Chapter VI: Mappings of a space onto the circle


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Chapter VI

MAPPINGS OF A SPACE ONTO THE CIRCLE

§ 24. Inessential mappings onto the circle

24.1. In this and in the following chapter we shall identify couples \((x, y)\) of real numbers with complex numbers \(x + iy\), so that \(E_2\) is the set of all the complex numbers and \(S_1\) (see 17.10) is the set of all complex numbers \(x + iy\) with absolute value

\[ |x + iy| = \sqrt{x^2 + y^2} \]

equal to one. The set \(E_2\) will be termed the plane, the set \(S_1\) will be termed the circle. Evidently

\[ \rho(a, b) = |a - b| \text{ for } a \in E_2, b \in E_2. \]

As is well known, for any \(t \in E_1\),

\[ e^{it} = \cos t + i \sin t \in S_1. \]

The following two theorems are well known:

24.1.1. Put \(f(t) = e^{it}\) for \(t \in E_1\). Then \(f\) is a continuous mapping of \(E_1\) onto \(S_1\).

24.1.2. Let \(\alpha \in E_1\), \(J = E[\alpha < t < \alpha + 2\pi]\). Put \(f(t) = e^{it}\) for \(t \in J\). Then \(f\) is a homeomorphic mapping of \(J\) onto \(S_1 - \{e^{i\alpha}\}\).

24.2. Let \(P\) be a metric space. The following two theorems are easy to prove:

24.2.1. Let \(f\) and \(g\) be continuous mappings of \(P\) into \(S_1\). Then \(f \cdot g\) is a continuous mapping of \(P\) into \(S_1\).

24.2.2. Let \(f\) be a continuous mapping of \(P\) into \(S_1\). Then \(1/f\) is a continuous mapping of \(P\) into \(S_1\).

It follows easily by 24.1.1:

24.2.3. Let \(\varphi\) be a continuous mapping of \(P\) into \(E_1\). Put \(f(x) = e^{i\varphi(x)}\) for every \(x \in P\). Then \(f\) is a continuous mapping of \(P\) into \(S_1\).

Let \(f\) be a continuous mapping of \(P\) into \(S_1\). We say that \(f\) is inessential, if there exists a continuous mapping \(\varphi\) of \(P\) into \(E_1\) such that \(f(x) = e^{i\varphi(x)}\) for every \(x \in P\). A mapping \(f\) is said to be essential, if it is not inessential.

The following three theorems are evident.
24.2.4. Let $f$ and $g$ be inessential continuous mappings of $P$ into $S_1$. Then $f \cdot g$ is an inessential continuous mapping of $P$ into $S_1$.

24.2.5. Let $f$ be an inessential continuous mapping of $P$ into $S_1$. Then $1/f$ is an inessential continuous mapping of $P$ into $S_1$.

24.2.6. Let $Q \subset P$. Let $f$ be an inessential continuous mapping of $P$ into $S_1$. Then the partial mapping $f_Q$ is also inessential.

24.2.7. Let $f$ be a continuous mapping of $P$ into $S_1$. If $S_1 - f(P) \neq \emptyset$, then $f$ is inessential.

Proof: There is an $\alpha \in E_1$ with $e^{i\alpha} \in S_1 - f(P)$. By 24.1.2 there exists a homeomorphic mapping $h$ of $S_1 - (e^{i\alpha})$ onto the interval $E[\alpha < t < \alpha + 2\pi]$ such that $e^{ih(z)} = z$ for every $z \in S_1 - (e^{i\alpha})$. Put $\varphi(x) = h\{f(x)\}$ for $x \in P$. Then $\varphi$ is a continuous mapping of $P$ into $E_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$.

24.2.8. Let $f$ and $g$ be continuous mappings of $P$ into $S_1$. Let $f$ be inessential. Let $|f(x) - g(x)| < 2$ for every $x \in P$. Then $g$ is also inessential.

Proof: Obviously $g(x)/f(x) \neq -1$ for any $x \in P$. Thus, the mapping $g = f \cdot (g/f)$ is inessential by 24.2.4 and 24.2.7.

24.2.9. Let $0 < \omega < 2\pi$. Let $f$ be a continuous mapping of $P$ into $S_1$. Let $\varphi$ be a mapping of $P$ into $E_1$. Let $f(x) = e^{i\varphi(x)}$ for every $x \in P$. Let $\varphi$ not be continuous in a point $a \in P$. Then there is a sequence $\{x_n\}$ in $P$ such that $\lim x_n = a$, $|\varphi(x_n) - \varphi(a)| > \omega$ for every $n$.

