P}$ is topological then $\mathscr{P}/\varrho$ is topological and the total character of $\mathscr{P}/\varrho$ is less than or equal to the total character of $\mathscr{P}$.*

(c) *If ϱ is a closed subset of $\mathscr{P} \times \mathscr{P}$, then $\mathscr{P}/\varrho$ is separated.*

Proof. Denote by f the canonical mapping of $\mathscr{P}$ onto $\mathscr{P}/\varrho$. I. If $\mathscr{U}$ is a local base at x, then clearly the set of all $f[U]$, $U \in \mathscr{U}$, is a local base at fx. – II. If $\mathscr{P}$ is topological then $\mathscr{P}/\varrho$ is topological (35 D.6). If $\mathscr{B}$ is an open base for $\mathscr{P}$, then clearly the set of all $f[B]$, $B \in \mathscr{B}$, is an open base for $\mathscr{P}/\varrho$. – III. The mapping $f \times f$ is continuous and inversely lower semi-continuous and ϱ is the inverse image under $f \times f$ of the diagonal of $\mathscr{P}/\varrho \times \mathscr{P}/\varrho$. Consequently the diagonal is closed which implies that $\mathscr{P}/\varrho$ is separated.

Remark. If $\mathscr{P}/\varrho$ is separated, then ϱ is closed because ϱ is the inverse image under $f \times f$ of the diagonal of $\mathscr{P}/\varrho \times \mathscr{P}/\varrho$, and the diagonal is closed because $\mathscr{P}/\varrho$ is separated.

35. CONVERGENCE

This section is devoted to questions related to the definition of a closure space by specifying which nets converge to which points.

We know that a closure space $\mathscr{P}$ is entirely determined by the relation **Lim** $\mathscr{P}$ (called the convergence class of $\mathscr{P}$) consisting of all $\langle N, x\rangle$ such that N is a net which converges to x in $\mathscr{P}$: namely $x \in \overline{X}$ if and only if there exists a net N in X which converges to x in $\mathscr{P}$. In subsection A we shall give a necessary and sufficient condition for a relation to be the convergence class of a closure space, and also a necessary and sufficient condition for a relation to be the convergence class of a topological space. A subclass $\mathscr{C}$ of **Lim** $\mathscr{P}$ is said to be a determining convergence relation for $\mathscr{P}$ provided that $x \in \overline{X}$ if and only if there exists a pair $\langle N, x\rangle$ in $\mathscr{C}$ such that N ranges in X and N converges to x in $\mathscr{P}$. In subsection A we shall also study the following question: given a determining convergence relation $\mathscr{C}$ for a closure space $\mathscr{P}$, how can one reconstruct the convergence class of $\mathscr{P}$ from $\mathscr{C}$.

Subsection B is concerned with the development of the properties of those closure spaces, the so-called **S**-spaces, which permit a determining convergence relation whose domain consists of sequences, and of a special kind of **S**-spaces, the so-called **L**-spaces, which are characterized among all **S**-spaces by the condition that each sequence has at most one limit point. Roughly speaking, **L**-spaces are related to **S**-spaces as separated closure spaces to closure spaces.

In subsection C the class **S** of all **S**-spaces will be studied. It turns out that the class **S** is inductive-stable and hence every space $\mathscr{P}$ has a lower modification in **S**, which is denoted by $\sigma\mathscr{P}$ and called the sequential modification of $\mathscr{P}$. In 33 B we investigated inductive constructions in a projective-stable class of spaces (e.g. topologically inductively generated closure operations). Here we are in an appropriate situation to introduce projective constructions in an inductive-stable class K, in particular, the product in K of a family of spaces in K. If $K = \mathbf{S}$ then the product in K is called the sequential product. It is easily seen that the sequential product of a family $\{\mathscr{P}_a\}$ of spaces is the sequential modification of the usual product, i.e. $\sigma\,\Pi\{\mathscr{P}_a\}$. The sequential product will be used in the definition of a sequential group, i.e. of a topologized group such that the inversion is continuous and the group multiplication is continuous as a mapping of $\sigma(\mathscr{G} \times \mathscr{G})$ into $\mathscr{G}$. This will be performed in subsection D.

In subsection E the spaces which are sequential modifications of uniformizable spaces are investigated. The class of all these spaces is projective-stable in the class **S** and consists of all spaces which are projectively generated by a mapping into $\sigma(\mathsf{R}^{\aleph})$.

Let $\langle N, \leqq \rangle$ be a net in a closure space $\mathscr{P}$ and let x be a point of $\mathscr{P}$. Let Q be a set consisting of the points of $\mathbf{D}N$ and a further point, say y, and let us consider the closure v on Q such that $\mathbf{D}N$ is a discrete open subspace of $\langle Q, v \rangle$ and $U \subset Q$ is a neighborhood of y if and only if $y \in U$ and $U \cap \mathbf{D}N$ is residual in $\langle \mathbf{D}N, \leqq \rangle$. Finally, let f be the mapping of $\langle Q, v \rangle$ into $\mathscr{P}$ which assigns to each $z \in \mathbf{D}N$ the point Nz, and $fy = x$. It is easily seen that N converges to x in $\mathscr{P}$ if and only if the mapping f is continuous. Consequently, subsections A–D can be regarded as a part of the theory of inductive constructions. This statement will be made more precise in subsection F, where also the spaces inductively generated by countable subspaces will be studied.

Recall that all nets are assumed to be directed.

A. CONVERGENCE CLASSES

35 A.1. Definition. The *convergence class of a closure space* $\mathscr{P}$ (*of a closure operation* u) denoted by $\mathbf{Lim}\,\mathscr{P}$ ($\mathbf{Lim}\, u$) is the relation consisting of all pairs $\langle N, x \rangle$ such that N is a net converging to x in $\mathscr{P}$ (relative to u). A *convergence class* is the convergence class of a space. A *convergence relation* is a relation $\mathscr{C}$ ranging in a set such that $\mathbf{D}\mathscr{C}$ is a class of nets.

Of course every convergence class is a convergence relation and $\mathbf{Lim}\,\langle P, u \rangle = \mathbf{Lim}\, u$ for any space $\langle P, u \rangle$. It is sometimes convenient to denote the convergence class of a space $\mathscr{P}$ by $\mathbf{Lim}_{\mathscr{P}}$ instead of by $\mathbf{Lim}\,\mathscr{P}$; then $\mathbf{Lim}_{\mathscr{P}}[N]$ denotes the set consisting of all limit points of N in $\mathscr{P}$, and if $\mathbf{Lim}_{\mathscr{P}}$ is single-valued, i.e. if $\mathscr{P}$ is separated, then we can write $\mathbf{Lim}_{\mathscr{P}} N$ to denote the unique element of $\mathbf{Lim}_{\mathscr{P}}[N]$, and this is the usual notation. Of course, if $\mathscr{P}$ is uniquely determined by the context, then $\mathscr{P}$ is not indicated and both notations coincide; we write $\mathbf{Lim}$, $\mathbf{Lim}\,[N]$ and $\mathbf{Lim}\, N$.

First let us recall the main properties of convergence classes.

35 A.2. Theorem. *Let* $\mathscr{C}$ *be the convergence class of a closure space* $\langle P, u \rangle$. *Then*

(a) *If* N *is a constant net in* $\mathbf{E}\mathscr{C}$ *and* x *is the only value of* N, *then* $\langle N, x \rangle \in \mathscr{C}$.

(b) $\mathbf{E}\mathscr{C} = P$.

(c) $x \in uX$ *if and only if* $X \subset \mathbf{E}\mathscr{C}$ *and there is a* $\langle N, x \rangle$ *in* $\mathscr{C}$ *such that* $\mathbf{E}N \subset X$.

(d) *If* M *is a generalized subnet of* N *and* $\langle N, x \rangle \in \mathscr{C}$, *then also* $\langle M, x \rangle \in \mathscr{C}$.

By statement (c) of this theorem a space can be reconstructed from its convergence class, that is, a space is completely determined by its convergence class. As a consequence, every concept based on the concept of a closure space can, in principle, be described in terms of convergence classes, and in particular a space $\mathscr{P}$ can be determined by specifying $\mathbf{Lim}\,\mathscr{P}$. Of course, the descriptions in terms of convergence classes are not always convenient and appropriate, though in some cases they are of great importance; recall, e.g., the simple description (cf. 32 A.6) of projectively generated closure operations. Further examples of such convenient descriptions are listed in the theorem which follows. The reader will easily find that all of these are merely restatements of earlier results.

35 A.3. Theorem. (a) *A mapping f of a space $\mathscr{P}$ into a space $\mathscr{Q}$ is continuous if and only if the relation* $\{\langle N, x\rangle \to \langle f \circ N, fx\rangle \mid \langle N, x\rangle \in \mathbf{Lim}\,\mathscr{P}\}$ *ranges in* $\mathbf{Lim}\,\mathscr{Q}$. *In particular, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are convergence classes such that $\mathbf{E}\mathscr{C}_1 = \mathbf{E}\mathscr{C}_2$, then $\mathscr{C}_1 \subset \mathscr{C}_2$ if and only if the closure corresponding to $\mathscr{C}_1$ is finer than that corresponding to $\mathscr{C}_2$; and more generally, the inclusion $\mathbf{Lim}\,\mathscr{P} \subset \mathbf{Lim}\,\mathscr{Q}$ is equivalent to this statement: the underlying set of $\mathscr{P}$ is a subset of that of $\mathscr{Q}$ and the closure of $\mathscr{P}$ is finer than the relativization of the closure of $\mathscr{Q}$.*

(b) *If $\{u_a\}$ is a non-void family of closures for a set P, then (see* 31 A.6*)* $\mathbf{Lim}(\inf\{u_a\})$ *consists of all $\langle N, x\rangle$ such that $\langle N, x\rangle \in \mathbf{Lim}\,u_a$ for each a; suggestively but incorrectly ($\mathbf{Lim}\,u_a$ is non-comprisable and hence $\{\mathbf{Lim}\,u_a\}$ does not exist),*

$$\mathbf{Lim}(\inf\{u_a\}) = \bigcap\{\mathbf{Lim}\,u_a\}\ .$$

(c) *A space $\mathscr{P}$ is separated if and only if the convergence class of $\mathscr{P}$ is single-valued (*27 A.6*).*

(d) *A space $\mathscr{P}$ is topological if and only if the convergence class of $\mathscr{P}$ satisfies the condition on iterated limits (*15 B.13*).*

If we want to define a concrete space by specifying the convergence class we must find a convenient sufficient condition for a given convergence relation to be a convergence class. Such conditions will be given in Theorem 35 A.17, and for topological spaces in Theorem 35 A.18. It is sometimes convenient to prescribe not the whole convergence class but only a sufficiently large subclass of the convergence class which completely determines the convergence class. The procedure is similar to that of the description of closure spaces by neighborhoods which has already been treated in subsection 14 B, and therefore we recall the main results in a formal fashion to point out the main ideas. The terminology introduced will be used only here.

35 A.4. Let us term the neighborhood relation of a closure u and denote by fu the relation consisting of all pairs $\langle U, x\rangle$ such that U is a neighborhood of x in the space $\langle P, u\rangle$ (P is uniquely determined by u). We know that the relation $\{u \to fu \mid u$ is a closure$\}$ is one-to-one. Next, if u is a closure then the relation fu has the following properties: (a) $x \in U$ for each $\langle U, x\rangle \in fu$, (b) $(fu)^{-1}[x]$ is a filter in $\mathbf{E}(fu)$ for each $x \in \mathbf{E}(fu)$, and (c) $\mathbf{E}(fu)$ is a set. Conversely, if a relation f possesses the properties (a), (b) and (c), then $f = fu$ for some u.

Notice that this fact has been often used to define a closure by specifying the neighborhood systems at points. We know that it is often convenient to specify not the entire neighborhood systems but local bases or local sub-bases only. The latter method is based on the following result: If σ is any relation ranging in a set such that $x \in U \subset \mathbf{E}\sigma$ for each $\langle U, x\rangle \in \sigma$, then there exists a smallest neighborhood relation

ϱ such that $\varrho \supset \sigma$ and $\mathbf{E}\varrho = \mathbf{E}\sigma$. This neighborhood relation can be described as follows: $\langle U, x\rangle \in \varrho$ if and only if U belongs to the smallest filter in $\mathbf{E}\sigma$ containing $\sigma^{-1}[x]$. The definition of spaces by specifying local bases is based on the following result: let σ be a relation as above, u the corresponding closure operation; hence fu is the corresponding smallest neighborhood relation containing σ. The following conditions are equivalent:

(a) $x \in uX \Leftrightarrow X \subset \mathbf{E}\sigma,\ x \in \mathbf{E}\sigma,\ \langle U, x\rangle \in \sigma \Rightarrow U \cap X \neq \emptyset$

(b) $fu = \{\langle U, x\rangle \mid V \subset U \subset \mathbf{E}\sigma$ for some $\langle V, x\rangle \in \sigma\}$

(c) $\sigma^{-1}[x]$ is a filter base in $\mathbf{E}\sigma$ for each $x \in \mathbf{E}\sigma$.

Thus the relation σ describes u "directly", that is, in the sense of (a), if and only if the neighborhood relation fu of u can be reconstructed from σ by (b). Condition (c), which is formulated in terms relating to σ only, is a characterization of those σ which directly describe a closure operation. We want to introduce a similar notion for convergence and obtain a similar result.

35 A.5. Definition. A *determining convergence relation for a closure space* $\langle P, u\rangle$ (*for a closure operation* u) is a convergence relation $\mathscr{C} \subset \mathbf{Lim}\,\langle P, u\rangle$ such that $x \in uX$ if and only if $X \subset P$ and $\langle N, x\rangle \in \mathscr{C}$ for some net N ranging in X. A *determining convergence relation* is a determining convergence relation for some space.

Every convergence class is a determining convergence relation, and a determining convergence relation $\mathscr{C}$ is a determining convergence relation for exactly one space $\langle P, u\rangle$; indeed, $P = \mathbf{E}\mathscr{C}$, and $x \in uX$ if and only if $\langle N, x\rangle \in \mathscr{C}$ for some N ranging in X. No characterization of determining convergence relations is known. Nevertheless, there are very simple and natural sufficient conditions which are given in the theorem which follows. Further comments will be given after the proof.

35 A.6. Theorem. *The following three conditions are sufficient for a convergence relation* $\mathscr{C}$ *to be a determining convergence relation:*

(a) $\mathbf{E}N \subset \mathbf{E}\mathscr{C}$ *for each* $N \in \mathbf{D}\mathscr{C}$.

(b) *If* $x \in \mathbf{E}\mathscr{C}$, *then there exists a net* N *in* (x) *such that* $\langle N, x\rangle \in \mathscr{C}$.

(c) *If* $\langle N, x\rangle \in \mathscr{C}$ *then* $\langle M, x\rangle \in \mathscr{C}$ *for each subnet* M *of* N.

Remark. Conditions (a) and (b) are also necessary.

Proof. Put $P = \mathbf{E}\mathscr{C}$ and consider the single-valued relation u which assigns to each subset X the set uX consisting of all points x such that $\langle N, x\rangle \in \mathscr{C}$ for some net N ranging in X, in symbols $uX = \mathbf{E}\{x \mid$ there exists $\langle N, x\rangle \in \mathscr{C}$ with $\mathbf{E}N \subset X\}$. It will be shown that u is a closure operation for the set P and $\mathscr{C}$ is a determining convergence relation for $\langle P, u\rangle$. It follows from the definition that $u\emptyset = \emptyset$. From condition (b) we obtain at once $X \subset uX$ for each $X \subset P$. Since clearly $X \subset Y$ implies $uX \subset uY$, we have $uX_1 \cup uX_2 \subset u(X_1 \cup X_2)$ for all $X_1, X_2 \subset P$. To prove the converse inclusion, suppose $x \in u(X_1 \cup X_2)$; it is to be proved that $x \in (uX_1 \cup uX_2)$. By definition of u there exists a net N in $(X_1 \cup X_2)$ such that $\langle N, x\rangle \in \mathscr{C}$. Consider the sets $A_i = \mathbf{E}\{a \mid a \in \mathbf{D}N,\ N_a \in X_i\}$, $i = 1, 2$. Since $A_1 \cup A_2 = \mathbf{D}N$, at least one of the sets A_1 or A_2, say A_i, must be cofinal in $\mathbf{D}N$, and hence directed by the relativized order. Now the restriction M of N to A_i is a subnet of N ranging in the set X_i. By condition (c) the pair $\langle M, x\rangle$ belongs to $\mathscr{C}$, and finally, by definition of u, $x \in uX_i$. Thus u is indeed a closure for P. To show that $\mathscr{C}$ is a determining convergence relation for $\langle P, u\rangle$ we must show that $\mathscr{C} \subset \mathbf{Lim}\ \langle P, u\rangle$ and $x \in uX$ if and only if $\langle N, x\rangle \in \in \mathscr{C}$ for some net N in X. The latter fact follows from the definition and the former one is proved as follows: Suppose that $\langle N, x\rangle \in \mathscr{C}$ but N does not converge to x in $\langle P, u\rangle$. By (a), $\mathbf{E}N \subset P$. There exists a neighborhood U of x such that N is not eventually in U, that is, N is frequently in $P - U$. It follows that the restriction M of N to the ordered subset $A = \mathbf{E}\{a \mid a \in \mathbf{D}N,\ N_a \in (P - U)\}$ of $\mathbf{D}N$ is a subnet of N, and M ranges in $P - U$. By condition (c) $\langle M, x\rangle \in \mathscr{C}$, and by definition of u we obtain $x \in u(P - U)$ which contradicts our assumption that U is a neighborhood of x. This concludes the proof; the remark is evident.

It may be noted that the condition (c) is not necessary. The reader can easily find a corresponding example. Obviously condition (c) cannot be omitted. Some weakened form of condition (c) will be discussed in the exercises.

A special sort of determining convergence relations which will be introduced in the definition which follows is very important in practice.

35 A.7. Definition. A convergence relation $\mathscr{C}$ will be termed a *convergence structure* if the following conditions are fulfilled: (a) $\mathbf{E}N \subset \mathbf{E}\mathscr{C}$ for each $N \in \mathbf{D}\mathscr{C}$; (b) If N is a net in $\mathbf{E}\mathscr{C}$ such that $N_a = x$ for each a in $\mathbf{D}N$, then $\langle N, x\rangle \in \mathscr{C}$; and (c) If $\langle N, x\rangle \in \mathscr{C}$ then $\langle M, x\rangle \in \mathscr{C}$ for each subnet M of N. A *convergence structure for a space* $\mathscr{P}$ is a convergence structure which is a determining convergence relation for $\mathscr{P}$.

As a corollary of 35 A.6 we obtain:

35 A.8. Theorem. *Every convergence structure is a determining convergence relation.*

Now let $\mathscr{C}$ be a determining convergence relation for a space $\mathscr{P}$. It is natural to inquire about a description of the convergence class of $\mathscr{P}$ in terms of $\mathscr{C}$. Such a description will be given for a special class of determining relations. First we shall give a sufficient condition for $\langle S, x\rangle \in \mathbf{Lim}\ \mathscr{P}$.

35 A.9. *Let $\mathscr{C}$ be a determining convergence relation for a space $\mathscr{P}$. If S is a net in $\mathscr{P}$ and $x \in \mathscr{P}$ is such that each subnet N of S has a generalized subnet M with $\langle M, x\rangle \in \mathscr{C}$, then S converges to x in $\mathscr{P}$, i.e. $\langle S, x\rangle \in \mathbf{Lim}\ \mathscr{P}$.*

Proof. Suppose that x and S fulfil the condition and S does not converge to x in $\mathscr{P}$. There exists a neighborhood U of x such that S is not eventually in U, and hence, S is frequently in $|\mathscr{P}| - U$. It follows that some subnet N of S ranges in $|\mathscr{P}| - U$. By the condition we can choose a generalized subnet M of N such that $\langle M, x\rangle \in \mathscr{C}$. Since N ranges in $|\mathscr{P}| - U$, M also ranges in $|\mathscr{P}| - U$. But M converges to x and hence x belongs to the closure of the set $|\mathscr{P}| - U$, which contradicts our assumption that U is a neighborhood of x.

Example. The condition of 35 A.9 is not necessary; e.g. let $\mathscr{P}$ be an infinite accrete space and let $\mathscr{C}$ be the class of all pairs $\langle N, x\rangle$ such that $x \in \mathscr{P}$ and N is a constant net in $\mathscr{P}$. Clearly $\mathscr{C}$ is a determining convergence relation for $\mathscr{P}$. Next, if S is a one-to-one net in $\mathscr{P}$ the domain of which does not have a greatest element, then no generalized subnet of S is constant. Since $\mathscr{P}$ is infinite, we can choose a one-to-one sequence $\{x_n\}$ in $\mathscr{P}$ and an x in $\mathscr{P}$. Since $\mathscr{P}$ is an accrete space, $\{x_n\}$ converges to x in $\mathscr{P}$. On the other hand, no generalized subnet of a subnet of $\{x_n\}$ is constant.

Let $\mathscr{C}$ be a determining convergence relation for a space $\mathscr{P}$ and let $\mathscr{B}$ be the class of all $\langle S, x\rangle$ satisfying the condition of 35 A.9. By 35 A.9 we have $\mathscr{B} \subset \mathbf{Lim}\,\mathscr{P}$, and the example following 35 A.9 shows that in general $\mathscr{B} \neq \mathbf{Lim}\,\mathscr{P}$. There is an important kind of pairs $\langle S, x\rangle \in \mathbf{Lim}\,\mathscr{P}$ which belong to $\mathscr{B}$ for each $\mathscr{C}$. The result is given as a corollary of 35 A.11. Recall that if N converges to x in $\langle P, u\rangle$ and A is cofinal in $\mathbf{D}N$, then $x \in uN[A]$.

35 A.10. Definition. We shall say that a net N *converges regularly to a point x in a space $\mathscr{P}$* if N converges to x and $x \in \overline{N[A]}$ implies that A is a cofinal subset of $\mathbf{D}N$.

Of course, not every net converging to a point x converges to x regularly. For example if a constant net converges regularly to a point then its domain is one-point. Next, in an accrete space, every net converges to each point but a net converges regularly if and only if its domain is one-point. In a discrete space a net N converges regularly to a point x if and only if the domain of N possesses a greatest element, say α, $N_\alpha = x$ and $N_a \neq x$ for $a \neq \alpha$. Evidently a generalized subnet M of a net regularly convergent to a point x need not converge regularly to x. On the other hand a subnet of a net regularly convergent to a point x converges regularly to x.

35 A.11. *Let $\mathscr{C}$ be a determining convergence relation for a space $\mathscr{P}$. If a net N regularly converges to a point x in $\mathscr{P}$, then there exists a generalized subnet M of N such that $\langle M, x\rangle \in \mathscr{C}$.*

Proof. Since $x \in \overline{\mathbf{E}N}$ we can choose a $\langle M, x\rangle$ in $\mathscr{C}$ such that $\mathbf{E}M \subset \mathbf{E}N$. We shall prove that M is a generalized subnet of N. Since $\mathbf{E}M \subset \mathbf{E}N$ we can choose a single-valued relation ϱ ranging in $\mathbf{D}N$ such that $M = N \circ \varrho$ and $\mathbf{D}\varrho = \mathbf{D}M$. It will be shown that M is a generalized subnet of N under ϱ, that is, given an α in $\mathbf{D}N$, there exists a β in $\mathbf{D}M$ such that $\beta \leqq b$ (in $\mathbf{D}M$) implies $\alpha \leqq \varrho b$ (in $\mathbf{D}N$).

The set A of all $a \in \mathbf{D}N$ such that α non $\leqq a$ is not cofinal in $\mathbf{D}N$. Since N converges regularly to x, $x \notin \overline{N[A]}$, and consequently there is a neighborhood U of x disjoint with $N[A]$. Thus $N_a \in U$ implies $\alpha \leqq a$. Since M converges to x in P, there exists a β in $\mathbf{D}M$ such that $M_b \in U$ for each $b \in \mathbf{D}M$, $\beta \leqq b$. If $\beta \leqq b$ in $\mathbf{D}M$, then $M_b = N_{\varrho b} \in U$ and hence $\varrho b \geqq \alpha$ in $\mathbf{D}N$.

Corollary. *If $\mathscr{C}$ is a determining convergence relation for a space $\mathscr{P}$ and a net S converges regularly to x in $\mathscr{P}$, then each subnet of S has a generalized subnet M such that $\langle M, x\rangle \in \mathscr{C}$. (Compare with* 35 A.2 (d).*)*

35 A.12. Example. Suppose that $\mathscr{P}$ is a semi-separated space and N is a sequence converging to x in $\mathscr{P}$. Then either the sequence N possesses a subsequence regularly convergent to x or N is constant for sufficiently large indexes. — Evident.
A similar result for nets is not true. The example which follows clarifies the situation.

35 A.13. Example. Let A be a non-void set and let $\{\mathscr{P}_a \mid a \in A\}$ be a family of spaces and $\{x_a \mid a \in A\}$ be a family such that $x_a \in |\mathscr{P}_a|$ for each a. Finally, let $\mathscr{P}$ be the space obtained from the family $\{\mathscr{P}_a \mid a \in A\}$ by identifying the points of the family $\{x_a\}$. That is, $\mathscr{P}$ is the quotient of the sum space $\Sigma\{\mathscr{P}_a\}$ under the smallest equivalence containing the set $X \times X$ where X is the set of all $\langle a, x_a \rangle$, $a \in A$.

Suppose that, for each a, S^a is a sequence in $\mathscr{P}_a$ converging to the point x_a, and $\leqq$ is an order directing the set A. Let S be the net the domain of which is the directed set $\langle A, \leqq \rangle \times \Pi\{\mathbf{D}S^a \mid a \in A\}$ and the value of which at a point $\langle \alpha, \{n_a\} \rangle$ is $\langle \alpha, S^{\alpha}_{n_\alpha} \rangle \in \in \mathscr{P}$. One can prove without difficulty that

(a) The net S converges to the point X in $\mathscr{P}$.

(b) The net S does not converge regularly to X provided that $\langle A, \leqq \rangle$ does not have a greatest element; moreover,

(c) no subnet of S converges regularly to X in $\mathscr{P}$ provided that $\langle A, \leqq \rangle$ does not have a greatest element.

(d) If all the spaces $\mathscr{P}_a$ are semi-separated, $S^a_n \neq x_a$ for each $a \in A$ and each $n \in \mathrm{N}$, and $\langle A, \leqq \rangle$ is the directed set of all natural numbers, then no diagonal sequence $\{\langle a, S^a_{n_a} \rangle \mid a \in A\}$ converges to the point X in $\mathscr{P}$.

Now we return to our subject. Under the notation of the remark preceding Definition 35 A.10, we want to find a sufficient condition on $\mathscr{C}$ for $\mathscr{B} = \mathbf{Lim}\mathscr{P}$ such that any convergence class always fulfils this condition. In 35 A.13 we have given, in a special case, a construction of non-regularly convergent nets from regularly convergents nets. Now we shall show that one such condition consists of requirements that $\mathscr{C}$ be stable under this construction.

35 A.14. Theorem. *Let $\mathscr{C}$ be a determining convergence relation satisfying the following condition, which will be called the* condition of diagonalization.

If A is a directed set and $\{M^a \mid a \in A\}$ is a family such that $\langle M^a, x \rangle \in \mathscr{C}$ for each a in A, then $\langle M, x \rangle \in \mathscr{C}$ where M is a net the domain of which is the product ordered set

$$A \times \Pi\{\mathbf{D}M^a \mid a \in A\}$$

and the value of which at a point $\langle \alpha, \{b_a\} \rangle$ is $M^{\alpha}_{b_\alpha}$.

Then, if S converges to x in $\mathscr{P}$ then each subnet of S has a generalized subnet M such that $\langle M, x \rangle \in \mathscr{C}$.

Remark. It should be noted that the condition of diagonalization is a weakened form of the condition on iterated limits. Nevertheless, although the condition on iterated limits characterizes convergence classes of topological spaces in the class of all convergence classes, the assumption of diagonalization is fulfilled by every convergence class, as will be proved in 35 A.16.

Proof. Suppose that a net S converges to x in $\mathscr{P}$ and that N is a subnet of S. Therefore N also converges to x in $\mathscr{P}$. Let A denote the directed domain of N. For each α in A let X_α be the set of all N_a, $\alpha \leqq a$. Since N converges to x, the point x belongs to the closure of each X_a, $a \in A$. Since $\mathscr{C}$ is a determining convergence relation for $\mathscr{P}$ we can choose a family $\{M^a \mid a \in A\}$ such that $\langle M^a, x\rangle \in \mathscr{C}$ and M^a ranges in X_a for each a in A. Let M be the net in the condition of diagonalization; thus $\langle M, x\rangle \in \mathscr{C}$. We shall prove that M is a generalized subnet of N. Let ϱ be a single-valued relation on $\mathbf{D}M$ into $\mathbf{D}N$ which assigns to each point $\langle \alpha, \{b_a\}\rangle$ a point a such that $\alpha \leqq a$, $N_a = M^\alpha_{b_\alpha}$. Clearly $M = N \circ \varrho$. To prove that M is a generalized subnet of N under ϱ it remains to show that for each α in A ($= \mathbf{D}N$) there exists $\gamma \in \mathbf{D}M$ such that $\gamma \leqq c$ in $\mathbf{D}M$ implies $\alpha \leqq \varrho c$ in A. But this is almost self-evident. Indeed, putting $\gamma = \langle \alpha, \{b_a\}\rangle$ where $\{b_a\}$ is arbitrarily chosen, then γ has the required property; since if $\gamma \leqq c$, $c = \langle a_1, \{b'_a\}\rangle$, then $\alpha \leqq a_1$ in A and $\varrho c \geqq a_1$ (by the choice of ϱ), and hence $\varrho c \geqq \alpha$.

Combining 35 A.14 and 35 A.9 we obtain at once the following description of convergent nets in terms of a determining convergence relation satisfying the condition of diagonalization.

35 A.15. Theorem. *Suppose that $\mathscr{P}$ is a space and $\mathscr{C}$ is a determining convergence relation for $\mathscr{P}$ satisfying the condition of diagonalization (see* 35 A.14*). Then a net S in $|\mathscr{P}|$ converges to a point x of $|\mathscr{P}|$ in $\mathscr{P}$ if and only if each subnet N of S possesses a generalized subnet M such that $\langle M, x\rangle \in \mathscr{C}$.*

Now we are prepared to give a characterization of convergence classes.

35 A.16. Theorem. *The following conditions are necessary and sufficient for a convergence relation $\mathscr{C}$ to be a convergence class.*

(a) *$\mathscr{C}$ is a determining convergence relation.*

(b) *$\mathscr{C}$ fulfils the condition of diagonalization.*

(c) *If S is a net in $\mathbf{E}\mathscr{C}$ and x is a point of $\mathbf{E}\mathscr{C}$ such that each subnet of S has a generalized subnet M with $\langle M, x\rangle \in \mathscr{C}$ then $\langle S, x\rangle \in \mathscr{C}$.*

Remark. Condition (c) is equivalent to the following condition

(c′) if S is a net in $\mathbf{E}\mathscr{C}$, x is a point of $\mathbf{E}\mathscr{C}$ and $\langle S, x\rangle \notin \mathscr{C}$, then there exists a subnet N of S such that $\langle M, x\rangle \notin \mathscr{C}$ for no generalized subnet M of N.

Proof. The sufficiency is a straightforward consequence of 35 A.14. We shall prove the necessity. Suppose that $\mathscr{C}$ is the convergence class of a space $\mathscr{P}$. Condition (c′) is fulfilled by 35 A.2 (d). Obviously $\mathscr{C}$ is a determining convergence relation for $\mathscr{P}$. It remains to show that $\mathscr{C}$ satisfies the condition of diagonalization. Suppose that A is a directed set, $\{M^a \mid a \in A\}$ is a family of nets each of which converges to x and M is the net from the assumption of diagonalization (see 35 A.14). We must show that M converges to x. Given a neighborhood U of x we can choose a family $\{b'_a \mid a \in A\}$ such that $b'_a \leqq b$ in $\mathbf{D}M^a$ implies $M^a_b \in U$. Now if α' is any element of A and $\langle \alpha', \{b'_a\}\rangle \leqq \langle \alpha, \{b_a\}\rangle = c$, then $M_c = M^\alpha_{b_\alpha}$ and $b'_\alpha \leqq b_\alpha$ in $\mathbf{D}M^\alpha$, and consequently $M_c \in U$ by the choice of $\{b'_a\}$.

35 A.17 Theorem (a characterization of convergence classes). *The following conditions are necessary and sufficient for a convergence relation $\mathscr{C}$ to be a convergence class:*

(a) *$\mathscr{C}$ is a convergence structure.*

(b) *$\mathscr{C}$ fulfils the condition of diagonalization.*

(c) *Condition* (c) *of* 35 A.16.

Proof. The sufficiency is a consequence of the previous theorem and of Theorem 35 A.8 asserting that a convergence structure is a convergence determining relation. The necessity of the condition again follows from the previous theorem and the fact that every convergence class is a convergence structure.

35 A.18. Theorem (characterization of convergence classes of topological spaces). *The following conditions are necessary and sufficient for a convergence relation $\mathscr{C}$ to be a convergence class of a topological space:*

(a) *$\mathscr{C}$ is a convergence structure.*

(b) *$\mathscr{C}$ fulfils the condition on iterated limits.*

(c) *Condition* (c) *of* 35 A.16.

Proof. Since condition (b) of 35 A.18 implies condition (b) of 35 A.17, and the other conditions are identical, a convergence relation satisfying the conditions of 35 A.18 is a convergence class of a space $\mathscr{P}$ (by 35 A.17) which is topological by 15 B.13, because it fulfils the condition on iterated limits. Conversely, if $\mathscr{C}$ is the convergence class of a topological space, then the condition (b) is fulfilled by 15 B.13, and the remaining are fulfilled by the foregoing theorem.

Subsection A is ended by examples. First we shall show that many of the results of previous sections can be easily proved by means of the results of this subsection.

35 A.19. (a) If $\{u_a\}$ is a non-void family in $\mathbf{C}(P)$ then $u = \inf \{u_a\}$ exists and $\mathscr{C} = \mathbf{E}\{\mathscr{N} \mid \mathscr{N} \in \mathbf{Lim}\, u_a$ for each $a\}$ is $\mathbf{Lim}\, u$; if all the u_a are topological, then so is u.

Proof. Let $\mathscr{C}$ be as above. It follows from 35 A.17 that $\mathscr{C}$ is a convergence class, say of a space $\langle P, u\rangle$; if all the u_a are topological then u is topological by 35 A.18 (because $\mathscr{C}$ fulfils the condition on iterated limits). It follows from the characterization of continuity by means of nets that $u = \inf \{u_a\}$.

(b) If f is a mapping of a set P into a space $\mathscr{Q}$, then there exists a unique closure projectively generated by f, say u, and $\mathbf{Lim}\, \langle P, u\rangle$ consists of all $\langle N, x\rangle$ such that $\langle f \circ N, fx\rangle \in \mathbf{Lim}\, \mathscr{Q}$; if $\mathscr{Q}$ is topological, then $\langle P, u\rangle$ is topological.

Proof. Consider the convergence relation $\mathscr{C}$ consisting of all $\langle N, x\rangle$ such that $\langle f \circ N, fx\rangle \in \mathbf{Lim}\, \mathscr{Q}$. It follows from 35 A.17 that $\mathscr{C}$ is a convergence class for a space $\langle P, u\rangle$ which is topological whenever $\mathscr{Q}$ is topological (by 35 A.18). It is easily seen that u is projectively generated by f.

It follows from (a), (b) that

(c) if $\{f_a\}$ is non-void family, each f_a being a mapping of a set P into a space $\mathscr{Q}_a$, then there exists a unique closure u projectively generated by $\{f_a\}$; the convergence class of $\langle P, u\rangle$ consists of all pairs $\langle N, x\rangle$ such that N is a net ranging in P, $x \in P$ and $\langle f_a \circ N, f_a x\rangle \in \mathbf{Lim}\ \mathcal{Q}_a$ for each a.

In particular, defining the product closure as the closure projectively generated by projections, we obtain that a net converges to a point x in a product space if and only if it converges pointwise to x.