Proof: Denote by $M$ the set of all $x \in P$ such that $|\varphi(x) - \varphi(a)| > \omega$. By 8.2.1, we have to prove that $a \in \overline{M}$. Let us assume the contrary. Then $U = P - \overline{M}$ is a neighborhood of $a$ such that $x \in U$ implies $|\varphi(x) - \varphi(a)| \leq \omega$. Evidently there is a neighborhood $V$ of $a$ such that $S_1 - f(V) \neq \emptyset$. By 24.2.7 there is a continuous mapping $\psi$ of $V$ into $E_1$ such that, for every $x \in V$

$$e^{i\psi(x)} = f(x) = e^{i\varphi(x)}.$$ 

In particular $e^{i\varphi(a)} = e^{i\varphi(a)}$, so that there is an integer $k$ with $\varphi(a) = \psi(a) + 2\pi k$. Since $\omega < 2\pi$ and since $\psi$ is continuous, there is obviously a neighborhood $W \subset U$ of $a$ such that $x \in W$ implies $|\psi(x) - \psi(a)| < 2\pi - \omega$. For $x \in U \cap W$ we have $|\varphi(x) - \psi(x) - 2k\pi| = |[\varphi(x) - \varphi(a)] - [\psi(x) - \psi(a)]| \leq |\varphi(x) - \varphi(a)| + |\psi(x) - \psi(a)| < 2\pi$. However, the number

$$\frac{\varphi(x) - \psi(x) - 2k\pi}{2\pi}$$

(1)
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is an integer, since

\[ e^{i\varphi(x)} = e^{i\psi(x)} = e^{i(\psi(x) + 2k\pi)}. \]

Thus, (1) is an integer and its absolute value is less than 1, hence \( \varphi(x) = \psi(x) + 2k\pi \) for every \( x \in U \cap W \). On the other hand, \( U \cap W \) is a neighborhood of \( a \) and \( \psi \) is continuous. Thus, \( \varphi \) is continuous in \( a \). This is a contradiction.

24.2.10. Let \( f \) be a continuous mapping of \( P \) into \( S_1 \). Let there exist an integer \( k \neq 0 \) such that the mapping \( f^k \) is inessential. Then \( f \) is also inessential.

Proof: There is a continuous mapping \( \varphi \) of \( P \) into \( E \) with

\[ [f(x)]^k = e^{i\varphi(x)} \]

for every \( x \in P \). For \( x \in P \) put

\[ g(x) = \exp\left[i\varphi(x)/k\right]. \]

Then \( g \) is an inessential continuous mapping of \( P \) into \( S_1 \). For every \( x \in P \) we have \([f(x)/g(x)]^k = 1\), so that \( f/g \) is inessential by 24.2.7. Thus, the mapping

\[ f = (f/g) \cdot g \]

is inessential by 24.2.4.

24.2.11. Let \( \varphi_1 \) and \( \varphi_2 \) be continuous mappings of a connected space \( P \) into \( E_1 \). Let

\[ e^{i\varphi_1(x)} = e^{i\varphi_2(x)} \]

for every \( x \in P \). Then there is an integer \( k \) such that

\[ \varphi_2(x) = \varphi_1(x) + 2k\pi \]

for every \( x \in P \).

Proof: \( \varphi = (2\pi)^{-1} \cdot (\varphi_2 - \varphi_1) \) is a continuous mapping of \( P \) into \( E_1 \) and the set \( \varphi(P) \) consists of integers, so that \( \varphi(P) \) is not an interval. Hence, \( \varphi(P) \) is a one-point set by 18.1.10 and 19.2.2.

24.2.12. Let \( K = 1, 2, 3, \ldots \). Let \( P = A \cup B \) and let \( A, B \) be either both closed or both open. Let \( A \cap B \) have at most \( k \) components. Let \( f_\lambda \) (\( 1 \leq \lambda \leq k \)) be continuous mappings of \( P \) into \( S_1 \). Let all the partial mappings

\[ (f_\lambda)_A, (f_\lambda)_B \quad (1 \leq \lambda \leq k) \]

be inessential. Then there are integers \( n_\lambda \) (\( 1 \leq \lambda \leq k \)) which are not all equal to zero such that the mapping

\[ \prod_{\lambda=1}^{k} (f_\lambda)^{n_\lambda} \]

is inessential.
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**Proof:** Let \( C_\mu (1 \leq \mu \leq h) \) be all the components of the set \( A \cap B \); thus, 
\( 0 \leq h \leq k \).

There are continuous mappings \( \varphi_\lambda (1 \leq \lambda \leq k) \) of \( A \) into \( E_1 \) and continuous mappings \( \psi_\lambda (1 \leq \lambda \leq k) \) of \( B \) into \( E_1 \) such that
\[
f_{A}(x) = e^{i\varphi_\lambda(x)} \quad \text{for} \quad x \in A, \\
f_{B}(x) = e^{i\psi_\lambda(x)} \quad \text{for} \quad x \in B.
\]

By 24.2.11 there are integers \( k_{\mu,\lambda} (1 \leq \mu \leq h, 1 \leq \lambda \leq k) \) such that
\[
\psi_\lambda(x) = \varphi_\lambda(x) + 2\pi k_{\mu,\lambda} \quad \text{for} \quad x \in C_\mu.
\]

Let us determine integers \( n, n_\lambda (1 \leq \lambda \leq k) \) satisfying the equations
\[
\sum_{\lambda=1}^{k} k_{\mu,\lambda} n_\lambda = n. \quad (1 \leq \mu \leq h) \tag{2}
\]
Since the number of the equations is less than the number of unknowns and since the coefficients are integers, there exists a solution of (2) such that we do not have \( n_1 = \ldots = n_k = 0 \).

Put \( f = \prod_{\lambda=1}^{k} (f_{\lambda})^{n_\lambda} \), so that \( f \) is a continuous mapping of \( P \) into \( S_1 \). We have to prove that \( f \) is inessential.