It follows immediately from (c) that:

(d) if $\{f_a\}$ is a projective generating family for a space $\mathscr{P}$, then a mapping f into $\mathscr{P}$ is continuous if and only if all the composites $f_a \circ f$ are continuous.

35 A.20. Convergence closure for the collection of all closed subsets of a closure space. *Let $\mathscr{F}$ be the collection of all closed subsets of a topological space $\langle P, u\rangle$. Let $\mathscr{C}$ be the set of all pairs $\langle S, X\rangle$ such that S is a net in $\mathscr{F}$ and X is the topological limit of S in $\langle P, u\rangle$ (for the definition see 15 ex. 10). The class $\mathscr{C}$ is a convergence structure for $\mathscr{F}$ satisfying the condition* (c) *of* 35 A.16.

35 A.21. Convergence closure for an ordered set. Suppose that $\langle P, \leqq\rangle$ is an ordered set (not necessarily monotone). An eventual upper (lower) bound of a net N in $\langle P, \leqq\rangle$ is an element x such that $N_a \leqq x$ $(x \leqq N_a)$ for all sufficiently large indices. A net N is said to be eventually bounded if it has both these bounds. If N is eventually bounded, then the infimum (supremum) of the set of all eventual upper (lower) bounds of N, if it exists, is called the upper limit (lower limit) of N, and is denoted by Lim sup N (Lim inf N). Evidently, always Lim inf $N \leqq$ Lim sup N. If equality holds, then we write Lim N for the common value and say that N is order-convergent to Lim N. Let $\mathscr{C}$ be the class of all pairs $\langle N, \text{Lim } N\rangle$ where N varies over all order-convergent nets. It is easily seen that $\mathscr{C}$ is a convergence structure. The closure determined by $\mathscr{C}$ will be called the convergence closure of $\langle P, \leqq\rangle$. If $\langle P, \leqq\rangle$ is order-complete, then Lim sup N and Lim inf N exist for each net N in $\langle P, \leqq\rangle$ and

$$\text{Lim sup } N = \inf\ \{\sup \{N_a \mid \alpha \leqq a\} \mid \alpha \in A\}$$
$$\text{Lim inf } N = \sup\ \{\inf\ \{N_a \mid \alpha \leqq a\} \mid \alpha \in A\}\,.$$

If $\langle P, \leqq\rangle$ is monotonically ordered then the convergence closure coincides with the order closure.

35 A.22. Convergence closure for the ordered set of subsets of a set. Let $\mathscr{P}$ be the collection of all subsets of a set P ordered by inclusion, and let $\mathscr{C}$ be the class described in the preceding example 35 A.21. Thus $\langle N, X\rangle \in \mathscr{C}$ if and only if N is a net in $\mathscr{P}$ and

$$X = \bigcap\{\bigcup\{N_a \mid \alpha \leqq a\} \mid \alpha \in \mathbf{D}N\} = \bigcup\{\bigcap\{N_a \mid \alpha \leqq a\} \mid \alpha \in \mathbf{D}N\}\,.$$

Let the symbols Lim N, Lim sup N and Lim inf N have the meaning from 35 A.21. We shall prove that then the convergence relation $\mathscr{C}$ fulfils condition (c) of 35 A.16, that is, if S is a net in $\mathscr{P}$ and $X \in \mathscr{P}$ such that $\langle S, X\rangle \notin \mathscr{C}$, then there exists a subnet N of S such that $\langle M, X\rangle \in \mathscr{C}$ for no generalized subnet M of N. If Lim S exists, then Lim M = Lim S for each generalized subnet M of S and hence we can put $N = S$. Now suppose that Lim S does not exist and pick a point x in Lim sup S −

$-$ Lim inf S. If $x \in X$ let N be the restriction of S to the set A of all $a \in \mathbf{D}S$ such that $x \notin N_a$, and if $x \notin X$ then let N be the restriction of S to the set A of all $a \in \mathbf{D}S$ such that $x \in N_a$. In both cases, by the choice of x, the set A is cofinal in $\mathbf{D}S$, and hence N is a subnet of S. If M is a generalized subnet of N, then $x \notin$ Lim sup M in the first case and $x \in$ Lim inf M in the second case by the choice of N. It follows N has the required property. It is easily seen that $\mathscr{C}$ fulfils the condition on iterated limits, and hence $\mathscr{C}$ is the convergence class of a space $\langle \exp P, u \rangle$. These results can also be derived from the fact that $\langle \exp P, u \rangle$ is a homeomorph of the product $\mathscr{Q}^P$ where $\mathscr{Q}$ is a two-point discrete space.

B. SEQUENTIAL DETERMINING RELATIONS

In this subsection we shall be concerned with questions related to the possibility of description of spaces in terms of convergent sequences. For convenience we shall introduce further terminology.

35 B.1. Definition. A *sequential relation* is a convergence relation the domain of which consists of sequences. A *sequential determining relation* is a determining convergence relation which is simultaneously a sequential relation. The *sequential convergence class of a space* $\mathscr{P}$ is the class (in fact a set) of all pairs $\langle S, x \rangle$ such that S is a sequence converging to x in $\mathscr{P}$. A *sequential convergence class* is the sequential convergence class of a space. Finally, a *sequential structure* is a sequential relation $\mathscr{C}$ satisfying the following three conditions:

(a) If $\langle S, x \rangle \in \mathscr{C}$, then $\mathbf{E}S \subset \mathbf{E}\mathscr{C}$.
(b) If $x \in \mathbf{E}\mathscr{C}$ and S is a constant sequence $\{x \mid n \in \mathbf{N}\}$, then $\langle S, x \rangle \in \mathscr{C}$.
(c) If $\langle S, x \rangle \in \mathscr{C}$ and N is a subsequence of S, then $\langle N, x \rangle \in \mathscr{C}$.

35 B.2. Theorem. *Every sequential convergence class is a sequential structure, and every sequential structure is a sequential determining relation.*

Proof. The first assertion is evident and the second one follows from 35 A.8.

35 B.3. Theorem. *Let u be a closure for a set P and let $\mathscr{C}$ be the sequential convergence class of the space $\langle P, u \rangle$. There exists a closure v for P such that $\mathscr{C}$ is a sequential determining relation for $\langle P, v \rangle$. The closure v is the finest closure such that S converges to x in $\langle P, v \rangle$ for each $\langle S, x \rangle \in \mathscr{C}$, and $\mathscr{C}$ is the sequential convergence class of $\langle P, v \rangle$.*

Proof. By 35 B.2 $\mathscr{C}$ is a sequential determining relation, say for a space $\langle Q, v \rangle$. Clearly $Q = P$. If w is any closure for P such that $\mathscr{C} \subset \mathbf{Lim} \langle P, w \rangle$, then the identity mapping of $\langle P, v \rangle$ onto $\langle P, w \rangle$ is continuous because $\mathscr{C}$ is a determining convergence relation for $\langle P, v \rangle$, and hence w is coarser than v; this proves the second statement. Finally, to prove the last statement, notice that $\mathscr{C}$ is contained in the sequential convergence class of $\langle P, v \rangle$ and, on the other hand, the sequential convergence class of $\langle P, v \rangle$ is contained in that of $\langle P, u \rangle$, which is $\mathscr{C}$.

35 B.4. Theorem. *Every space with a countable local character admits a sequential determining relation, in particular, every semi-pseudometrizable space admits a sequential determining relation.*

Proof. Suppose that x belongs to the closure of a set X and $\{U_n \mid n \in \mathsf{N}\}$ is a local base at x. Thus $\{X \cap U_n \mid n \in \mathsf{N}\}$ is a filter base and we can choose a sequence $\{x_n\}$ so that $x_n \in \bigcap\{X \cap U_k \mid k \leqq n\}$. The sequence $\{x_n\}$ ranges in X and converges to x.

The following example shows that the converse of this theorem is not true, that is, there exists a space with an uncountable local character which admits a sequential determining relation.

35 B.5. Example. Let P be a set consisting of all points of Q and a further point x. Let us define a closure operation u for P such that the space Q is a subspace of $\langle P, u\rangle$ and $x \in uX$ if and only if either $x \in X$ or the closure of $X \cap \mathsf{Q}$ in the space R intersects $\mathsf{R} - \mathsf{Q}$. Finally, let $\mathscr{C}$ be the sequential convergence class of the space $\langle P, u\rangle$. It can be verified that

(a) $\langle P, u\rangle$ is a semi-separated topological space;

(b) $\langle P, u\rangle$ is not separated because the closure of each neighborhood of each point of Q contains x;

(c) $\langle P, u\rangle$ is of a countable local character at each point of Q but not at x;

(d) $\mathscr{C}$ is determining convergence relation for $\langle P, u\rangle$;

(e) $\mathscr{C}$ is single-valued.

The verification of all assertions, perhaps except of the fact that $\langle P, u\rangle$ is not of a countable local character at x, is straightforward and may be left to the reader. The uncountability of the local character at x will follow from Theorem 35 B.12 asserting that a space $\mathscr{P}$ satisfying conditions (d) and (e) is separated provided that it is of a countable local character. Nevertheless, a direct proof is not difficult.

Now we shall proceed to the following problem. Given a sequential determining relation $\mathscr{C}$ for a space $\mathscr{P}$, to find a reasonable description of the sequential convergence class of $\mathscr{P}$ in terms of $\mathscr{C}$. We shall restrict ourselves to the case when $\mathscr{P}$ is semi-separated, or equivalently, as will be shown, $\mathscr{C}$ is single-valued at constant nets. Under this assumption no form of the condition of diagonalization will be needed, and in fact the resulting theorem will follow by a slight modification of the result 35 A.11 for regularly convergent nets.

35 B.6. Theorem. *Let $\mathscr{C}$ be a determining sequential structure for a semi-separated space $\mathscr{P}$. Then a sequence S in $|\mathscr{P}|$ converges to a point x in $\mathscr{P}$ if and only if every subsequence N of S has a subsequence M such that $\langle M, x\rangle \in \mathscr{C}$.*

Proof. If a sequence S fulfils the condition, then S converges to x by 35 A.9. Conversely, suppose that S converges to x in $\mathscr{P}$ and N is a subsequence of S. It follows that the sequence N also converges to x in $\mathscr{P}$. According to 35 A.12, either N possesses a constant subsequence or some subsequence of N converges to x regularly. In the former case choose a constant subsequence M of N. Since P is semi-separated,

$M_n = x$ for each n, and consequently $\langle M, x\rangle \in \mathscr{C}$ because $\mathscr{C}$ is a sequential determining relation. In the latter case choose a subsequence M of N which regularly converges to x. The point x belongs to $\overline{\mathbf{E}M}^{\mathscr{P}} - \mathbf{E}M$, and therefore there exists a pair $\langle M', x\rangle$ in $\mathscr{C}$ such that $\mathbf{E}M' \subset \mathbf{E}M$. Since M converges regularly to x, one can easily construct a subsequence M'' of both M' and M. Since $\mathscr{C}$ is a sequential structure, $\langle M'', x\rangle \in \mathscr{C}$. Clearly M'' is a subsequence of N.

The fact the a space $\mathscr{P}$ admitting a sequential determining relation $\mathscr{C}$ is semi-separated can be expressed in terms of $\mathscr{C}$, as the proposition which follows asserts; this will enable us to restate the foregoing theorem without any reference to the space $\mathscr{P}$.

35 B.7. *Let $\mathscr{C}$ be a sequential determining relation for a space $\mathscr{P}$. In order that $\mathscr{P}$ be semi-separated it is necessary and sufficient that $\mathscr{C}$ be single-valued at each constant sequence.*

Proof. The space $\mathscr{P}$ is semi-separated if and only if $y \in \overline{(x)}$ implies $y = x$. On the other hand, for each x in $\mathscr{P}$ there exists exactly one sequence ranging in (x), say S, and consequently, if y belongs to the closure of (x) then $\langle S, y\rangle \in \mathscr{C}$. The proposition follows.

By virtue of 35 B.7, Theorem 35 B.6 can be restated as follows.

35 B.8. Theorem. *Let us suppose that $\mathscr{C}$ is a sequential structure, and let $\mathscr{D}$ be the smallest sequential convergence class containing $\mathscr{C}$ (that is, $\mathscr{D}$ is the sequential convergence class of the space determined by $\mathscr{C}$). If $\mathscr{C}$ is single-valued at each constant sequence, then $\langle S, x\rangle \in \mathscr{D}$ if and only if S is a sequence ranging in $\mathbf{E}\mathscr{C}$ and each subsequence N of S possesses a subsequence M such that $\langle M, x\rangle \in \mathscr{C}$.*

35 B.9. Theorem. *The following conditions are necessary and sufficient for a sequential relation $\mathscr{C}$ to be the sequential convergence class of a semi-separated space:*

(a) *$\mathscr{C}$ is a sequential structure single-valued at constant sequences.*

(b) *If S is a sequence in $\mathbf{E}\mathscr{C}$, $x \in \mathbf{E}\mathscr{C}$ and $\langle S, x\rangle \notin \mathscr{C}$, then there exists a subsequence N of S such that $\langle M, x\rangle \in \mathscr{C}$ for no subsequence M of N.*

Proof. I. First suppose that $\mathscr{C}$ is the sequential convergence class of a semi-separated space $\mathscr{P}$. Then condition (a) follows from 35 B.2 and 35 B.7, and condition (b) follows from 15 B.21. — II. Now suppose that $\mathscr{C}$ fulfils the conditions. By (a) the sequential relation $\mathscr{C}$ is a sequential determining relation for a space $\mathscr{P}$ which is semi-separated by 35 B.7. By 35 B.8 condition (b) implies that $\mathscr{C}$ is the sequential convergence class of $\mathscr{P}$.

We know that a closure space is semi-separated provided that it admits a sequential determining relation which is single-valued at constant sequences and, on the other hand, a space is separated provided that its convergence class is single-valued. It is natural to ask whether a space is separated provided that it admits a single-valued determining convergence relation, and if not, whether a space is separated provided that it admits a single-valued sequential determining relation. The answer is in the negative. Indeed, the sequential class of the space $\langle P, u\rangle$ in 35 B.5 is a single-valued determining convergence relation for $\langle P, u\rangle$ but $\langle P, u\rangle$ is not separated. It should be noted that every space admits a single-valued determining convergence relation.

35 B.10. Definition. An **L**-*space* is a space $\mathscr{P}$ the sequential convergence class of which is a single-valued determining convergence relation for $\mathscr{P}$. An **L**-*structure* is a single-valued sequential structure.

35 B.11. *Every* **L**-*space is semi-separated, but not necessarily separated.*

An important class of separated **L**-spaces is described in the theorem which follows.

35 B.12. *Every* **L**-*space with a countable local character is separated.*

Proof. The proof is similar to that of the fact that a space is separated whenever its convergence class is single-valued. Suppose that points x and y are not separated and the space is of a countable local character at both points x and y. There exist local bases $\{U_n \mid n \in \mathrm{N}\}$ and $\{V_n \mid n \in \mathrm{N}\}$ at x and y respectively, and $U_n \cap V_m \neq \emptyset$ for each n and m. Since both $\{U_n\}$ and $\{V_n\}$ are filter bases, we can choose a sequence $\{x_n\}$ so that $x_n \in \bigcap\{U_l \cap V_k \mid k \leqq n, l \leqq n\}$ for each $n \in \mathrm{N}$. Clearly the sequence $\{x_n\}$ converges to both x and y.

35 B.13. Theorem. *The following two conditions are necessary and sufficient for a sequential relation $\mathscr{C}$ to be the sequential convergence class of an* **L**-*space:*

(a) *$\mathscr{C}$ is an* **L**-*structure.*

(b) *If S is a sequence ranging in $\mathbf{E}\mathscr{C}$, $x \in \mathbf{E}\mathscr{C}$ and $\langle S, x\rangle \notin \mathscr{C}$, then there exists a subsequence N of S such that $\langle M, x\rangle \in \mathscr{C}$ for no subsequence M of N.*

Proof. The conditions are sufficient by 35 B.9. The necessity is obvious.

35 B.14. Theorem. *A space $\mathscr{P}$ is an* **L**-*space if and only if it admits a determining* **L**-*structure.*

Proof. The necessity is obvious. We shall prove the sufficiency. Suppose that an **L**-structure $\mathscr{C}$ is a determining convergence relation for a space $\mathscr{P}$. We must show that each sequence S in $\mathscr{P}$ possesses at most one limit point. Suppose that x and y are limit points of a sequence S. We shall prove $x = y$. It is enough to find a subsequence M of S such that $\langle M, x\rangle \in \mathscr{C}$ and $\langle M, y\rangle \in \mathscr{C}$ because $\mathscr{C}$ is single-valued. First notice that $\mathscr{P}$ is semi-separated, by 35 B.7. Now, since S converges to x, it follows from 35 B.6 that there exists a subsequence N of S such that $\langle N, x\rangle \in \mathscr{C}$. Since S converges to y, N also converges to y. Applying once more 35 B.6 we obtain a subsequence M of N such that $\langle M, y\rangle \in \mathscr{C}$. But $\mathscr{C}$ is a sequential structure, and hence $\langle M, x\rangle \in \mathscr{C}$ because $\langle N, x\rangle \in \mathscr{C}$ and M is a subsequence of N.

It is to be noted that "**L**-structure" cannot be replaced by "sequential relation" in the foregoing theorem. The corresponding example is given in 35 B.17.

35 B.15. Definition. We shall say that a *net N converges strictly to a point x in a space $\mathscr{P}$* if it converges to x in $\mathscr{P}$ and no $y \in |\mathscr{P}| - (x)$ is a cluster point of N, that is, if it is eventually in each neighborhood of x and each point $y \neq x$ has a neighborhood V such that N is eventually in $|\mathscr{P}| - V$.

35 B.16. *In a separated space every convergent net is strictly convergent. In an* **L**-*space every convergent sequence is strictly convergent.*

Proof. I. Suppose that a net N converges to a point x in a separated space $\mathscr{P}$ and $y \in |\mathscr{P}| - (x)$. There exist neighborhoods U and V of x and y such that $U \cap V = \emptyset$. Since N converges to x, N is eventually in $U \subset P - V$. — II. Now suppose that a sequence S converges to x in an **L**-space $\mathscr{P}$ and $y \in |\mathscr{P}| - (x)$ is a cluster point of S. Since $\mathscr{P}$ is semi-separated, $|\mathscr{P}| - (y)$ is a neighborhood of x and hence S is eventually in $|\mathscr{P}| - (y)$, say $n \geqq n_0$ implies that $S_n \neq y$. Consider the set X of all S_n, $n \geqq n_0$. Since y is a cluster point of S, y belongs to the closure of X in $\mathscr{P}$. Thus we can choose a sequence N ranging in X and converging to y. Since $y \in |\mathscr{P}| - X$ and $\mathscr{P}$ is semi-separated, the sequence N contains no constant subsequence. Therefore we can construct a common subsequence M of both S and N. This sequence M must converge to both x and y, which contradicts our assumption that $\mathscr{P}$ is an **L**-space.

The following examples show that the assumption $\mathscr{P}$ is separated and $\mathscr{P}$ is an **L**-space are essential in the foregoing theorem.

35 B.17. Examples. (a) Let P be the set consisting of all natural numbers and two further points x_1 and x_2. Let us define a closure operation u for the set P so that N is an open discrete subspace of $\langle P, u\rangle$ and $x_i \in uX$ if and only if $x_i \in X$ or $X \cap \mathsf{N}$ is infinite. It is easily shown that

(1) $\langle P, u\rangle$ is a semi-separated space.

(2) The sequential convergence class of $\langle P, u\rangle$ is a determining convergence relation for $\langle P, u\rangle$.

(3) If a sequence S ranging in N converges to x_1, then it converges to x_2 as well.

(b) Let P be the set consisting of all points of the ultrafilter space $\beta\mathsf{N}$ (see 14B.12) and a further point x. Define a closure u for P so that $\beta\mathsf{N}$ is an open subspace of P and x belongs to the closure of a set X if and only if either $x \in X$ or X contains an infinite number of elements of N. It is easy to show that

(1) $\langle P, u\rangle$ is a semi-separated space.

(2) A sequence S in P is convergent if and only if either it is eventually constant or it converges to x.

(3) Any sequence S in $\langle P, u\rangle$ is either eventually constant or it has an infinite number of cluster points.

(4) Only the eventually constant sequences converge strictly.

C. SEQUENTIAL MODIFICATION

Now we proceed to an investigation of properties of the class of all spaces admitting a sequential determining relation and the class of all **L**-spaces. For convenience we shall introduce further terminology.

35 C.1. Definition. An **S**-*space* is a closure space $\mathscr{P}$ such that some sequential relation is a determining relation for $\mathscr{P}$. An **S**-*closure* is the closure structure of some **S**-space. The ordered class of all **S**-closures will be denoted by **S**. If P is a set then $\mathbf{S}(P)$ denotes the ordered set of all **S**-closures for P. The class of all **S**-spaces will also be denoted by **S**. The ordered class of all **L**-closures will be denoted by **L**, and $\mathbf{L}(P)$ will denote the ordered set of all **L**-closures for P. The letter **L** also denotes the class of all **L**-spaces.

By 35 B.4 every space with a countable local character is an **S**-space. By 35 B.5 the local character of an **S**-space need not be countable. Recall that an **L**-space is an **S**-space $\mathscr{P}$ such that the sequential convergence class of $\mathscr{P}$ is single-valued (35 B.10).

35 C.2. *A closure space $\mathscr{P}$ is an **S**-space if and only if the sequential convergence class of $\mathscr{P}$ is a determining convergence relation for $\mathscr{P}$. Next, a mapping f of an **S**-space $\mathscr{Q}$ into a closure space $\mathscr{P}$ is continuous if and only if $f \circ S$ converges to fx in $\mathscr{P}$ whenever S is a sequence which converges to x in $\mathscr{Q}$; in particular, an **S**-closure v is finer than a closure u if and only if the sequential convergence class of v is contained in the sequential convergence class of u. The class **S** is hereditary.*

35 C.3. Theorem. *The class **S** is inductive-stable and contains all accrete spaces.*

Of course every accrete space is an **S**-space. The fact that **S** is inductive-stable will be proved in a more general situation.

35 C.4. *Let $\mathcal{O}$ be a non-void class of directed sets and let K be the class of all the spaces $\mathscr{P}$ which admit a determining convergence relation $\mathscr{C}$ such that the ordered domains of nets from $\mathbf{D}\mathscr{C}$ belong to $\mathcal{O}$ (thus $K = \mathbf{S}$ provided that $\mathcal{O} = (\mathsf{N})$). Then K is an inductive-stable class of spaces.*

If $\mathscr{P}$ is any space, $X \subset |\mathscr{P}|$ and $x \in X$, then we can choose an $\langle A, \leqq\rangle \in \mathcal{O}$ (because $\mathcal{O} \neq \emptyset$), and the constant net $\{a \to x\}$ ranges in X and converges to x in $\mathscr{P}$. Now 35 C.4 will follow from a somewhat more general result 35 C.6 for which we need the important concept of a generating convergence relation.

35 C.5. Definition. A *generating convergence relation* for a closure space $\mathscr{P} = \langle P, u\rangle$ is a convergence relation $\mathscr{C} \subset \mathbf{Lim}\,\mathscr{P}$ such that if $x \in uX - X$ then there exists a $\langle N, x\rangle \in \mathscr{C}$ with N ranging in X.

Thus a determining convergence relation for a space $\mathscr{P}$ is a generating convergence relation for $\mathscr{P}$. A generating convergence relation for $\mathscr{P}$ need not be a determining convergence relation for $\mathscr{P}$, e.g. $\emptyset$ is a generating convergence relation for each discrete space. This example shows that a generating convergence relation for a space $\mathscr{P}$ does not determine $\mathscr{P}$.

35 C.6. *Suppose that a space $\mathscr{P} = \langle P, u\rangle$ is inductively generated by a family of mappings $\{f_a \mid a \in A\}$ and $\{\mathscr{C}_a\}$ is a family such that $\mathscr{C}_a$ is a generating convergence relation for $\mathbf{D}^*f_a$ for each a in A. Then the class $\mathscr{C}$ of all $\langle f_a \circ N, f_a x\rangle$, $a \in A$, $\langle N, x\rangle \in \mathscr{C}_a$, is a generating convergence relation for $\mathscr{P}$.*

Proof. Clearly $\mathscr{C} \subset \mathbf{Lim}\,\mathscr{P}$. Assuming $y \in uY - Y$ we can pick an a in A and x in $\mathbf{D}f_a$ such that $f_a x = y$ and x belongs to the closure of $f_a^{-1}[Y]$ in $\mathbf{D}^*f_a$ (by the description 33 A.4 of inductively generated closures). Since $\mathscr{C}_a$ is a generating convergence relation for $\mathbf{D}^*f_a$ we can pick $\langle N, x\rangle$ in $\mathscr{C}_a$ such that N ranges in $f_a^{-1}[Y]$. Clearly $f_a \circ N$ ranges in Y.

According to 33 C.11, Theorem 35 C.3 implies the following proposition.

35 C.7. *Let P be a set. Then*

(a) *For each u in $\mathbf{C}(P)$ there exists the lower modification of u in $\mathbf{S}(P)$.*

(b) *The ordered set $\mathbf{S}(P)$ is order-complete and contains the discrete and the accrete closure for P.*

(c) *$\mathbf{S}(P)$ is completely join-stable and completely join-preserving in $\mathbf{C}(P)$.*

35 C.8. Definition. The lower modification of a closure u in $\mathbf{S}$ will be termed the *sequential modification of u* and will be denoted by $\boldsymbol{\sigma} u$. The *sequential modification of a closure space $\mathscr{P} = \langle P, u\rangle$*, denoted by $\boldsymbol{\sigma}\mathscr{P}$, is defined to be the space $\langle P, \boldsymbol{\sigma} u\rangle$. The letter $\boldsymbol{\sigma}$ will be used to denote the relation $\{u \to \boldsymbol{\sigma} u \mid u \in \mathbf{C}\}$ as well as the relation $\{\mathscr{P} \to \boldsymbol{\sigma}\mathscr{P} \mid \mathscr{P} \in \mathbf{C}\}$.

Since the class $\mathbf{S}$ is inductive-stable, by 35 D.8 (or ex 7) the sequential modification of a closure operation can be characterized as follows.

35 C.9. *Let u be a closure for a set P. The sequential modification of u is the unique closure v for P which satisfies the following condition:*

A mapping f of an $\mathbf{S}$-space into $\langle P, u\rangle$ is continuous if and only if the mapping $f : \mathbf{D}^*f \to \langle P, v\rangle$ is continuous.

35 C.10. *The relation $\{\mathscr{P} \to \mathscr{Q} \mid \boldsymbol{\sigma}\mathscr{P} = \boldsymbol{\sigma}\mathscr{Q}\}$ is an equivalence on the class of all closure spaces, and $\boldsymbol{\sigma}\mathscr{P} = \boldsymbol{\sigma}\mathscr{Q}$ if and only if the sequential convergence classes of $\mathscr{P}$ and $\mathscr{Q}$ coincide. In particular, for every space $\mathscr{P}$ the sequential modification of $\mathscr{P}$ is the unique $\mathbf{S}$-space $\mathscr{Q}$ such that the sequential convergence classes of $\mathscr{P}$ and $\mathscr{Q}$ coincide.*

Proof. Clearly it will be sufficient to prove the last statement. Let $\mathscr{C}$ be the sequential convergence class of a space $\mathscr{P}$. By 35 B.3, $\mathscr{C}$ is a determining convergence relation for a space $\mathscr{Q}$ which is an $\mathbf{S}$-space by definition, and the identity mapping of $\mathscr{Q}$ into $\mathscr{P}$ is continuous because $\mathscr{C} \subset \mathbf{Lim}\,\mathscr{P}$. Evidently $\mathscr{C}$ is the sequential convergence class of $\mathscr{Q}$. If $\mathscr{R}$ is an $\mathbf{S}$-space such that the identity mapping of $\mathscr{R}$ onto $\mathscr{P}$ is continuous, then the sequential convergence class $\mathscr{D}$ of $\mathscr{R}$ is contained in $\mathbf{Lim}\,\mathscr{P}$, and hence in $\mathscr{C}$, which implies that the identity mapping $\mathscr{R}$ onto $\mathscr{Q}$ is continuous because $\mathscr{Q}$ is an $\mathbf{S}$-space. Thus $\mathscr{Q} = \boldsymbol{\sigma}\mathscr{P}$. The uniqueness is evident.

Remark. Notice that the proof of 35 C.10 does not depend on the results 35 C.3 – 35 C.9. In fact we have given an alternate proof of the existence of sequential modification and hence a new proof of 35 C.7 (but not a new proof of 35 C.9).

The following example shows that the class $\mathbf{S}$ is not meet-preserving in $\mathbf{C}$.

35 C.11. Example. Let P be the ordered set of all ordinals less than or equal to the first uncountable ordinal ω_1, u be the order closure for P, and let $\mathscr{C}$ be the sequential convergence class of $\langle P, u\rangle$. For each ordinal $\alpha < \omega_1$ let u_α be the finest closure for P coarser than u and such that the subspace of all $a \geqq \alpha$ is an accrete subspace of $\langle P, u_\alpha\rangle$. One can easily show that

(a) u is the infimum of $\{u_a \mid a < \omega_1\}$ (taken in $\mathbf{C}(P)$);

(b) each closure u_α is of a countable local character;

(c) the sequential convergence class $\mathscr{C}_\alpha$ of $\langle P_\alpha, u_\alpha\rangle$ consists of all pairs from $\mathscr{C}$, and all pairs $\langle N, x\rangle$ such that $x \geqq \alpha$ and each $\beta < \alpha$ is an eventual strict lower bound of N;

(d) $\mathscr{C} = \bigcap\{\mathscr{C}_\alpha \mid \alpha < \omega_1\}$;

(e) the closure v for which $\mathscr{C}$ is a determining convergence class, is the infimum of $\{u_a\}$ in $\mathbf{S}(P)$ and simultaneously it is the lower modification of u in $\mathbf{S}(P)$, and consequently v is finer than u;

(f) the closure u does not belong to $\mathbf{S}(P)$ (no sequence ranging in $P - (\omega_1)$ converges to ω_1 in $\langle P, u\rangle$, and hence ω_1 is isolated in $\langle P, v\rangle$), which shows that v is strictly finer than u. Consequently, u has no upper modification in $\mathbf{S}(P)$.

Now we shall give a rather interesting result. First let us recall that each closure has an upper modification in the class of all topological closures (because $\boldsymbol{\tau}\mathbf{C}$ is projective-stable), and a closure need not possess a lower modification in $\boldsymbol{\tau}\mathbf{C}$ (because $\boldsymbol{\tau}\mathbf{C}$ is not join-stable in $\mathbf{C}$), and moreover a closure is topological provided it has a lower modification in $\boldsymbol{\tau}\mathbf{C}$ (31 ex. 3). On the other hand, there exists a closure which is not uniformizable but has a lower modification in $\boldsymbol{\upsilon}\mathbf{C}$.

35 C.12. Theorem. *If a closure operation has an upper modification in* $\mathbf{S}$, *then it is an* $\mathbf{S}$*-closure.*

Proof. Let u be a closure for P having an upper modification v_1 in $\mathbf{S}(P)$. Consider also its lower modification v, and denote by $\mathscr{C}_1$ and $\mathscr{C}$ the sequential convergence classes of v_1 and v respectively. By 35 C.10, $\mathscr{C}$ is also the sequential convergence class of u. Suppose that $v_1 \neq u$. We shall derive a contradiction by constructing a $v_2 \in \mathbf{S}(P)$ coarser than u and strictly finer than v_1. Clearly also $v_1 \neq v$ and hence there exists a subset X of P and an $x \in P$ such that $x \in v_1 X$ but $x \notin vX$. Choose a sequence S ranging in X and converging to x in $\langle P, v_1\rangle$. Clearly $\langle S, x\rangle \notin \mathscr{C}$. In consequence, there exists a subsequence N of S such that $\langle M, x\rangle \in \mathscr{C}$ for no subsequence M of N. Since $\mathscr{C}$ is also the sequential convergence class of $\langle P, u\rangle$, no subsequence M of N converges to x in $\langle P, u\rangle$, in particular, N does not converge to x in $\langle P, u\rangle$. Therefore there exists a neighborhood U of x in $\langle P, u\rangle$ such that N is not eventually in U, and hence, some subsequence M of N ranges in $P - U$. Remove from $\mathscr{C}_1$ all pairs $\langle R, x\rangle$ such that R is a subsequence of M and denote the resulting set by $\mathscr{C}_2$. Obviously $\mathscr{C}_2 \supset \mathscr{C}$, $\mathscr{C}_2$ is a sequential structure and the closure v_2, for which $\mathscr{C}_2$ is a determining convergence relation, belongs to $\mathbf{S}(P)$ and is strictly finer than v_1 (because $\mathscr{C}_2 \subset \mathscr{C}_1$, $\mathscr{C}_2 \neq \mathscr{C}_1$). On the other hand v_2 is coarser than u. We shall prove that $y \in uY - Y$

implies that $y \in v_2 Y$. If $y \neq x$, then evidently $y \in v_1 Y$ if and only if $y \in v_2 Y$, and therefore $y \in v_2 Y$ because v_1 is coarser than u and hence $y \in v_1 Y$. If $y = x$, then clearly $x \in u(Y - \mathbf{E}M)$, hence $x \in v_1(Y - \mathbf{E}M)$ which implies $x \in v_2(Y - \mathbf{E}M)$ because the relativization of v_2 to $P - \mathbf{E}M$ coincides with the relativization of v_1 to $P - \mathbf{E}M$. The proof is complete.

Alternate proof. Clearly any quasi-discrete closure is an **S**-closure. By 31 D.3 every closure is a greatest lower bound of quasi-discrete closures. Hence $v_1 = u$.

By 35 C.11 or 35 C.12 the class **S** is not completely meet-stable in **C**. Now we shall prove that the class **S** is not meet-stable in **C**.

35 C.13. *There exist* **L**-*closures u and v for a countable set P such that the infimum w of u and v is not an* **S**-*closure and the sequential modification of w is the discrete closure for P, i.e. the infimum of u and v in* **S**(P) *is the discrete closure. The space* $\langle P, w\rangle$ *(and hence both spaces* $\langle P, u\rangle$ *and* $\langle P, v\rangle$*) can have the following property: no two points are separated in* $\langle P, w\rangle$*, that is, any neighborhood of any point is dense in* $\langle P, w\rangle$*. Consequently, no point of* $\langle P, u\rangle$ *or* $\langle P, v\rangle$ *is of a countable local character.*

Both spaces will be constructed by the same method (related to that used in 35 B.5).

(a) *Let P be a countable dense subset of the space* R *of reals and let* $\{X_x \mid x \in P\}$ *be a disjoint family of dense subsets of* R *such that* $P \cap X_x = \emptyset$ *for each x (such P and* X_x *can be chosen by 22 ex. 7). Let us define a closure operation u for P as follows:* $y \in uY$ *if and only if either* $y \in Y$ *or the closure of Y in* R *intersects the set* X_y*. It is easily seen that u is indeed a closure for the set P.*

(α) *A subset U of P is a neighborhood of a point y in* $\langle P, u\rangle$ *if and only if* $y \in U$ *and* $U \supset P \cap G$ *for some open set G in* R *containing* X_y *(thus G is dense in* R*).*

Indeed, if U fulfils the condition then clearly $y \notin u(P - U)$; and conversely, if $y \notin u(P - U)$ then $y \in U$ and the closure F of $P - U$ in R does not intersect the X_y and consequently we can put $G = \mathrm{R} - F$. From (α) we will derive:

(β) *If* $\{Y_a \mid a \in A\}$ *is a finite family of subsets of P such that each* Y_a *is a neighborhood of at least one point, then the intersection Y of* $\{Y_a\}$ *is an infinite set.*

Indeed, let Y_a be a neighborhood of y_a in $\langle P, u\rangle$ and let $G_a \supset X_{y_a}$ be an open subset of R such that $P \cap G_a \subset Y_a$. Since X_{y_a} is dense in R for each a, each G_a is also dense, and R being a topological space and G_a being open, the intersection G of $\{G_a \mid a \in A\}$ is also an open dense subset of R. Clearly $P \cap G \subset Y$. Since P is dense in R, $P \cap G$ is an open dense subset of the subspace $\langle P, v\rangle$ of R. But $\langle P, v\rangle$ is infinite and semi-separated and hence the set $P \cap G$ is infinite.