Equations (2) yield that \( x \in A \cap B \) implies \( \sum_{\lambda=1}^{k} n_\lambda \psi_\lambda(x) = \sum_{\lambda=1}^{k} n_\lambda \varphi_\lambda(x) + 2\pi n \).

Thus, we may define a mapping \( \chi \) of \( P \) into \( E_1 \) by
\[
\chi(x) = \sum_{\lambda=1}^{k} n_\lambda \varphi_\lambda(x) + 2\pi n \quad \text{for} \quad x \in A, \\
\chi(x) = \sum_{\lambda=1}^{k} n_\lambda \psi_\lambda(x) \quad \text{for} \quad x \in B.
\]

Evidently \( f(x) = e^{ix(x)} \) for every \( x \in P \), so that it suffices to prove that \( \chi \) is continuous. This follows easily from the continuity of the partial mappings \( \chi_A, \chi_B \) (see ex. 9.5).

24.2.13. Let \( P = A \cup B \) and let \( A, B \) be either both closed or both open. Let \( A \cap B \) be either void or connected. Let \( f \) be a continuous mapping of \( P \) into \( S_1 \). Let both partial mappings \( f_A, f_B \) be inessential. Then also \( f \) is inessential.

This follows immediately from 24.2.10 and 24.2.12.*

24.2.14. Let \( P = \bigcup_{n=1}^{\infty} A_n \). Let \( A_n \subset A_{n+1} \) \( (n = 1, 2, 3, \ldots) \). Let the sets \( A_n \) be connected. For every \( x \in P \) let there be an index \( n \) such that \( x \) is an interior point (see 8.6) of \( A_n \).

*) 24.2.13 is a particular case of theorem 24.2.12. If the proof is carried out for this particular case, we see easily that we do not need theorem 24.2.10.
Let $f$ be a continuous mapping of $P$ into $S_1$. Let the partial mappings $f_{A_n}$ be inessential $(n = 1, 2, 3, \ldots)$. Then $f$ is inessential.

Proof: Choose an $a \in A_1$, so that $a \in A_n$ for every $n$. For $n = 1, 2, 3, \ldots$ there is a continuous mapping $\psi_n$ of $A_n$ into $E_1$ such that $f(x) = e^{i\psi_n(x)}$ for every $x \in A_n$. If $m < n$, then, by 24.2.11, there exists an integer $k_{mn}$ such that $x \in A_m$ implies $\psi_n(x) = \psi_m(x) + 2\pi k_{mn}$. Put $h_n = k_1 n$. We have

$$
\begin{align*}
\psi_n(x) &= \psi_m(x) + 2\pi k_{mn}, \\
\psi_n(a) &= \psi_1(a) + 2\pi h_n, \\
\psi_m(a) &= \psi_1(a) + 2\pi h_m,
\end{align*}
$$

hence, $k_{mn} = h_n - h_m$. Thus, we may define a mapping $\varphi$ of $P$ into $E_1$ by

$$
\varphi(x) = \psi_n(x) - 2\pi h_n \quad \text{for} \quad x \in A_n.
$$

Evidently $f(x) = e^{i\varphi(x)}$ for every $x \in P$. Since for every $x \in P$ there is an index $n$ such that $x$ is an interior point of $A_n$ and since the mappings $\psi_n$ are continuous, $\varphi$ is also continuous. Thus, $f$ is inessential.

24.2.15. Let $Q \subset P$. Let either $T = E_1$ or $T = S_1$. Let $\varepsilon > 0$. Let $\varphi$ be a continuous mapping of $Q$ into $T$. Then there is a neighborhood $G$ of $Q$ and a continuous mapping $\psi$ of $G$ into $T$ such that $|\psi(x) - \varphi(x)| \leq \varepsilon$ for every $x \in Q$.

Proof: I. First, let $T = E_1$. We may assume that $Q \neq \emptyset$.

II. Let $\Gamma$ be the set of all $x \in \bar{Q}$ such that there is a number $\eta_x > 0$ with

$$(a) \cup (b) \subset Q \cap \Omega(x, \eta_x) \Rightarrow |\varphi(a) - \varphi(b)| < \frac{1}{2} \varepsilon.$$ 

As $\varphi$ is continuous, we have obviously

$$Q \subset \Gamma \subset \bar{Q}.$$ 

Moreover, it is easy to prove that

$$x \in \Gamma \Rightarrow \bar{Q} \cap \Omega(x, \eta_x) \subset \Gamma$$

so that $\Gamma$ is open in $\bar{Q}$.

III. For $n = 0, \pm 1, \pm 2, \ldots$ denote by $A_n$ the set of all $x \in Q$ with

$$n \varepsilon \leq \varphi(x) \leq (n + 1) \varepsilon,$$

so that

$$Q = \bigcup_{n=-\infty}^{\infty} A_n.$$ 

IV. We have

$$\Gamma \subset \bigcup_{n=-\infty}^{\infty} \bar{A}_n.$$
To prove this, we choose an \( x \in \Gamma \). Since \( \Gamma \subset \overline{Q} \), we have \( 0 = q(x, Q) < \eta_x \), so that there is an \( a \in Q \) with \( q(a, x) < \eta_x \). Choose such an \( a \) and determine an integer \( m \) with \( |\varphi(a) - me| \leq \frac{1}{6} \varepsilon \). If \( 0 < \delta \leq \eta_x \), then \( 0 = q(x, Q) < \delta \), so that there is a point \( b \in Q \) with \( q(b, x) < \delta \). By II, \(|\varphi(a) - \varphi(b)| < \frac{1}{6} \varepsilon \), so that \(|\varphi(b) - me| < \varepsilon \), hence \( b \in A_{m-1} \cup A_m \). Thus, \( q(x, A_{m-1} \cup A_m) < \delta \) for every \( \delta > 0, \delta \leq \eta_x \), so that \( q(x, A_{m-1} \cup A_m) = 0 \), hence \( x \in A_{m-1} \cup A_m = \overline{A}_{m-1} \cup \overline{A}_m \).