(γ) *No two points of* $\langle P, u\rangle$ *are separated (see* (β)*).*

(δ) *If a sequence S converges to a point y in* $\langle P, u\rangle$*, then the set Z of all accumulation points of S in the space* R *is non-void and is contained in the set* $(y) \cup X_y$*.*

First we shall prove that $Z \subset (y) \cup X_y$. Let z be any accumulation point of S in

R; there exists a subsequence N of S converging to z in R. Moreover, N can be chosen constant or one-to-one. If N is constant, then obviously $N_n = z$ for each n and consequently $z = y$. If N is one-to-one, then z is the unique accumulation point in R of the set Y of all N_n. On the other hand, since S converges to y in $\langle P, u \rangle$, the sequence N also converges to y in $\langle P, u \rangle$, and consequently $y \in uY_n$ for each n where Y_n is the set of all N_k, $n \leqq k$. Since N is one-to-one, there exists an n so that $y \notin Y_n$. Since $y \in uY_n$, the closure of Y_n in R must intersect the set X_y. But the closure of Y_n in R is contained in that of Y which is contained in $Y \cup (z) \subset P \cup (z)$. Since $P \cap X_y = \emptyset$, $z \in X_y$; this completes the proof of the fact $Z \subset (y) \cup X_y$.

It remains to show that $Z \neq \emptyset$. Suppose that $Z = \emptyset$. Then the set Y of all S_n, $n \in \mathrm{N}$, is discrete. Consider the sequence $\{Y_n\}$ where Y_n is the set of all S_k, $n \leqq k$. Since S converges to y in $\langle P, u \rangle$ we have $y \in uY_n$ for each n. The closure of Y_n in R is contained in that of Y in R, which equals Y since Y is discrete in R. But $Y \subset P \subset \subset \mathrm{R} - X_y$ and consequently, by definition of u, $y \in Y_n$. But this implies that y is an accumulation point of S in R, which contradicts our assumption $Z = \emptyset$.

(ε) *Suppose that a sequence S converges to a point y in $\langle P, u \rangle$ and y is the only cluster point of S in the space* R. *Then $S_n = y$ for sufficiently large n.*

Proof. I. Consider the sequence $\{Y_n\}$ where Y_n is the set of all S_k, $k \geqq n$. Since y is the unique accumulation point of S in R, the closure of Y_n in R is $Y_n \cup (y)$ for each n. Since S converges to y in $\langle P, u \rangle$ we have $y \in uY_n$ for each n, and hence $y \in Y_n$ for each n because $(Y_n \cup (y)) \cap X_y = \emptyset$. Thus S frequently equals y.

II. If N is any subsequence of S, then N converges to y in $\langle P, u \rangle$ as well and by our assumption y is the unique accumulation point of N in R. By the first part of the proof the sequence N frequently equals y.

III. By II every subsequence of S frequently equals y. By a simple argument the reader shows that S eventually equals y.

(η) *$\langle P, u \rangle$ is an* **L**-*space.*

By (δ) the sequential convergence class $\mathscr{C}$ of $\langle P, u \rangle$ is single-valued. It remains to show that $\mathscr{C}$ is a determining convergence relation of $\langle P, u \rangle$. Suppose $y \in uY - Y$. By definition of u the closure of Y in R intersects X_y, and consequently we can choose a sequence S ranging in Y and converging to a point of X_y. It is easily seen that S converges to y in $\langle P, u \rangle$.

(b) Let us choose a countable dense subset P of R and disjoint families $\{X_y \mid y \in P\}$ and $\{Z_y \mid y \in P\}$ of dense subsets of R such that $P \cap X_y = \emptyset$, $P \cap Z_y = \emptyset$ and $X_{y_1} \cap Z_{y_2} = \emptyset$ for each y, y_1 and y_2 in P. Let u be the closure for P constructed in (a) and let v be the closure for P constructed in (a) with X_y replaced by Z_y. From the assertions (δ) and (ε) of (a) it follows that the infimum of u and v in $\mathbf{S}(P)$ is the discrete closure. Indeed, if a sequence S converges to a point y in both $\langle P, u \rangle$ and $\langle P, v \rangle$ then the point y is the unique cluster point of S in the space R of reals by (δ), and by virtue of (ε) the sequence S eventually equals y. On the other hand, by virtue of (β) the infimum w of u and v in $\mathbf{C}(P)$ is not the discrete closure for P. It is sufficient to show that no two points of $\langle P, w\rangle$ are separated in $\langle P, w\rangle$ and to prove this it will suffice to show that any neighborhood of every point contains a set of the form $G \cap P$, where G is an open dense set in the space R. Indeed, if G_1 and G_2 are open and dense in R, then $G_1 \cap G_2$ is also open and dense in R, and hence, P being dense in R, $(G_1 \cap P) \cap (G_2 \cap P) = G_1 \cap G_2 \cap P \neq \emptyset$. Let y be any point of $\langle P, w\rangle$. The collection $\mathscr{W}$ of all sets of the form $U \cap V$, U and V being neighborhoods of y in $\langle P, u\rangle$ and $\langle P, v\rangle$ respectively, is a local base at y in $\langle P, w\rangle$. Given U and V, by (α) we can choose open dense sets G_1 and G_2 in R such that $G_1 \cap P \subset U$ and $G_2 \cap P \subset V$. Put $G = G_1 \cap G_2$. The set G is open and dense in R, and clearly $G \cap P \subset U \cap V$.

Remark. Choose a point z in P. Consider the closure u_1 such that $P - (z)$ is discrete and the u_1-neighborhoods and u-neighborhoods coincide, and also the closure v_1 obtained similarly from v. If $w_1 = \inf(u_1, v_1)$, then w_1 is not discrete and the **S**-modification of w_1 is discrete. Of course, u_1 and v_1 are **S**-closures. Since z is the unique cluster point of u_1 as well as of v_1, both spaces $\langle P, u_1\rangle$ and $\langle P, v_1\rangle$ are paracompact. It should be noted that a direct construction of u_1 and v_1 is essentially simpler than that of u and v.

35 C.14. Corollary. *The product of two* **L**-*spaces need not be an* **S**-*space.*

Proof. Consider the product $\mathscr{R} = \langle P, u\rangle \times \langle P, v\rangle$ where u and v are the closures constructed in 35 C.13. The mapping $f = \{x \to \langle x, x\rangle\} : \langle P, \inf(u, v)\rangle \to \mathscr{R}$ is an embedding and $\mathbf{D}^*f$ is not an **S**-space by 35 C.13. Therefore the subspace $\mathbf{E}f$ of $\mathscr{R}$ is not an **S**-space and hence $\mathscr{R}$ is not an **S**-space because the class **S** is hereditary.

The infimum of two **L**-closures need not be an **S**-closure. On the other hand the following proposition follows immediately from the description of neighborhoods relative to the infimum of a family of closures.

35 C.15. *The greatest lower bound of a countable family of closures of countable local character is a closure of a countable local character, in particular, an* **S**-*closure.*

Now we proceed to an examination of the class **L**. First we shall state trivialities.

35 C.16. *The class* **L** *of all* **L**-*closures is down-saturated in the class* **S**. *The class of all* **L**-*spaces is hereditary and closed under sums.*

By definition an **L**-space is an **S**-space whose sequential convergence class is single-valued. Remember that a closure space $\mathscr{P}$ is separated if and only if the convergence class of $\mathscr{P}$ is single-valued. Thus **L**-spaces are characterized among all **S**-spaces similarly as separated spaces among all closure spaces. It turns out that **L**-spaces have properties similar to separated spaces, e.g. 35 B.16. On the other hand, in some points the properties are rather different. E.g. we know that the supremum of a monotone family of separated closures need not be a separated closure and a separated closure may be finer than no coarse separated closure. For **L**-closures we shall prove

35 C.17. Theorem. *The supremum of a non-void monotone family of* **L**-*closures is an* **L**-*closure.*

Proof. Let $\{u_a\}$ be a monotone family in $\mathbf{L}(P)$, that is, $\{u_a\}$ is non-void and for each a_1 and a_2 either u_{a_1} is finer than u_{a_2} or u_{a_2} is finer than u_{a_1}, and let u be the supremum of $\{u_a\}$ in $\mathbf{C}(P)$. Since $\mathbf{S}$ is completely join-stable, u is an $\mathbf{S}$-closure. Now, to prove that u is an $\mathbf{L}$-closure it is enough to show (by 35 B.14) that there exists a determining $\mathbf{L}$-structure $\mathscr{C}$ for $\langle P, u\rangle$, that is, a single-valued sequential convergence structure $\mathscr{C}$ for $\langle P, u\rangle$. Let $\mathscr{C}$ be the union of $\{\mathscr{C}_a\}$ where $\mathscr{C}_a$ is the sequential convergence class of $\langle P, u_a\rangle$ for each a. Evidently $\mathscr{C}$ is a determining sequential structure for $\langle P, u\rangle$. The fact that $\mathscr{C}$ is single-valued follows from the monotonicity of $\{u_a\}$. First notice that the family $\{\mathscr{C}_a\}$ is also monotone (relative to the inclusion); now if $\langle N, x\rangle$, $\langle N, y\rangle \in \mathscr{C}$, then both $\langle N, x\rangle$ and $\langle N, y\rangle$ belong to some $\mathscr{C}_a$, and since $\mathscr{C}_a$ is single-valued, we obtain $x = y$.

35 C.18. Corollary. *For any $\mathbf{L}$-closure u there exists a maximal $\mathbf{L}$-closure coarser than u, i.e. an $\mathbf{L}$-closure v such that v is coarser than u and that if an $\mathbf{L}$-closure w is coarser than v, then $w = v$.*

35 C.19. A closure u is said be a *coarse $\mathbf{L}$-closure* if u is an $\mathbf{L}$-closure and each $\mathbf{L}$-closure coarser than u coincides with u.

The class of all coarse $\mathbf{L}$-closures will be studied in the exercises. It is to be noted that the supremum of two $\mathbf{L}$-closures need not be an $\mathbf{L}$-closure. For example, consider the space $\langle P, u\rangle$ from 35 B.17 (a) and let u_i be the closure for P obtained from u by declaring the set (x_i) to be open. Obviously both u_1 and u_2 are (separated) $\mathbf{L}$-closures, but the supremum of u_1 and u_2 is the closure u which is not an $\mathbf{L}$-closure. This example also shows that there exists no coarsest $\mathbf{L}$-closure finer than u. One can show that, more generally, a closure u for a set P admits a lower modification in $\mathbf{L}(P)$ if and only if its sequential modification is an $\mathbf{L}$-closure.

In conclusion we shall introduce three important classes of $\mathbf{L}$-spaces the properties of which will be discussed in the exercises.

35 C.20. (a) Sequential convergence closure for an ordered set $\langle P, \leqq\rangle$. In 15 B.16 we defined the order-limit of a net N in $\langle P, \leqq\rangle$ which was denoted by $\lim N$. The set $\mathscr{C}$ of all pairs $\langle S, \lim S\rangle$ where S runs over all order-convergent sequences in $\langle P, \leqq\rangle$, is a single-valued sequential convergence structure for $\mathscr{C}$, i.e. an $\mathbf{L}$-structure for P. The $\mathbf{L}$-structure $\mathscr{C}$ determines an $\mathbf{L}$-closure for $\langle P, \leqq\rangle$ which will be called the *sequential closure for $\langle P, \leqq\rangle$*.

(b) Sequential closure for $\exp P$. The sequential closure for $\exp P$, where P is a set, is defined to be the sequential convergence closure for the ordered set $\langle \exp P, \subset\rangle$.

(c) Sequential closure for closed subsets of a closure space. Let $\mathscr{E}$ be the set of all closed subsets of a space $\langle P, u\rangle$ and let $\mathscr{C}$ be the set of all pairs $\langle S, F\rangle$ such that S is a sequence in $\mathscr{E}$ topologically convergent to F. The relation $\mathscr{C}$ is an $\mathbf{L}$-structure for P. The closure determined by $\mathscr{C}$ will be called the *sequential convergence closure for closed subsets of $\langle P, u\rangle$*.

D. PROJECTIVE GENERATION IN A GIVEN CLASS

In subsection 33 B we studied inductive constructions in a given class K of closure spaces (Definition 33 B.3), in particular, in a projective-stable class K. The theory developed was applied to the inductive construction in the class of all topological spaces. Up to present section we have considered no important and interesting inductive-stable class which is not projective-stable. The class $\mathbf{S}$ is very interesting, and possibly important, and it seems to be appropriate for illustrating general theorems on projective constructions in an inductive-stable class.

In what follows let K be a class of closure spaces, L be the class consisting of closure structures of spaces of K, λ be the relation consisting of all pairs $\langle u, v\rangle$ such that v is the lower modification of u in L, and finally, $\kappa = \{\langle P, u\rangle \to \langle P, \lambda u\rangle\}$.

Notice that the classes K and L were usually denoted by the same symbol, e.g. $\tau\mathbf{C}$ denotes both the class of all topological spaces and the class of all topological closure operations, the symbols $\mathfrak{v}\mathbf{C}$, $\mathbf{S}$ and $\mathbf{L}$ were used similarly, and the relations λ and κ were usually denoted by the same symbol, e.g. τ, $\mathfrak{v}$, σ.

We want to introduce all definitions without any assumption on K. Nevertheless, the main theorems require some of the following additional assumptions on K which are fulfilled if $K = \mathbf{S}$: K is inductive-stable, $\mathbf{D}\kappa = K$, K is hereditary. By 33 C.11 if K is inductive-stable, then $\mathbf{D}\kappa = K$. If K is the class of all semi-separated closures, then $\mathbf{D}\kappa = \mathbf{C}$ but K is not inductive-stable. If K is the class of all locally connected spaces, then K is inductive-stable, but it is not hereditary.

35 D.1. Definition. A closure u for a set P is said to be *K-projectively generated* (or *projectively generated in K*) by a family of mappings $\{f_a\}$ if $\{f_a\}$ is a projective family of mappings for closure spaces with a common domain carrier P or $\langle P, u\rangle$, and u is the coarsest closure such that $\langle P, u\rangle \in K$ and all the mappings $f_a : \langle P, u\rangle \to \to \mathbf{E}^*f_a$ are continuous; the family $\{f_a : \langle P, u\rangle \to \mathbf{E}^*f_a\}$ is said to be a *K-projective generating family for $\langle P, u\rangle$*. The definitions just stated are carried over to collections of mappings and to single mappings in a standard manner. A class K_1 is said to be *K-projective-stable* or *projective-stable in K* if $K_1 \subset K$ and the common domain carrier of any K-projective generating family with range carriers in K_1 belongs to K_1.

From the definition we obtain immediately

35 D.2. *Let f_a be a projective family of mappings for closure spaces with a common domain carrier $\langle P, u\rangle$. Then u is K-projectively generated by the family $\{f_a\}$ if and only if u is the lower modification in L of the closure operation projectively generated by the family $\{f_a: P \to \mathbf{E}^*f_a\}$.*

35 D.3. Corollary. *If $\{f_a\}$ is a projective generating family for closure spaces with a common domain carrier $\mathscr{P}$ and if $\kappa\mathscr{P}$ exists, then $\{f_a : \kappa\mathscr{P} \to \mathbf{E}^*f_a\}$ is a K-projective generating family.*

Of course, a non-void family can K-projectively generate at most one closure operation. If $K = \emptyset$, then no family K-projectively generates a closure operation. From the existence of projectively generated closures and from 35 D.2 we obtain at once the following theorem.

35 D.4. Theorem. $\mathbf{D}\kappa = \mathbf{C}$ *(i.e. every closure has a lower modification in L) if and only if every projective family for closure spaces K-projectively generates a closure operation.*

35 D.5. If f is a projective generating mapping for closure spaces then $f : \kappa\mathbf{D}^*f \to \to \kappa\mathbf{E}^*f$ need not be a K-projective mapping. For example consider a class K which is not inductive-stable but which has the property $\mathbf{D}\kappa = \mathbf{C}$, e.g. one may take the class of all semi-separated spaces. It is easily seen from 33 C.11 that there exists an inductive generating mapping f such that $\mathbf{D}^*f \in K$ but $\mathbf{E}^*f \notin K$. Thus the closure structure of $\kappa\mathbf{E}^*f$ is strictly finer than the closure structure of $\mathbf{E}^*f$ and consequently the mapping $f : \mathbf{D}^*f \to \kappa\mathbf{E}^*f$ is not continuous; now only recall that $\mathbf{D}^*f \in K$, i.e. $\kappa\mathbf{D}^*f = \mathbf{D}^*f$. Let $\mathscr{P}$ be the space projectively generated by the mapping $f : \mathbf{D}f \to \mathbf{E}^*f$. Since f is continuous, the closure structure of $\kappa\mathscr{P}$ is coarser than the closure structure of $\kappa\mathbf{D}^*f = \mathbf{D}^*f$. It follows that the mapping $f : \kappa\mathscr{P} \to \kappa\mathbf{E}^*f$ is not continuous. On the other hand, $f : \kappa\mathscr{P} \to \mathbf{E}^*f$ is a K-projective generating mapping.

35 D.6. Theorem. *If K is inductive-stable and if $\{f_a\}$ is a projective generating family for closure spaces, then $\{\kappa f_a\}$ is a K-projective generating family, where κf_a denotes the transpose of f_a to a mapping for spaces from K, i.e. $\kappa f_a = f_a : : \kappa\mathbf{D}^*f_a \to \kappa\mathbf{E}^*f_a$.*

Proof. Let $\{f_a\}$ be a projective generating family for a closure space $\mathscr{P}$. By 35 D.3 $\{f_a : \kappa\mathscr{P} \to \mathbf{E}^*f_a\}$ is a K-projective generating family. Since the closure structure of $\kappa\mathbf{E}^*f_a$ is finer than that of $\mathbf{E}^*f_a$ for each a, to prove that $\{\kappa f_a\}$ is a K-projective generating family it is sufficient to show that each κf_a is continuous. Fix an arbitrary a and consider the space $\mathscr{Q}$ inductively generated by the mapping $f_a : \kappa\mathscr{P} \to |\mathbf{E}^*f_a|$. Evidently the closure structure of $\mathscr{Q}$ is finer than that of $\mathbf{E}^*f_a$ and thus finer than that of $\kappa\mathbf{E}^*f$ because K is inductive-stable and $\kappa\mathscr{P} \in K$; this implies that κf_a is continuous. It should be remarked that the main step of the proof consisted in showing that κf is continuous whenever f is continuous and K is inductive-stable.