V. Further, we prove that

\[
x \in \Gamma \cap \overline{A}_n, y \in \Gamma \cap \overline{A}_m, \quad q(x, y) < \eta_x \Rightarrow |m - n| \leq 1.
\]

(In particular, \( x \in \Gamma \cap \overline{A}_n \cap \overline{A}_m \Rightarrow |m - n| \leq 1 \.)

Since \( x \in \Gamma \cap \overline{A}_n \), there exists a point \( a \in A_n \cap \Omega(x, \eta_x) \). Choose a \( \delta > 0 \) with \( \delta < \eta_x \), \( q(x, y) + \delta < \eta_x \). Since \( y \in \Gamma \cap \overline{A}_m \), there exists a point \( b \in A_m \cap \Omega(y, \delta) \). We have \( q(b, x) \leq q(x, y) + q(b, y) \leq q(x, y) + \delta < \eta_x \). Hence, \( (a) \cup (b) \subset Q \cap \Omega(x, \eta_x) \), so that \(|\varphi(a) - \varphi(b)| < \frac{1}{7} \varepsilon \). Since \( a \in A_n \), \( b \in A_m \), we have \( ne \leq \varphi(a) \leq (n + 1) \varepsilon \), \( me \leq \varphi(b) \leq (m + 1) \varepsilon \). Since \(|\varphi(a) - \varphi(b)| < \varepsilon \), we have \(|m - n| \leq 1 \).

VI. Let us define a mapping \( \chi \) of \( \Gamma \) into \( E_1 \) as follows:

If \( x \in \Gamma \cap \overline{A}_n \) (\( n = 0, \pm 1, \pm 2, \ldots \)) then \({}^{*} \)

\[
\chi(x) = ne + \varepsilon - \frac{q(x, A_{n-1})}{q(x, A_{n-1}) + q(x, A_{n+1})}
\]

(the ratio on the right-hand side is always defined, since \( q(x, A_{n-1}) + q(x, A_{n+1}) = 0 \) implies \( x \in \overline{A}_{n-1} \cap \overline{A}_{n+1} \), which is, for \( x \in \Gamma \), impossible by V). By IV, the number \( \chi(x) \) is defined for any \( x \in \Gamma \) at least in one way. If \( x \in \Gamma \cap \overline{A}_n \), \( x \in \Gamma \cap \overline{A}_n \) provided \( m = n \), then, by V, \( m = n \pm 1 \). Then we obtain two formally different definitions, which, however, both lead to the same value, namely \( \chi(x) = ne \) provided \( m = n - 1 \), \( \chi(x) = (n + 1) \varepsilon \) provided \( m = n + 1 \).

VII. \( x \in Q \Rightarrow |\chi(x) - \varphi(x)| \leq \varepsilon \).

In fact, there is an index \( n \) with \( x \in A_n \subset \Gamma \cap \overline{A}_n \). By III, \( ne \leq \varphi(x) \leq (n + 1) \varepsilon \), by VI, \( ne \leq \chi(x) \leq (n + 1) \varepsilon \), hence \(|\chi(x) - \varphi(x)| \leq \varepsilon \).

VIII. The mapping \( \chi \) is continuous. Let \( x_r \in \Gamma \) (\( r = 1, 2, 3, \ldots \)), \( x \in \Gamma \), \( \lim x_r = x \).

We have to prove that \( \lim \chi(x_r) = \chi(x) \). Let us assume the contrary. Then there is a number \( \delta > 0 \) and a subsequence \( \{y_r\} \) of \( \{x_r\} \) such that \(|\chi(y_r) - \chi(x)| > \delta \) for every \( r \). By IV there is an index \( n \) such that \( x \in \Gamma \cap \overline{A}_n \). There is an index \( p \) such that \( r > p \) implies \( q(x, y_r) < \eta_x \).

By V, \( y_r \in \Gamma \cap (\overline{A}_{n-1} \cup \overline{A}_n \cup \overline{A}_{n+1}) \) for every \( r > p \). If \( y_r \in \Gamma \cap \overline{A}_{n-1} \) for infinitely many indices \( r \), then \( q(x, \overline{A}_{n-1}) \leq q(x, y_r) \to 0 \), hence \( q(x, \overline{A}_{n-1}) = 0 \), i.e. \( x \in \Gamma \cap \overline{A}_{n-1} \). Similarly, \( x \in \Gamma \cap \overline{A}_{n+1} \) provided there exist infinitely many

*) We arrange to set \( q(x, 0) = 1 \) for every point \( x \).
indices \( r \) with \( y_r \in \Gamma \cap \overline{A}_{n+1} \). Thus, there exists an index \( m \) (\( m = n \) or \( m = n - 1 \) or \( m = n + 1 \)) such that \( x \in \Gamma \cap \overline{A}_m \) and \( \{y_r\} \) contains a subsequence \( \{z_r\} \) such that \( z_r \in \Gamma \cap \overline{A}_m \) for every \( r \). On the other hand, \( z_r \to x \) and the partial mapping \( \chi_{\Gamma \cap \overline{A}_m} \) is continuous (see ex. 9.10). Hence, \( \chi(z_r) \to \chi(x) \). This is a contradiction, since \( |\chi(z_r) - \chi(x)| > \delta > 0 \) for every \( r \).