The following characterization of K-projective generating families of mappings in an inductive-stable class is a generalization of Theorem 32 A.10.

35 D.7. Theorem. *Suppose that K is an inductive-stable class of closure spaces. Then every projective family of mappings for closure spaces K-projectively generates a closure operation, and in order that a projective family of mappings for closure spaces $\{f_a\}$ with a common domain carrier $\langle P, v\rangle \in K$ be a K-projective generating family it is necessary and sufficient that a mapping f of a space $\mathscr{Q} \in K$ into $\langle P, v\rangle$ be continuous if and only if all the mappings $f_a \circ f$ are continuous.*

We shall need the following characterization of the lower modification in an inductive-stable class.

35 D.8. Theorem. *Let u be a closure for a set P and let v be a closure from $L \cap \cap \mathbf{C}(P)$ such that the following condition is fulfilled:*

(*) *A mapping $f : \mathcal{Q} \to \langle P, u \rangle$, $\mathcal{Q} \in K$, is continuous if and only if the mapping $g = f : \mathcal{Q} \to \langle P, v \rangle$ is continuous.*

Then v is the lower modification of u in L, i.e. $v = \lambda u$. Conversely, if K is inductive-stable, i.e. $\operatorname{ind} K = K$, *then the lower modification in L of any closure u for P is the unique closure v satisfying condition* (*).

Proof. I. Assume that a closure $v \in L$ fulfils (*). Since $\mathsf{J} : \langle P, v \rangle \to \langle P, v \rangle$ is continuous and $v \in L$, by (*) the mapping $\mathsf{J} : \langle P, v \rangle \to \langle P, u \rangle$ is continuous and hence v is finer than u. If $w \in L$ is finer than u, then the mapping $\mathsf{J} : \langle P, w \rangle \to \langle P, u \rangle$ is continuous, and by (*) the mapping $\mathsf{J} : \langle P, w \rangle \to \langle P, v \rangle$ is continuous, i.e. v is coarser than w. Thus $v = \lambda u$.

II. Suppose that K is inductive-stable. By 33 C.11 each closure has a lower modification in L, i.e. $\mathbf{D}\lambda = \mathbf{C}$. Given a closure u for P we shall prove that $v = \lambda u$ fulfils condition (*). If g is continuous, then f is continuous because v is finer than u. Conversely, assuming that f is continuous, let us consider the closure w inductively generated by the mapping $f : \mathcal{Q} \to P$. Thus w is finer than u, and K being inductive-stable, w belongs to L. Consequently w is finer than $v = \lambda u$. Since $f : \mathcal{Q} \to \langle P, w \rangle$ is continuous, g is also continuous.

35 D.9. Proof of 35 D.7. Let us consider the closure u projectively generated by the family $\{f_a : P \to \mathbf{E}^* f_a\}$. Since K is inductive-stable, by 33 C.11 each closure has a lower modification in L (i.e. $\mathbf{D}\lambda = \mathbf{C}$). By 35 D.3 the closure $v = \lambda u$ is K-projectively generated by the family $\{f_a : P \to \mathbf{E}^* f_a\}$. By 32 A.10 a mapping f into $\langle P, u \rangle$ is continuous if and only if all the mappings $f_a \circ f$ are continuous. By 35 D.8 a mapping f of a space of K into $\langle P, u \rangle$ is continuous if and only if the mapping $f : \mathbf{D}^* f \to \langle P, v \rangle$ is continuous. Combining these two results we find that the conditions is necessary. It is evident that the condition is sufficient.

35 D.10. Remark. Proposition 35 D.8 can be extended as follows. A class K is inductive-stable if and only if $\mathbf{D}\lambda = \mathbf{C}$ and $u \in \mathbf{C}(P)$, $v = \lambda u$ imply (*). Indeed, if K is not inductive-stable and if $\mathbf{D}\lambda = \mathbf{C}$, then there exists an inductive generating mapping f such that $\mathbf{D}^* f \in K$, $\mathbf{E}^* f \notin K$; clearly $\kappa \mathbf{E}^* f \neq \mathbf{E}^* f$. Thus the mapping f is continuous but the mapping $f : \mathbf{D}^* f \to \kappa \mathbf{E}^* f$ is not continuous, which shows that condition (*) is not fulfilled. It is evident that Theorem 35 D.6 can be proved in a similar way.

Now we shall prove that 35 D.7 implies the associativity of projective construction in an inductive-stable class K. Notice that Theorem 32 A.9 is obtained for $K = \mathbf{C}$.

35 D.11. Theorem on associativity. *Suppose that K is an inductive-stable class of closure spaces and $\{f_a \mid a \in A\}$ is a projective family for closure spaces with common domain carrier $\mathcal{P}$. For each a let $\mathbf{E}^* f_a$ be K-projectively generated by a family $\{g_{ab} \mid b \in B_a\}$. Then $\{f_a\}$ is a K-projective generating family if and only if the family $\{g_{ab} \circ f_a \mid a \in A, b \in B_a\}$ is a K-projective generating family.*

Proof. Both the projective families in question have the same common domain carrier. By 35 D.7 the statement that $\{f_a\}$ or $\{f_a \circ g_{ab}\}$ is a K-projective generating family is equivalent to the statement that if f is a mapping of a space of K into $\mathscr{P}$, then f is continuous if and only if all the mappings $f_a \circ f$ or $(g_{ab} \circ f_a) \circ f$ respectively, are continuous. Since each $\{g_{ab} \mid b \in B_a\}$, $a \in A$, is a K-projective generating family, again by 35 D.7, all $g_{ab} \circ (f_a \circ f)$, $b \in B_a$, are continuous if and only if the mapping $f_a \circ f$ is continuous. Since $g_{ab} \circ (f_a \circ f) = (g_{ab} \circ f_a) \circ f$ the two statements are equivalent.

Corollary. *If K is inductive-stable then a projective family $\{f_a\}$ is a K-projective generating family if and only if $\{f_a : \mathbf{D}^* f_a \to \kappa \mathbf{E}^* f_a\}$ is such.*

Remark. Neither "if" nor "only if" hold in 35 D.11 whenever $\mathbf{D}\kappa = \mathbf{C}$ and K is not inductive-stable. Indeed, under these assumptions there exists a projective generating mapping f of a space $\mathscr{P}$ into a space $\mathscr{Q}$ such that the mapping $\kappa f = f : \kappa\mathscr{P} \to \kappa\mathscr{Q}$ is not continuous (by 35 D.5). Now $\mathsf{J} : \kappa\mathscr{Q} \to \mathscr{Q}$ is a K-projective generating mapping, the composite $(\mathsf{J} : \kappa\mathscr{Q} \to \mathscr{Q}) \circ \kappa f = f : \kappa\mathscr{P} \to \mathscr{Q}$ is a K-projective generating mapping, but κf is not a K-projective generating mapping, which shows that the "if" is not true. Let us consider the space $\mathscr{R}$ which is K-projectively generated by the mapping $f : |\mathscr{P}| \to \kappa\mathscr{Q}$; thus $f : \mathscr{R} \to \kappa\mathscr{Q}$ and $\mathsf{J} : \kappa\mathscr{Q} \to \mathscr{Q}$ are K-projective generating mappings but their composite is not a K-projective generating mapping because $\mathscr{R} \neq \kappa\mathscr{P}$ (remember that $f : \kappa\mathscr{P} \to \kappa\mathscr{Q}$ is not continuous).

A subspace of closure space $\mathscr{P}$ can be defined to be a space $\mathscr{Q}$ such that the identity mapping of $\mathscr{Q}$ into $\mathscr{P}$ is a projective generating mapping (in particular $|\mathscr{Q}| \subset |\mathscr{P}|$).

35 D.12. Definition. A *K-subspace of a space* $\mathscr{P}$ or a subspace of $\mathscr{P}$ in K is a space $\mathscr{Q}$ such that $|\mathscr{Q}| \subset |\mathscr{P}|$ and the identity mapping of $\mathscr{Q}$ into $\mathscr{P}$ is a K-projective generating mapping.

35 D.13. *If $\mathscr{Q}$ is a subspace of a space $\mathscr{P}$, then $\kappa\mathscr{Q}$ is a K-subspace of $\mathscr{P}$. If K is inductive-stable and $\mathscr{Q}$ is a subspace of $\mathscr{P}$, then $\kappa\mathscr{Q}$ is a K-subspace of the space $\kappa\mathscr{P}$.*

Proof. If $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ is a projective generating mapping, then $\mathsf{J} : \kappa\mathscr{Q} \to \mathscr{P}$ is a K-projective generating mapping (by 35 D.3), and if, in addition, K is inductive-stable, $\mathsf{J} : \kappa\mathscr{Q} \to \kappa\mathscr{P}$ is a K-projective generating mapping (by 35 D.6).

If $\mathscr{Q}$ is a subspace of a space $\mathscr{P}$ then $\kappa\mathscr{Q}$ need not be a subspace of $\kappa\mathscr{P}$ even if K is inductive-stable. It is sufficient to show that a subspace of a space of K need not belong to K even if K is inductive-stable; e.g. the class of all locally connected spaces is inductive-stable but not hereditary.

35 D.14. *If K is hereditary and $\mathscr{P} \in K$ then $\mathscr{Q}$ is a subspace of $\mathscr{P}$ if and only if $\mathscr{Q}$ is a K-subspace of $\mathscr{P}$.* – Evident.

35 D.15. Corollary. *Let K be a hereditary inductive-stable class. If $\mathscr{Q}$ is a subspace of $\mathscr{P}$, then $\kappa\mathscr{Q}$ is a subspace of $\kappa\mathscr{P}$.*

One can define a *K-restriction of a mapping* f for closure spaces to be a mapping $f : \mathscr{P} \to \mathscr{Q}$ such that $\mathscr{P}$ is a K-subspace of $\mathbf{D}^* f$ and $\mathscr{Q}$ is a K-subspace of $\mathbf{E}^* f$. We leave the discussion of K-restrictions to the exercises. Here we shall only prove the following generalization of theorem 32 A.13 on commutativity of projective constructions with the operation of taking subspaces.

35 D.16. *If K is inductive-stable, $\{f_a\}$ is a K-projective generating family for $\mathscr{P}$, and $\mathscr{Q}$ is a K-subspace of $\mathscr{P}$, then $\{f_a : \mathscr{Q} \to \mathsf{E}^*f_a\}$ is a K-projective generating family for $\mathscr{Q}$.*

Proof. Notice that

$$\{f_a : \mathscr{Q} \to \mathsf{E}^*f_a\} = f_a \circ (\mathsf{J} : \mathscr{Q} \to \mathscr{P})$$

and $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ is a K-projective generating mapping, and apply 35 D.11.

Recall that the product of a family $\{\mathscr{P}_a\}$ of closure spaces is the set $P = \Pi\{|\mathscr{P}_a|\}$ endowed with the closure operation projectively generated by the family $\{\mathrm{pr}_a : P \to \mathscr{P}_a\}$.

35 D.17. Definition. The *K-product of a family $\{\mathscr{P}_a\}$ of closure spaces* is the set $P = \Pi\{|\mathscr{P}_a|\}$ endowed with the closure operation K-projectively generated by the family of mappings $\{\mathrm{pr}_a : P \to \mathscr{P}_a\}$.

Of course, the K-product of a family need not exist. On the other hand, from 35 D.2 we obtain immediately the following description of K-products by means of products and κ.

35 D.18. Theorem. The *K-product of a family $\{\mathscr{P}_a\}$ exists if and only if $\kappa\,\Pi\{\mathscr{P}_a\}$ exists. If $\mathscr{P} = \kappa\,\Pi\{\mathscr{P}_a\}$ exists, then $\mathscr{P}$ is the K-product of $\{\mathscr{P}_a\}$. If K is inductive-stable then the K-product exists and $\kappa\,\Pi\{\mathscr{P}_a\} = \kappa\,\Pi\{\kappa\mathscr{P}_a\}$.*

Proof. The second assertion follows from 35 D.11.

If K is inductive-stable then the K-projective construction can be reduced to construction of K-products and the closure K-projectively generated by a single mapping. The K-reduced product, which is needed for this, will not be introduced.

35 D.19. Theorem. *Suppose that K is inductive-stable and $\{f_a\}$ is a projective family of mappings with a common domain carrier $\mathscr{P}$, which is a space. Then $\{f_a\}$ is a K-projective generating family if and only if the mapping*

$$f = \{x \to \{f_a x\}\} : \mathscr{P} \to \kappa\,\Pi\{\mathsf{E}^*f_a\}$$

is a K-projective generating mapping.

Proof. Let $\mathscr{Q}$ be the space projectively generated by the family $\{f_a : |\mathscr{P}| \to \mathsf{E}^*f_a\}$, $g_a = f_a : \mathscr{Q} \to \mathsf{E}^*f_a$ and let g be the reduced product of $\{g_a\}$. By 32 A.12, g is a projective generating mapping if and only if $\{g_a\}$ is a projective generating family. It follows from 35 D.2 that $g = f : \mathscr{P} \to \Pi\{\mathsf{E}^*f_a\}$ is a K-projective generating mapping if and only if $\{f_a\}$ is a K-projective generating family. It remains to show that f is a K-projective generating family if and only if g is such: apply 35 D.11, Corollary.

Alternate proof. $f_a = (\mathrm{pr}_a : \kappa\,\Pi\{\mathsf{E}^*f_a\} \to \mathsf{E}^*f_a) \circ f$ for each a. Apply 35 D.11.

35 D.20. Example. A *K-continuous internal composition* is a topologized internal composition $\langle \sigma, u \rangle$ such that the mapping σ of the K-product of $\langle \mathbf{DD}\sigma, u \rangle$ with $\langle \mathbf{DD}\sigma, u \rangle$ into $\langle \mathbf{DD}\sigma, u \rangle$ is continuous, i.e. the mapping

$$\sigma : \langle \mathbf{D}\sigma, \lambda(u \times u) \rangle \to \langle \mathbf{DD}\sigma, u \rangle$$

is continuous. A *K-group* is a topologized group $\langle G, \sigma, u \rangle$ with continuous inversion such that the composition $\langle \sigma, u \rangle$ is K-continuous. If $K = \mathbf{S}$ then we shall speak about *sequentially continuous compositions* and *sequential groups*. If $\langle \sigma, u \rangle$ is a continuous internal composition, then $\langle \sigma, u \rangle$ is K-continuous provided that the K-product of u with u exists. If $\langle \sigma, u \rangle$ is sequentially continuous and u is an $\mathbf{S}$-closure, then $\langle \sigma, u \rangle$ is inductively continuous. Indeed, it is easily seen that ind $(u \times u)$ is finer than $\sigma(u \times u)$. For properties of $\mathbf{S}$-groups see the exercises.

35 D.21. Definition. The *K-projective progeny of a class H of spaces* is the class of all spaces K-projectively generated by a family of mappings with range carriers in H. The class H is *K-projective-stable* if it coincides with its K-projective progeny.

35 D.22. Theorem. *If H_2 is the projective progeny of a class H, then the K-projective progeny H_1 of H consists of all spaces $\kappa\mathscr{P}$, $\mathscr{P} \in H_2$, i.e. $H_1 = \kappa[H_2]$.* – 35 D.2.

We wish to describe the K-projective progeny of a given class without any reference to the projective progeny. The following concept will be needed.

35 D.23. Definition. An *L-accrete closure for a set P* is the greatest L-closure for P, i.e. the lower modification in L of the accrete closure for P. A *K-accrete space* is a space whose closure structure is L-accrete, i.e. a space $\mathscr{P}$ is a K-accrete space if and only if $\mathscr{P} = \kappa\mathscr{Q}$ for some accrete space $\mathscr{Q}$.

Example. If K is the class of all discrete spaces, then a space $\mathscr{P}$ is K-accrete if and only if $\mathscr{P}$ belongs to K.

35 D.24. Theorem. *Let K be an inductive-stable class and H be any class of spaces. Then the K-projective progeny H_1 of H consists of all K-accrete spaces and all K-subspaces of K-products of the form $\kappa(\mathscr{P} \times \kappa \Pi\{\mathscr{P}_a\})$ where $\mathscr{P}$ is a K-accrete space and $\mathscr{P}_a \in H$ for each a. If, in addition, K is hereditary, then H_1 consists of all K-accrete spaces and all subspaces of K-products of the form $\kappa(\mathscr{P} \times \kappa \Pi\{\mathscr{P}_a\})$ where $\mathscr{P}$ is a K-accrete space and $\mathscr{P}_a \in H$ for each a.*

Proof. Let H_2 be the K-projective progeny of H. By 32 B.5 H_2 consists of all accrete spaces and all subspaces of products of the form $\mathscr{R} \times \Pi\{\mathscr{P}_a\}$ where $\mathscr{R}$ is an accrete space and $\mathscr{P}_a \in H$ for each a. If K is inductive-stable then $\kappa(\mathscr{R} \times \Pi\{\mathscr{P}_a\}) = \kappa(\kappa\mathscr{R} \times \kappa \Pi\{\mathscr{P}_a\})$ (by 35 D.18), and if f is a K-projective generating mapping, then κf is a K-projective generating mapping (by 35 D.6). Now the first statement follows from 35 D.22. The second statement follows from the first one and from 35 D.15.

35 D.25. Theorem. *Let K be hereditary and inductive-stable. Each of the following two conditions is necessary and sufficient for a space $\mathscr{Q}$ to be an element of the K-projective progeny of $(\mathscr{R})$, where $\mathscr{R}$ is a non-void space:*

(a) $\mathcal{Q}$ *is K-projectively generated by a mapping into a cube* $\mathcal{R}^{\aleph}$.

(b) $\mathcal{Q}$ *is K-projectively generated by a mapping into* $\kappa\mathcal{R}^{\aleph}$.