IX. The set \( \overline{Q} - \Gamma \) is closed by II and 8.7.3, so that the set \( G = P - (\overline{Q} - \Gamma) \) is open. Moreover, \( \Gamma = \overline{Q} \cap G \), so that \( \Gamma \) is closed in \( G \) by 8.7.2. Hence, by VIII and 14.8.3, there exists a continuous mapping \( \psi \) of \( G \) into \( E_1 \) such that \( \psi(x) = \chi(x) \) for \( x \in \Gamma \). As \( Q \subset \Gamma \), \( x \in Q \) implies \( |\psi(x) - \varphi(x)| \leq \varepsilon \) by VII.

X. The proof is finished for \( T = E_1 \). Now, let us turn to the case of \( T = S_1 \). We may assume that \( \varepsilon < 1 \). For \( x \in Q \) put \( \varphi(x) = \varphi_1(x) + i\varphi_2(x) \). Then \( \varphi_1 \), \( \varphi_2 \) are continuous mappings of \( Q \) into \( E_1 \), and, for every \( x \in Q \) we have \( [\varphi_1(x)]^2 + [\varphi_2(x)]^2 = 1 \). Hence, there exist neighborhoods \( G_1 \), \( G_2 \) of \( Q \), a continuous mapping \( \psi_1 \) of \( G \) into \( E_1 \) and a continuous mapping \( \psi_2 \) of \( G_2 \) into \( E_1 \) such that for every \( x \in Q \) we have \( |\varphi_1(x) - \psi_1(x)| < \frac{1}{2}\varepsilon \), \( |\varphi_2(x) - \psi_2(x)| < \frac{1}{4}\varepsilon \), and hence also

\[
|\psi_1(x) + i\psi_2(x) - 1| = |\psi_1(x) + i\psi_2(x)| - |\varphi_1(x) + i\varphi_2(x)| \leq \\
\leq |[\varphi_1(x) - \psi_1(x)] + i[\varphi_2(x) - \psi_2(x)]| < \frac{1}{3}\varepsilon.
\]

Let us denote by \( G \) the set of all \( x \in G_1 \cap G_2 \) with \( |\psi_1(x) + i\psi_2(x) - 1| < \frac{1}{3}\varepsilon \). We see easily that \( G \) is a neighborhood of \( Q \), that

\[
\psi = \frac{\psi_1 + i\psi_2}{|\psi_1 + i\psi_2|}
\]

is a continuous mapping of \( G \) into \( S_1 \), and that \( |\psi(x) - \varphi(x)| \leq \varepsilon \) for every \( x \in Q \).

24.2.16. Let \( f \) be a continuous mapping of \( P \) into \( S_1 \). Let \( Q \subset P \). Let the partial mapping \( f_Q \) be inessential. Then there exists a neighborhood \( G \) of the set \( Q \) such that the partial mapping \( f_G \) is inessential.

Proof: There is a continuous mapping \( \varphi \) of \( Q \) into \( E_1 \) such that \( f(x) = e^{i\varphi(x)} \) for every \( x \in Q \). By 24.2.15 there is an open set \( G_0 \supset Q \) and a continuous mapping \( \psi \) of \( G_0 \) into \( E_1 \) such that \( |\psi(x) - \varphi(x)| < \pi \) for every \( x \in Q \). Let \( G \) be the set of all \( x \in G_0 \) with \( f(x) \cdot e^{-i\psi(x)} \neq -1 \). Then \( G \) is, by 9.2, open in \( G_0 \), hence, open in \( P \) by 8.7.7. It is easy to prove that \( Q \subset G \). If \( x \in G \), then \( f(x) = f(x) \cdot e^{-i\psi(x)} \cdot e^{i\varphi(x)} \), \( f(x) \cdot e^{-i\psi(x)} \neq -1 \), so that the partial mapping \( f_G \) is inessential by 24.2.4 and 24.2.7.

24.2.17. Let a space \( P \) be either compact or locally connected. Let \( f \) be a continuous mapping of \( P \) into \( S_1 \). Let \( f_K \) be inessential for every component \( K \) of \( P \). Then the mapping \( f \) is inessential.
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Proof: We may assume that $P \neq \emptyset$.

I. Let $P$ be locally connected. By 18.2.1 there exists a mapping $\varphi$ of $P$ into $\mathbb{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$, and such that $\varphi_K$ is continuous for every component $K$ of $P$. Since the sets $K$ are open (see 22.1.4), we prove easily that $\varphi$ is continuous.

II. Let $P$ be compact. Let $\mathcal{K}$ be the system of all components of $P$. Every $K \in \mathcal{K}$ has, by 24.2.16, a neighborhood $\Gamma(K)$ such that the partial mapping $f_{\Gamma(K)}$ is inessential. By 19.1.4 (see also 19.1.5) there is a neighborhood $\Delta(K) \subset \Gamma(K)$ of $K$ such that $\Delta(K)$ is both closed and open. Since the sets $\Delta(K)$ are open and since $\bigcup\limits_{K \in \mathcal{K}} \Delta(K) = P$, we prove easily that $\varphi$ is continuous.

24.2.18.* Let $P$ be a separable, locally compact and locally connected space. Let $f$ be a continuous mapping of $P$ into $\mathbb{S}_1$. If $f$ is essential, there is a continuum $K \subset P$ such that the partial mapping $f_K$ is essential.

Proof: By 24.2.17 there exists a component $Q$ of the space $P$ such that the partial mapping $f_Q$ is essential. By 16.1.2, ex. 17.20, 22.1.4 and 22.1.6, $Q$ is a connected, separable, locally compact and locally connected space. Since $Q$ is locally compact, we may associate with every $z \in Q$ a neighborhood $U(z)$ of $z$ in $Q$ such that $U(z)$ is compact. Since $Q$ is locally connected, we may find (for every $z \in Q$) a connected neighborhood $V(z)$ of $z$ in $Q$ such that $V(z) \subset U(z)$. The set $\bigcup\limits_{z \in Q} V(z)$ is connected by 18.1.6 and compact by 17.2.2. By 16.2.2 we may find a sequence $\{z_n\}$ such that

$$
\bigcup\limits_{n=1}^{\infty} V(z_n) = Q.
$$

By 18.4.2 (see also 18.3.1), for every $m = 1, 2, 3, \ldots$ there is a finite subsequence $\{u^{(m)}_\lambda\}_{\lambda=1}^{k_m}$ of $\{z_n\}$ such that $u^{(m)}_0 = z_1$, $u^{(m)}_{k_m} = z_m$, $V(u^{(m)}_{\lambda-1}) \cap V(u^{(m)}_\lambda) = \emptyset$ for $1 \leq \lambda \leq k_m$. Put

$$
H_m = \bigcup_{\lambda=0}^{k_m} V(u^{(m)}_\lambda), \quad G_n = \bigcup_{m=1}^{n} H_m.
$$

*) This is a particular case of theorem 24.4.2. The proof of the more general theorem is, of course, more complicated.
It is easy to prove that the sets $G_n$ are connected and open in $Q$. Moreover, $G_n \subseteq G_{n+1}$, $\bigcup_{n=1}^{\infty} G_n = Q$ and the mapping $f_Q$ is essential. Hence, by 24.2.14, there exists an index $n$ such that $f_{G_n}$ is essential. Hence (see 24.2.6) the mapping $f_K$ is also essential, if $K = G_n$. It is easy to prove (see ex. 24.8) that $K$ is a continuum.

24.2.19. Let $Q$ be a connected dense subset of a space $P$. Let $f$ be a continuous mapping of $P$ into $S_t$. Let the partial mapping $f_Q$ be inessential. Then there exists a set $M \subseteq P$ such that

1. $M$ is closed,
2. $M \cap Q = \emptyset$,
3. if $Q \subseteq X \subseteq P$, $M \cap X = \emptyset$, then the partial mapping $f_X$ is inessential, [4] if $Q \subseteq X \subseteq P$, $M \cap X \neq \emptyset$, then the partial mapping $f_X$ is essential.

Proof: I. There exists a continuous mapping $\varphi$ of the set $Q$ into $E_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in Q$. Let $G$ be the set of all $x \in P$ which have the following property: There is a number $\varphi(x)$ such that, if $a_n \to x$ and $a_n \in Q$ for every $n$, then $\varphi(a_n) = \varphi(x)$.

Evidently $Q \subseteq G$ and $\varphi(x) = \varphi(x)$ for $x \in Q$.

Put $M = P - G$, so that $M \cap Q = \emptyset$. By ex. 12.2, for every $x \in G$ there is a sequence $\{a_n\}$ such that $a_n \in Q$ for every $n$, $a_n \to x$, so that obviously $f(x) = e^{i\varphi(x)}$ for every $x \in G$.

II. $\psi$ is a continuous mapping of $G$ into $E_1$, so that $f_X$ is inessential whenever $Q \subseteq X \subseteq P$, $M \cap X = \emptyset$. Let $x \in G$, $x_n \in G$, $x_n \to x$. We have to prove that $\psi(x_n) \to \psi(x)$. There exist sequences $\{a_{nv}\}_{v=1}^{\infty}$ such that $a_{nv} \in Q$, $\lim_{v \to \infty} a_{nv} = x_n$. As $x_n \in G$, we have $\lim_{v \to \infty} \varphi(a_{nv}) = \varphi(x_n)$. For every $n$ there is an index $v_n$ with $\varphi(a_{nv_n}, x_n) < n^{-1}$, $|\varphi(a_{nv_n}) - \varphi(x_n)| < n^{-1}$. Thus, $\lim_{n \to \infty} a_{nv_n} = x$, $a_{nv_n} \in Q$, hence $\lim_{n \to \infty} \varphi(a_{nv_n}) = \varphi(x)$, so that $\lim_{n \to \infty} \psi(x_n) = \psi(x)$.