(c) $\mathcal{Q}$ *is a K-subspace of a space of the form* $\mathcal{Q}' \times \mathcal{R}^{\aleph}$, *where* $\mathcal{Q}'$ *is an appropriate accrete space (which can be chosen so that* $|\mathcal{Q}'| = |\mathcal{Q}|$*) and* $\aleph$ *is an appropriate cardinal.*

(d) $\mathcal{Q}$ *is a subspace of a space of the form* $\kappa(\mathcal{Q}' \times \kappa\mathcal{R}^{\aleph})$, *where* $\mathcal{Q}'$ *is an appropriate K-accrete space (which can be chosen so that* $|\mathcal{Q}| = |\mathcal{Q}'|$*) and* $\aleph$ *is an appropriate cardinal.*

Proof: 35 D.24.

35 D.26. Theorem. *Let K be an inductive-stable class and let* H_1 *be the K-projective progeny of a class H. Then* H_1 *is K-projective-stable, in particular,* H_1 *is closed under K-products, and K-subspaces of spaces of* H_1 *belong to* H_1 *(i.e.* H_1 *is K-hereditary). If H is hereditary, then* H_1 *is also hereditary.*

Proof: 35 D.11, 35 D.13.

E. SEQUENTIAL MODIFICATIONS OF UNIFORMIZABLE SPACES

Here we shall examine the basic properties of the class consisting of sequential modifications of uniformizable spaces, i.e. the class $\sigma[\mathfrak{v}\mathbf{C}]$ which will be denoted by $\sigma\mathfrak{v}\mathbf{C}$. Since $\mathfrak{v}\mathbf{C}$ is the projective progeny of the space R of reals, by 35 D.22 the class $\sigma\mathfrak{v}\mathbf{C}$ is the **S**-projective progeny of R; thus the theory developed in 35 D applies. Since **S** is hereditary and inductive-stable, from the results of 35 D.22, 24, 25 we obtain directly the following assertions.

35 E.1. Theorem. *The class* $\sigma\mathfrak{v}\mathbf{C}$ *is hereditary and* **S***-projective-stable, in particular, closed under* **S***-products.* – 35 D.26.

Since every accrete space is an **S**-space, every **S**-accrete space is necessarily an accrete space.

35 E.2. Theorem. *The following conditions on a space* $\mathcal{P}$ *are equivalent:*

(a) $\mathcal{P}$ *is the sequential modification of a uniformizable space.*

(b) $\mathcal{P}$ *is a subspace of a space of the form* $\sigma(\mathcal{Q} \times \sigma\mathrm{R}^{\aleph})$, *where* $\mathcal{Q}$ *is an appropriate accrete space (which can be chosen so that* $|\mathcal{Q}| = |\mathcal{P}|$*) and* $\aleph$ *is an appropriate cardinal.*

(c) $\mathcal{P}$ *is the sequential modification of a subspace of* $\mathcal{Q} \times \mathrm{R}^{\aleph}$ *where* $\mathcal{Q}$ *is an appropriate accrete space (which can be chosen so that* $|\mathcal{P}| = |\mathcal{Q}|$*) and* $\aleph$ *is an appropriate cardinal.*

(d) $\mathcal{P}$ *is* **S***-projectively generated by a mapping into a cube* $\mathrm{R}^{\aleph}$.

(e) $\mathcal{P}$ *is* **S***-projectively generated by a mapping into* $\sigma\mathrm{R}^{\aleph}$ *for some appropriate cardinal* $\aleph$.

Proof. 35 D.25.

From 35 E.2 (b) we shall derive the following characterization.

35 E.3. Theorem. *An* **S**-*space* $\mathscr{P}$ *is the sequential modification of a uniformizable space if and only if* $\mathscr{P}$ *is projectively generated by a mapping into* $\sigma\mathsf{R}^{\aleph}$ *for some appropriate cardinal* $\aleph$.

Proof. Clearly the conditions imply 35 E.2(e). Conversely, suppose that $\mathscr{P}$ belongs to $\boldsymbol{\sigma\upsilon}\mathbf{C}$. By 35 E.2(b), there exists an embedding f of $\mathscr{P}$ into $\sigma(\mathscr{Q} \times \sigma\mathsf{R}^{\aleph})$ where $\mathscr{Q}$ is an accrete space and $\aleph$ is a cardinal. It is sufficient to prove that $\sigma(\mathscr{Q} \times \sigma\mathsf{R}^{\aleph}) = \mathscr{Q} \times \sigma\mathsf{R}^{\aleph}$, because then f followed by the projection of $\mathscr{Q} \times \sigma\mathsf{R}^{\aleph}$ into $\sigma\mathsf{R}^{\aleph}$ will be a projective generating mapping by 32 B.4. To prove the equality it is sufficient to show that $\mathscr{Q} \times \sigma\mathsf{R}^{\aleph}$ is an **S**-space, i.e. $\mathscr{Q} \times \sigma\mathsf{R}^{\aleph}$ admits a sequential determining relation, and this will be proved in a more general situation.

35 E.4. *The product of an* **S**-*space and an accrete space is an* **S**-*space.*

Proof. Let $\mathscr{R} = \mathscr{Q} \times \mathscr{P}$, where $\mathscr{Q}$ is an accrete space and $\mathscr{P}$ is an **S**-space. Consider the projection π of $\mathscr{R}$ into $\mathscr{P}$. If x belongs to the closure of a set X in $\mathscr{R}$, then πx belongs to the closure of $\pi[X]$ in $\mathscr{P}$, and therefore there exists a sequence $\{y_n\}$ in $\pi[X]$ which converges to πx in $\mathscr{P}$ (because $\mathscr{P}$ is an **S**-space). Let $\{x_n\}$ be any sequence in X such that $\pi x_n = y_n$ for each n. Since $\mathscr{Q}$ is an accrete space, the sequence $\{x_n\}$ converges to any point z such that $\{\pi x_n\}$ converges to πz, in particular, $\{x_n\}$ converges to x.

Now we shall give a direct description of spaces which are sequential modifications of uniformizable spaces.

35 E.5. Theorem. *Each of the following conditions is necessary and sufficient for an* **S**-*space* $\mathscr{P}$ *to be the sequential modification of a uniformizable space:*

(a) *There exists a set* $\mathscr{C}$ *of functions on* $\mathscr{P}$ *with* $0 \leqq f \leqq 1$ *for each* f *in* $\mathscr{C}$ *such that a sequence* S *in* $\mathscr{P}$ *converges to a point* x *in* $\mathscr{P}$ *if and only if the sequence* $f \circ S$ *converges to* fx *in* R *for each* $f \in \mathscr{C}$.

(b) *If a sequence* S *in* $\mathscr{P}$ *does not converge to a point* $x \in |\mathscr{P}|$ *in* $\mathscr{P}$, *then there exists a bounded continuous function* f *on* $\mathscr{P}$ *such that the sequence* $f \circ S$ *does not converge to the point* fx *in* R.

Proof. I. If (a) is fulfilled, then each function of $\mathscr{C}$ is continuous (because $\mathscr{P}$ is an **S**-space) and therefore (a) implies (b). Next, it is easy to show that (b) implies (a). Indeed, assuming (b), take the set $\mathscr{C}$ of all continuous functions f on $\mathscr{P}$ such that $0 \leqq f \leqq 1$ for each f in $\mathscr{C}$. If a sequence S converges to x in P, then $f \circ S$ converges to fx in R for each f in $\mathscr{C}$ because each such f is continuous. If a sequence S in $\mathscr{P}$ does not converge to x in $\mathscr{P}$, then $f \circ S$ does not converge to fx for some f in $\mathscr{C}$ (by (b)). Thus the two conditions are equivalent.

II. We shall prove that condition (a) is necessary and sufficient. If $\mathscr{P}$ is the sequential modification of a uniformizable space $\mathscr{Q}$, then $|\mathscr{P}| = |\mathscr{Q}|$ and a sequence S converges to x in $\mathscr{P}$ if and only if the sequence S converges to x in $\mathscr{Q}$. Since $\mathscr{Q}$ is uniformizable, $\mathscr{Q}$ is projectively generated by the collection $\mathscr{C}'$ of all continuous functions on $\mathscr{Q}$ whose values lie between 0 and 1, and hence a sequence S converges to x in $\mathscr{Q}$ if and only if $f \circ S$ converges to fx in R for each f in $\mathscr{C}'$. Let $\mathscr{C}$ be the collection of all

$f : \mathscr{P} \to \mathrm{R}$, $f \in \mathscr{C}'$. Clearly condition (a) is fulfilled. Conversely, assuming (a) let us consider the space $\mathscr{Q}$ projectively generated by the family of mappings $\{f : |\mathscr{P}| \to \mathrm{R} \mid f \in \mathscr{C}\}$. The space $\mathscr{Q}$ is uniformizable and a sequence S converges to x in $\mathscr{Q}$ if and only if the sequence $f \circ S$ converges to fx in R for each f in $\mathscr{C}$. By condition (a) a sequence S converges to x in $\mathscr{Q}$ if and only if it converges to x in $\mathscr{P}$. Since $\mathscr{P}$ is an **S**-space, $\mathscr{P}$ is the sequential modification of $\mathscr{Q}$.

35 E.6. *If a closure space $\langle P, v \rangle$ is the sequential modification of a uniformizable space, then $\langle P, v \rangle$ is the sequential modification of its uniformizable modification.*

Proof. The statement was proved, in fact, in the last part of the proof of 35 E.5; however, a direct proof may be in place. Suppose that v is the sequential modification of a uniformizable closure u. Clearly v is finer than u. Then the uniformizable modification w of v, which is the finest uniformizable closure coarser than v, is also finer than u and consequently σw is finer than σu. Since w is coarser than σu, σw is coarser than $\sigma\sigma u = \sigma u$. Thus $\sigma w = \sigma u$.

Remark. In general, there are many uniformizable closures u such that $\sigma u = v$, and moreover a uniformizable **S**-space $\mathscr{P}$ may be the sequential modification of a uniformizable space $\mathscr{Q} \neq \mathscr{P}$. For example let $\langle P, u \rangle$ be a separated non-discrete uniformizable space such that only the eventually constant sequences are convergent (for example, one may take the ultrafilter space βN of an infinite set N), then σu is the discrete closure for P which is, obviously, uniformizable.

35 E.7. *If $\mathscr{Q}$ is a subspace of a space $\mathscr{P}$, then $\sigma\mathscr{Q}$ is a subspace of $\sigma\mathscr{P}$. If $\{\mathscr{P}_a\}$ is any family of closure spaces then*

$$\sigma\,\Pi\{\mathscr{P}_a\} = \sigma\,\Pi\{\sigma\mathscr{P}_a\}\,.$$

Proof. The first statement follows from 35 D.15 because **S** is hereditary and inductive-stable. On the other hand, a direct proof is evident. The second statement follows from 35 D.18 because **S** is inductive-stable. A direct proof is almost evident. Indeed, a sequence S converges to x in $\sigma\,\Pi\{\mathscr{P}_a\}$ ($\sigma\,\Pi\{\sigma\mathscr{P}_a\}$, respectively) if and only if the sequence $\mathrm{pr}_a \circ S$ converges to $\mathrm{pr}_a\,x$ in $\mathscr{P}_a$ (in $\sigma\mathscr{P}_a$, respectively) for each a; on the other hand, a sequence S' converges to x' in $\mathscr{P}_a$ if and only if S' converges to x' in $\sigma\mathscr{P}_a$.

From 35 E.3 we shall derive the following result:

35 E.8. Theorem. *The following conditions on a space $\mathscr{P}$ are equivalent:*

(a) *$\mathscr{P}$ is separated and $\mathscr{P}$ is the sequential modification of a uniformizable space.*

(b) *$\mathscr{P}$ is an **L**-space and $\mathscr{P}$ is the sequential modification of a uniformizable space.*

(c) *$\mathscr{P}$ is feebly semi-separated and $\mathscr{P}$ is the sequential modification of a uniformizable space.*

(d) *$\mathscr{P}$ is a homeomorph of a subspace of $\sigma\mathrm{R}^{\aleph}$ for some cardinal $\aleph$.*

Proof. Evidently (a) $\Rightarrow$ (b) $\Rightarrow$ (c). The closure structure of $\sigma\mathsf{R}^{\aleph}$ is finer than the closure structure of the cube $\mathsf{R}^{\aleph}$ which is separated, and therefore each subspace of $\sigma\mathsf{R}^{\aleph}$ is separated. Next, the sequential modification of a subspace is the subspace of the sequential modification (by 35 E.7) and therefore (d) implies (a). It remains to show that (c) implies (d). But this follows from 35 E.3 and the following simple lemma.

35 E.9. *A projective generating mapping whose domain carrier is feebly semi-separated is injective and hence an embedding.*

Proof. If $fx = fy$, then x belongs to the closure of (y) and y belongs to the closure of (x).

Remark. It is interesting to notice that if an **L**-space $\mathscr{P}$ is the sequential modification of a uniformizable space $\mathscr{Q}$, then $\mathscr{Q}$ is separated. Indeed, if $\mathscr{Q}$ is not separated, then some point x belongs to the closure of a singleton (y) in $\mathscr{Q}$, $y \neq x$ and hence the constant sequence $\{y \mid n \in \mathsf{N}\}$ converges to x in $\mathscr{Q}$ and hence in $\mathscr{P}$; this shows that $\mathscr{P}$ is not an **L**-space.

35 E.10. Corollary. *The class* $\mathbf{L} \cap \sigma\upsilon\mathbf{C}$ *is hereditary and closed under* **S**-*products.*

Proof. The class $\mathbf{L} \cap \sigma\upsilon\mathbf{C}$ is the intersection of the class of all separated spaces and of the class $\sigma\upsilon\mathbf{C}$.

35 E.11. In conclusion we shall give an alternate proof of 35 E.3 (which does not depend on the results of 35 D). First suppose that h is a projective generating mapping of an **S**-space $\mathscr{P}$ into $\sigma\mathsf{R}^{\aleph}$.

If a sequence S does not converge to a point x in $\mathscr{P}$ then the sequence $h \circ S$ does not converge to hx in $\sigma\mathsf{R}^{\aleph}$ (because h is a projective generating mapping), and therefore there exists a projection π of $\sigma\mathsf{R}^{\aleph}$ onto one of coordinate spaces such that $\pi \circ$ $\circ (h \circ S)$ does not converge to πhx in R. Put $g = \pi \circ h$. Then g is a continuous function on $\mathscr{P}$ and $g \circ S$ does not converge to gx. Now it is easy to construct a bounded continuous function with this property. Thus condition (b) of 35 E.5 is fulfilled. Conversely, assuming (a) of 35 E.5, let us take the set $\mathscr{C}$ from (a) and consider the mapping $h = \{x \to \{fx \mid f \in \mathscr{C}\}\}$ of $\mathscr{P}$ into $\mathscr{Q} = \sigma\mathsf{R}^{\mathscr{C}}$. It is evident that a sequence S converges to x in $\mathscr{P}$ if and only if the sequence $h \circ S$ converges to hx in $\mathscr{Q}$. Since both $\mathscr{P}$ and $\mathscr{Q}$ are **S**-spaces, h is necessarily a projective generating mapping.

F. REMARKS AND EXAMPLES

If $\langle P, u\rangle$ is an **S**-space and $x \in uX$, then $x \in uY$ for some countable subset Y of X, and hence $\langle P, u\rangle$ is inductively generated by its countable subspaces (see 33 D.3). If $\langle P, u\rangle$ is an **L**-space and $x \in uX - X$, then there exists a countable subset Y of X such that x is the only cluster point of Y in $\langle P, u\rangle$, i.e. $\langle P, u\rangle$ is inductively generated by countable subspaces with only one cluster point in $\langle P, u\rangle$, or, equivalently, by closed countable subspaces with exactly one cluster point. In this subsection we examine the class of all spaces inductively generated by countable subspaces and the class of all spaces $\langle P, u\rangle$ inductively generated by closed countable subspaces with only one cluster point. We begin with several rather general remarks which will enable the reader to formulate and prove the results of this subsection in a more general situation.

By 16 ex. 6 a net $\langle N, \leqq\rangle$ ranging in a space $\mathscr{P}$ converges to a point x of $\mathscr{P}$ if and only if the mapping f defined by $\operatorname{gr} f = N \cup (\langle \mathscr{A}, x\rangle)$ is continuous; here $\mathscr{A}$ is the filter of all residual sets in $\mathbf{D}N$, $\mathscr{A}$ is the unique cluster point of $\mathbf{D}^*f$ and $(\mathscr{A}) \cup [\mathscr{A}]$ is the neighborhood system at $\mathscr{A}$. The theory of convergence can be reduced to the theory of inductive generation of spaces by assigning to each $\mathscr{P}$, $\langle N, \leqq\rangle$ and x the mapping $\operatorname{gr} f : \mathbf{D}^*f \to \mathscr{P}$ with f defined above.

In agreement with the current approach to similar situations we have implicitely assumed $\mathscr{A} \notin \mathbf{D}N$; however it may well happen that $\mathscr{A} \in \mathbf{D}N$. A similar situation has occurred in 14 B where βX had been defined, and will also appear in 41 C, where a completion of a uniform space is constructed. Here one of the simplest correct approaches to such situations will be described. For further possibilities see Remark in the following item.

35 F. 1. Definition. For each set X let λX be the set $\exp \bigcup \mathbf{E}\{x \mid x \in X, x \text{ is a set}\}$. It is easily seen that if $x \in X$ is a set, then card $x <$ card λX, and hence $\lambda X \notin X$. Now, given an arbitrary directed set $\mathscr{A} = \langle A, \leqq\rangle$, let $\alpha\mathscr{A}$ be the filter α' of all residual sets provided that $\alpha' \notin A$ and let $\alpha\mathscr{A} = \lambda\mathscr{A} \times \alpha'$ otherwise. It follows that $\alpha\mathscr{A} \notin A$, and $\mathbf{E}(\alpha\mathscr{A}) = \alpha'$ if $\alpha' \in A$. Let sp $\mathscr{A}$ be the closure space with underlying set $|\mathscr{A}| \cup \alpha\mathscr{A}$ such that the point $\alpha\mathscr{A}$, called the ideal point of sp $\mathscr{A}$, has the collection $(\alpha\mathscr{A}) \cup [\alpha']$ for the neighborhood system, while all the other points are isolated. The space sp $\mathscr{A}$ will be termed the *space associated with* $\mathscr{A}$.

Remark. It may be noted that this approach differs from the current one precisely in those cases in which the latter is not correct. Next, observe that $\alpha\mathscr{A} \notin |\mathscr{A}|$ is the only property of $\alpha\mathscr{A}$ needed, and therefore $\alpha\mathscr{A}$ might be introduced e.g. on the basis of the Axiom of Choice. It should also be remarked that there are further very reasonable approaches. If one resigns from the assumption that $|\mathscr{A}| \subset |\text{sp}\,\mathscr{A}|$, $X \subset |\beta X|$, then one can take sp $\mathscr{A}$ such that $|\text{sp}\,\mathscr{A}| = (\alpha') \cup \mathbf{E}\{(x) \mid x \in |\mathscr{A}|\}$ with α' an ideal point and similarly for βX.

First let us notice that

(a) if ϱ is a single-valued relation on a directed set $\langle B, \prec\rangle$ ranging in a directed set $\langle A, \leqq\rangle$ and if f is the mapping of sp $\langle B, \prec\rangle$ into sp $\langle A, \leqq\rangle$ which assigns to each $b \in B$ the point ϱb, and to the ideal point of sp $\langle B, \prec\rangle$ assigns the ideal point of sp $\langle A, \leqq\rangle$, then f is continuous if and only if, for each residual set X in $\langle A, \leqq\rangle$, there exists a residual set Y in $\langle B, \prec\rangle$ such that $\varrho[Y] \subset X$.

It follows immediately from (a) that, under the assumptions of (a),

(b) if f is continuous, then $\varrho[Y]$ is cofinal in $\langle A, \leqq\rangle$ for each residual set Y in $\langle B, \prec\rangle$.

The mapping f just defined will be termed the *mapping of associated spaces associated with* ϱ.

For each pair $c = \langle N, x\rangle$ where N is a net and x is an element, let f_c be the single-valued relation on the underlying set of the space associated with the ordered domain of N such that N is a restriction of f_c and the value of f_c at the ideal point is x. It is almost self-evident that

(c) a net N converges to x in a space $\mathscr{P}$ if and only if the mapping $f_{\langle N,x\rangle}$ of the space associated with the ordered domain of N into $\mathscr{P}$ is continuous (i.e. the mapping $f_{\langle N,x\rangle} : \operatorname{sp} \mathbf{D}N \to \mathscr{P}$ is continuous).

If M is a generalized subnet of a net N under the relation ϱ and if f is the mapping of spaces associated with ordered domains of M and N which is associated with ϱ, then $f_{\langle M,x\rangle} = f_{\langle N,x\rangle} \circ f$ and therefore we obtain a new proof of the fact that if N converges to x in $\mathscr{P}$, then M converges to x in $\mathscr{P}$. Indeed, $f : \operatorname{sp} \mathbf{D}M \to \operatorname{sp} \mathbf{D}N$ is always continuous (by (a)); if N converges to x in $\mathscr{P}$, then $f_{\langle N,x\rangle} : \operatorname{sp} \mathbf{D}N \to \mathscr{P}$ is continuous (by (c)) and hence $f_{\langle M,x\rangle} : \operatorname{sp} \mathbf{D}M \to \mathscr{P}$ is continuous as the composite of two continuous mappings, which implies that M converges to x (again by (c)).

We know that M may converge to x in a space $\mathscr{P}$ even if N does not converge to x in $\mathscr{P}$. On the other hand, if f is an inductive generating mapping, then $f_{\langle M,x\rangle} : \operatorname{sp} \mathbf{D}M \to \mathscr{P}$ is continuous if and only if $f_{\langle N,x\rangle} : \operatorname{sp} \mathbf{D}N \to \mathscr{P}$ is continuous, and therefore, N converges to x in $\mathscr{P}$ if and only if M converges to x in $\mathscr{P}$. It is easy to find necessary and sufficient conditions for f to be an inductive generating mapping.

35 F.1. Let $\mathscr{C}$ be a generating convergence relation for a space $\mathscr{P}$, and for each $c = \langle N, x\rangle$ in $\mathscr{C}$ let g_c be the mapping $f_c : \operatorname{sp} \mathbf{D}N \to \mathscr{P}$.

The following result is evident:

(a) If $\mathscr{C}$ is a generating convergence relation for $\mathscr{P}$, then $\{g_c \mid c \in \mathscr{C}\}$ is an inductive generating family of mappings for $\mathscr{P}$.

If $\{g_c \mid c \in \mathscr{C}\}$ is an inductive generating family of mappings for $\mathscr{P}$, then $\mathscr{C}$ need not be a generating convergence relation. E.g., if $\langle A, \leqq\rangle$ is any directed set, α is the ideal point of $\mathscr{P} = \operatorname{sp} \langle A, \leqq\rangle$ and $N = \mathsf{J}_A$, then $f_{\langle N,\alpha\rangle} = \mathsf{J}_{|\mathscr{P}|}$ and $g_{\langle N,\alpha\rangle} = \mathsf{J} : \mathscr{P} \to \mathscr{P}$. Thus $g_{\langle N,\alpha\rangle}$ is an inductive generating mapping for $\mathscr{P}$ but, evidently, $(\langle N, \alpha\rangle)$ need not be a generating convergence relation for $\mathscr{P}$ (e.g. if $\langle A, \leqq\rangle$ is the ordered set of integers). On the other hand it is easily seen that

(b) If $\{g_c \mid c \in \mathscr{C}\}$ is an inductive generating family for $\mathscr{P}$, then $\{g_c \mid c \in \mathscr{C}\}$ is a generating convergence relation for $\mathscr{P}$ provided that the following condition is fulfilled: If $\langle N, x\rangle \in \mathscr{C}$ and M is a subnet of N, then $\langle M, x\rangle \in \mathscr{C}$.

Of course, the condition can be replaced by the following weaker condition: If $\langle N, x\rangle \in \mathscr{C}$ and M is a subnet of N, then there exists a subnet M' of M such that $\langle M', x\rangle \in \mathscr{C}$.

35 F.3. Definition. A *feeble generating convergence relation* for a space $\mathscr{P}$ is a convergence relation $\mathscr{C}$ such that the closure structure of $\mathscr{P}$ is the finest closure for $|\mathscr{P}|$ with N converging to x for each $\langle N, x\rangle \in \mathscr{C}$; stated in other words, if u is a closure for $|\mathscr{P}|$ coarser than the closure structure of $\mathscr{P}$, then $\mathscr{C} \subset \mathbf{Lim}\,\langle|\mathscr{P}|, u\rangle$.

35 F.4. Theorem. *A convergence relation $\mathscr{C}$ is a feeble generating convergence relation for $\mathscr{P}$ if and only if $\{g_c \mid c \in \mathscr{C}\}$ is an inductive generating family of mappings for $\mathscr{P}$.*

The proof is very simple and therefore left to the reader.

35 F.5. Theorem. *Let $\mathscr{D}$ be a non-void class of directed sets and let K be the class of all spaces associated with directed sets of $\mathscr{D}$. Then the inductive progeny of K coincides with the class of all spaces admitting a determining relation $\mathscr{C}$ such that the ordered domains of nets from $\mathbf{D}\mathscr{C}$ belong to $\mathscr{D}$.*

Proof. Let K_1 be the class of all spaces admitting a determining convergence relation $\mathscr{C}$ such that the ordered domains of nets of $\mathbf{D}\mathscr{C}$ belong to $\mathscr{D}$. By 35 C.4 the class K_1 is inductive-stable, i.e. $\operatorname{ind} K_1 = K_1$. By 35 F.2 (a) the class K_1 is contained in the inductive progeny of K, i.e. $K_1 \subset \operatorname{ind} K$. To prove the inverse inclusion it is sufficient to show that $K \subset K_1$. Indeed, $K \subset K_1$ implies $\operatorname{ind} K \subset \operatorname{ind} K_1$ $(= K_1)$. We shall prove somewhat more (remember that $\mathscr{D} \neq \emptyset$).

35 F.6. *If $\langle A, \leqq\rangle$ is a directed set, then the space $\langle A', u\rangle$ associated with $\langle A, \leqq\rangle$ and also each of its subspaces admits a determining convergence relation $\mathscr{C}$ such that the ordered domain of each net of $\mathbf{D}\mathscr{C}$ is $\langle A, \leqq\rangle$.*

Proof. If $x \in A$ then the constant net $\{a \to x \mid a \in A\}$ converges to x in $\langle A', u\rangle$. If $x \in uX - X$, then x is the ideal point of $\langle A', u\rangle$ and $X \subset A$; since $A' - X$ is not a neighborhood of x in $\langle A', u\rangle$, X is cofinal in $\langle A, \leqq\rangle$ and we can choose a single-valued relation N on A ranging in X such that $a \leqq N_a$ for each a. Clearly the net $\langle N, \leqq\rangle$ converges to x in $\langle A', u\rangle$.

35 F.7. Corollary. *The class $\mathbf{S}$ is the inductive progeny of $(\mathscr{Q})$, where $\mathscr{Q}$ is the space of ordinals less than $\omega_0 + 1$.*

Indeed, $\mathscr{Q}$ is a homeomorph of the space associated with the ordered set of natural numbers.

35 F.8. Theorem. *Let $\mathscr{D}$ be a class of directed sets and let K_1 be the class of all spaces admitting a determining convergence relation $\mathscr{C}$ such that the ordered domains of nets of $\mathbf{D}\mathscr{C}$ belong to $\mathscr{D}$. Then K_1 is hereditary.*

Proof. If $\mathscr{D} = \emptyset$, then K_1 is the class of all discrete spaces which is hereditary. Assuming $\mathscr{D} \neq \emptyset$ let us consider the class K consisting of spaces associated with directed sets of $\mathscr{D}$. By 35 F.6 the subspaces of spaces of K belong to K_1. Since $K_1 = \operatorname{ind} K$ and $\operatorname{ind} K$ contains subspaces of spaces of K, $\operatorname{ind} K$ is hereditary.

According to the foregoing results the theory of convergence can be reduced, in a certain sense, to the theory of inductive generating families of mappings such that the domain carriers are spaces associated with directed sets. Each space associated with a directed set has exactly one cluster point. On the other hand, a space with exactly one cluster point need not be a homeomorph of a space associated with a directed set. For example, the local character of a space $\mathscr{P}$ associated with a directed set is at most card $|\mathscr{P}|$ whereas the local character of a space $\mathscr{P}$ with one cluster point may be exp card $|\mathscr{P}|$, e.g. if X is countable infinite and $\mathscr{P}$ is a subspace of βX such that $|\mathscr{P}| = X \cup (x)$, $x \notin X$.

Now we turn to the proper subject of this subsection.

By 35 F.7 the class $\mathbf{S}$ of all $\mathbf{S}$-spaces is the inductive progeny of $(\mathscr{Q})$ where $\mathscr{Q}$ is the ordered space of ordinals less than $\omega_0 + 1$. It is easily seen that $\mathscr{Q}$ is a coarse separated space (see definition 31 D. 7). Indeed, the complements of neighborhoods of ω_0 are finite, and therefore, if u is any separated closure coarser than the closure structure of $\mathscr{Q}$, then each point $\alpha \neq \omega_0$ must be isolated in $\langle|\mathscr{Q}|, u\rangle$, and $|\mathscr{Q}| - (\alpha)$, $\alpha \neq \omega_0$, must be a neighborhood of ω_0 in $\langle|\mathscr{Q}|, u\rangle$; thus $\langle|\mathscr{Q}|, u\rangle = \mathscr{Q}$. One can show that every countable coarse separated space is an $\mathbf{S}$-space. On the other hand there exists a countable space which is not an $\mathbf{S}$-space. E.g. let $\mathscr{P}$ be a subspace of βX such that $|\mathscr{P}| = X \cup (x)$, $x \in \beta X - X$, where X is a countable infinite set. We know that no sequence ranging in X converges to x in $\mathscr{P}$ (15 B.7) and hence $\mathscr{P}$ is not an $\mathbf{S}$-space. Now we shall examine the basic properties of the inductive progeny of the class of all countable spaces.

35 F.9. Definition. A *fine non-discrete closure operation* is a closure u for a set P such that the discrete closure for P is the unique closure for P strictly finer than u. A *fine non-discrete space* is a space whose closure structure is a fine non-discrete closure.

35 F.10. Theorem. *Let P be a set. A closure u for P is a fine non-discrete closure if and only if $\langle P, u\rangle$ has exactly one accumulation point, say x, and if $\mathscr{U}$ is the neighborhood system at x, then $\mathscr{V} = [\mathscr{U}] \cap (P - (x))$ is an ultrafilter on $P - (x)$.*

Remark. The subspace $P - (x)$ of $\langle P, u\rangle$ is discrete and u is separated if and only if $\mathscr{V}$ is a free ultrafilter ($\bigcap \mathscr{V} = \emptyset$). If u is not separated, then $\mathscr{V}$ is fixed, and if $(y) = \bigcap \mathscr{V}$, then $x \in u(z)$ if and only if $z = y$.

Proof. I. First let u be a closure such that a point x is the unique accumulation point of $\langle P, u\rangle$ and the neighborhood system $\mathscr{U}$ at x has the property that $\mathscr{V} = [\mathscr{U}] \cap (P - (x))$ is an ultrafilter on $P - (x)$. If u_1 is a closure finer than u and $\mathscr{U}_1$ is the neighborhood system at x in $\langle P, u_1\rangle$, then $\mathscr{U}_1 \supset \mathscr{U}$; and if $\mathscr{V}_1 = [\mathscr{U}_1] \cap (P - (x))$ is a proper filter, i.e. if x is a cluster point of $\langle P, u_1\rangle$, then necessarily $\mathscr{V}_1 = \mathscr{V}$ because $\mathscr{V}$ is an ultrafilter. But, evidently, u_1 is not discrete if and only if x is a cluster point of $\langle P, u_1\rangle$. – II. Now let u be a fine non-discrete closure for P. Obviously, there exists exactly one accumulation point of $\langle P, u\rangle$, say x. Let $\mathscr{U}$ be the neighborhood system at x, $\mathscr{V} = [\mathscr{U}] \cap (P - (x))$. Since $x \in u(P - (x))$, $\mathscr{V}$ is a filter. Choose an ultrafilter $\mathscr{V}_1$ on $P - (x)$ containing $\mathscr{V}$ and let $\mathscr{U}_1 = (x) \cup [\mathscr{V}_1]$. Consider the closure u_1 for P such that all points $y \in P - (x)$ are isolated and $\mathscr{U}_1$ is the neighborhood system at x. Clearly u_1 is a non-discrete closure finer than u and $u_1 = u$ if and only if $\mathscr{U} = \mathscr{U}_1$, i.e. $\mathscr{V} = \mathscr{V}_1$. But $\mathscr{V} = \mathscr{V}_1$ if and only if $\mathscr{V}_1$ is an ultrafilter in $P - (x)$. – The remark is obvious.

35 F.11. Corollary. *Any separated non-discrete closure is coarser than a separated fine non-discrete closure. If P is an infinite set then there exists a separated fine non-discrete closure for P.*

35 F.12. Theorem. *Let K be the inductive progeny of all countable spaces. Then K is the inductive progeny of all countable separated fine non-discrete spaces, and a space $\mathcal{P}$ belongs to K if and only if $\mathcal{P}$ is inductively generated by its countable subspaces.*

Proof. Evidently it is sufficient to prove that any space of K is inductively generated by a family of mappings the domain carriers of which are separated fine non-discrete spaces, and to prove this it is sufficient to show that any countable space is inductively generated by a family of mappings the domain carriers of which are separated fine non-discrete spaces. Let $\langle P, u\rangle$ be a countable space. For each pair $\langle X, x\rangle$ such that $x \in uX - X$ we shall construct a continuous mapping $f_{X,x}$, or simply f, of a separated fine non-discrete space $\mathcal{Q}_{X,x}$, or simply $\mathcal{Q}$, into $\langle P, u\rangle$ which assigns to the only cluster point of $\mathcal{Q}$ the point x and which maps the remaining points into X. If $x \in u(x_1)$ for some $x_1 \in X$, then we take any countable separated fine non-discrete space $\mathcal{Q}$ and the mapping f which maps the only cluster point of $\mathcal{Q}$ into x and the remaining points into x_1. Evidently f is continuous. Notice that this is the case if $\langle P, u\rangle$ is not semi-separated. In the other case, if $\mathcal{U}$ is the neighborhood system at x, then $\mathcal{V} = [\mathcal{U}] \cap X$ is free proper filter on X and we can take an ultrafilter $\mathcal{W}$ on X containing $\mathcal{V}$ and then the separated fine non-discrete closure v for $Q = X \cup x$ such that $(x) \cup [\mathcal{W}]$ is the neighborhood system at x. By definition $f = \mathrm{J} : \langle Q, v\rangle \to \langle P, u\rangle$ is continuous and $x \in vX$. Evidently, $\{f_{X,x} \mid x \in (uX - X)\}$ is the required inductive generating family for $\langle P, u\rangle$.

35 F.13. *Let K be the inductive progeny of the class of all countable spaces and L be the class consisting of closure structures of spaces of K. Since K is inductive-stable, the class L is order-complete and completely join-stable in $\mathscr{C}$. On the other hand L is not completely meet-stable.*

E.g. the closure structure of the space of all ordinals less than $\omega_1 + 1$ does not belong to L but it is the infimum of **S**-closures (see 35 C.11). Next, any closure, say u, for a set P, has a lower modification v in L and clearly

$$vX = \bigcup\{uY \mid Y \subset X\,,\ Y \text{ countable}\}$$

for each $X \subset P$. Finally, the class K contains the class **S** but $K \neq$ **S** because no separated fine non-discrete space belongs to **S** (in such a space each sequence is eventually constant).

35 F.14. The class of all semi-separated **S**-closures coincides with the class of all spaces inductively generated by a family of injective mappings of the space of all ordinals less than $\omega_0 + 1$. The class of all semi-separated spaces of the inductive progeny of countable spaces coincides with the class of all spaces inductively generated by families of injective mappings of countable separated fine non-discrete spaces.