III. Let $Q \subseteq X \subseteq P$ and let the partial mapping $f_X$ be inessential. We have to prove that $M \cap X = \emptyset$, i.e. that $X \subseteq G$. There exists a continuous mapping $\chi$ of $X \supset Q$ into $E_1$ such that $f(x) = e^{i\chi(x)}$ for every $x \in X$, so that $e^{i\varphi(x)} = e^{i\chi(x)}$ for every $x \in Q$. By 24.2.11 there exists an integer $k$ with $\varphi(x) = \chi(x) + 2k\pi$ for every $x \in Q$. Choose an $x \in X$. Let $a_n \in Q$, $a_n \to x$ (see ex. 12.2). Then we have $\chi(a_n) \to \chi(x)$, hence $\varphi(a_n) \to \chi(x) + 2k\pi$. Thus, $x \in G$, $\psi(x) = \chi(x) + 2k\pi$, so that in fact $X \subseteq G$.

IV. It remains to be proved that $M$ is closed, i.e. that $G$ is open. Choose an $a \in G$. By 24.1.2 there is a homeomorphic mapping $h$ of $S_1 - \{ -f(a) \}$ onto the interval $J = E[\psi(a) - \pi < t < \psi(a) + \pi]$ such that $e^{ih(y)} = y$ for every $y \in S_1 - \{ -f(a) \}$.

Evidently $h[f(a)] = \psi(a)$. There is a neighborhood $U$ of $a$ such that $f(x) \neq -f(a)$ for every $x \in U$. For $x \in U$ put $\Phi(x) = h[f(x)]$. Then $\Phi$ is a continuous mapping
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of $U$ into $E_1$; we have $\Phi(a) = \psi(a)$, and $f(x) = e^{i\psi(x)}$ for every $x \in U$. There is a neighborhood $U_1 \subseteq U$ of $a$ such that $x \in U_1$ implies $| \Phi(x) - \psi(a) | < \frac{1}{2} \pi$. By II there is a neighborhood $U_2 \subseteq U_1$ of $a$ such that $x \in G \cap U_2$ implies $| \psi(x) - \psi(a) | < \frac{1}{2} \pi$. Thus, $x \in Q \cap U_2$ implies $| \Phi(x) - \phi(x) | < \pi$. On the other hand, we have $e^{i\Phi(x)} = f(x) = e^{i\phi(x)}$ for every $x \in Q \cap U_2$. Hence, $x \in Q \cap U_2$ implies $\Phi(x) = \phi(x)$. If $x \in U_2$ and if $a_n \in Q$, $a_n \rightarrow x$, there exists an index $p$ such that $n > p$ implies $a_n \in U_2$, which implies $\Phi(a_n) = \phi(a_n)$. We have $\Phi(a_n) \rightarrow \Phi(x)$. Thus, $\phi(a_n) \rightarrow \phi(x)$, i.e. $x \in G$, $\psi(x) = \Phi(x)$. Thus, every $x \in G$ has a neighborhood $U_2 \subseteq G$ so that the set $G$ is open.

24.3. 24.3.1. Let $P$ be a simple arc. Then every continuous mapping $f$ of $P$ into $S_1$ is inessential.

Proof: By 17.4.4 (see also 9.6.1), there exists an $\varepsilon > 0$ such that

$$x \in P, \ y \in P, \ \phi(x, y) < \varepsilon \ \text{implies} \ |f(x) - f(y)| < 2. \quad (1)$$

By 20.1.12 there is a finite point sequence $\{c_i\}_{i=1}^{m-1}$ and a finite sequence $\{C_i\}_{i=1}^{m}$ of point sets such that [1] $C_i$ are simple arcs and, hence (see 17.2.2), they are closed sets, [2] $\bigcup_{i=1}^{m} C_i = P$, [3] $C_i \cap C_{i+1} = (c_i)$ ($1 \leq i \leq m - 1$), [4] $C_i \cap C_j = \emptyset$ ($1 \leq i \leq m, 1 \leq j \leq m, |i - j| \leq 2$), [5] $d(C_i) < \varepsilon$ ($1 \leq i \leq m$), so that, by (1), $S_1 - f(C_i) \neq \emptyset$. Thus, the partial mappings $f_{C_i}$ are inessential by 24.2.7. Put $A_i = \bigcup_{j=1}^{i} C_j$ ($1 \leq i \leq m$). Then $A_1 = C_1$ and for $1 \leq i \leq m - 1$ we have $A_{i+1} = A_i \cup C_{i+1}$ with closed summands, $A_i \cap C_{i+1} = (c_i)$. Thus, by 24.2.13, it follows by induction that the partial mappings $f_{A_i}$ ($1 \leq i \leq m$) are inessential. We have $P = A_m$, so that $f$ is inessential.

Now, let $P$ be a simple loop and let $f$ be a continuous mapping of $P$ into $S_1$. Choose an orientation of $P$ (see 21.2). Choose $a \in P, b \in P, a \neq b$. By 21.2.2 (see also 21.1.2) we have $P = P(a, b) \cup P(b, a), P(a, b) \cap P(b, a) = (a) \cup (b)$. The sets $P(a, b), P(b, a)$ are simple arcs, so that, by 24.3.1, there exists a continuous mapping $\phi_1$ of $P(a, b)$ into $E_1$ and a continuous mapping $\phi_2$ of $P(b, a)$ into $E_1$ such that

$$x \in P(a, b) \ \text{implies} \ e^{i\phi_1(x)} = f(x), \quad (2)$$
$$x \in P(b, a) \ \text{implies} \ e^{i\phi_2(x)} = f(x).$$

We have $e^{i\phi_1(a)} = e^{i\phi_2(a)}$, $e^{i\phi_1(b)} = e^{i\phi_2(b)}$, so that there are integers $n_1, n_2$ with

$$\phi_2(a) = \phi_1(a) + 2n_1 \pi, \quad (3)$$
$$\phi_2(b) = \phi_1(b) + 2n_2 \pi.$$ 

Put

$$n = n_1 - n_2,$$

so that $n$ is an integer.
Preserving the points $a$, $b$ and the chosen orientation of the simple loop $P$, replace the mappings $\varphi_1$, $\varphi_2$ by other mappings $\psi_1$, $\psi_2$ having the same properties. We obtain integers $m_1$, $m_2$ instead of the integers $n_1$, $n_2$. By 20.1.1 and 24.2.11 there are integers $k_1$, $k_2$ such that

$$ x \in P(a, b) \implies \psi_1(x) = \varphi_1(x) + 2k_1\pi, $$
$$ x \in P(b, a) \implies \psi_2(x) = \varphi_2(x) + 2k_2\pi. $$

Thus,

$$ \psi_2(a) = \varphi_2(a) + 2k_2\pi = \varphi_1(a) + 2(n_1 + k_2)\pi = \psi_1(a) + 2(n_1 + k_2 - k_1)\pi, $$

so that $m_1 = n_1 + k_2 - k_1$ and similarly $m_2 = n_2 + k_2 - k_1$. Hence,

$$ n = n_1 - n_2 = m_1 - m_2. $$

Thus, the number $n$ does not depend on the choice of $\varphi_1$, $\varphi_2$. Let us write, more precisely, $n = n(a, b)$. We are going to prove that (with the orientation of $P$ given) the number $n$ does not depend on the choice of $a$, $b$. It suffices to prove that the number $n$ remains unchanged whenever we preserve one of the points—a and replace the point $b$ by another point $c$; i.e. we prove that $n(a, b) = n(a, c)$ for distinct $a$, $b$, $c$.

For certainty, let $c \in P(a, b)$. It is easy to prove that

$$ P(a, c) \cup P(c, b) = P(a, b), \quad P(a, c) \cap P(c, b) = (c), $$
$$ P(c, b) \cup P(b, a) = P(c, a), \quad P(c, b) \cap P(b, a) = (b). $$

By 24.3.1 there are continuous mappings $\varphi_1$, $\varphi_2$, $\varphi_3$ of the simple arcs $P(a, c)$, $P(c, b)$, $P(b, a)$ into $\mathbb{E}_1$ such that

$$ x \in P(a, c) \implies e^{i\varphi_1(x)} = f(x), $$
$$ x \in P(c, b) \implies e^{i\varphi_2(x)} = f(x), $$
$$ x \in P(b, a) \implies e^{i\varphi_3(x)} = f(x). $$

There are integers $h_1$, $h_2$, $h_3$ with

$$ \varphi_3(a) = \varphi_1(a) + 2h_1\pi, $$
$$ \varphi_3(b) = \varphi_2(b) + 2h_2\pi, $$
$$ \varphi_2(c) = \varphi_1(c) + 2h_3\pi. $$

There exist (see ex. 9.5) continuous mappings $\varphi_4$, $\varphi_5$ of the simple arcs $P(a, b)$, $P(c, a)$ into $\mathbb{E}_1$ such that

$$ x \in P(a, c) \Rightarrow \varphi_4(x) = \varphi_1(x), \quad x \in P(c, b) \Rightarrow \varphi_4(x) = \varphi_2(x) - 2h_3\pi, $$
$$ x \in P(c, b) \Rightarrow \varphi_5(x) = \varphi_2(x), \quad x \in P(b, a) \Rightarrow \varphi_5(x) = \varphi_3(x) - 2h_2\pi. $$

Evidently

$$ n(a, b) = n_1 - n_2, \quad n(a, c) = m_1 - m_2. $$
where
\[\begin{align*}
2n_1\pi &= \varphi_3(a) - \varphi_4(a) = \varphi_3(a) - \varphi_1(a) = 2h_1\pi \\
2n_2\pi &= \varphi_3(b) - \varphi_4(b) = \varphi_3(b) - [\varphi_2(b) - 2h_3\pi] = 2(h_2 + h_3)\pi, \\
2m_1\pi &= \varphi_5(a) - \varphi_1(a) = [\varphi_3(a) - 2h_2\pi] - \varphi_1(a) = 2(h_1 - h_2)\pi, \\
2m_2\pi &= \varphi_5(c) - \varphi_1(c) = \varphi_2(c) - \varphi_1(c) = 2h_3\pi,
\end{align*}\]
so that
\[n_1 - n_2 = h_1 - (h_2 + h_3) = (h_1 - h_2) - h_3 = m_1 - m_2,\]
i.e., \(n(a, b) = n(a, c)\).

Thus, the number \(n\) — for a given mapping \(f\) — depends on the orientation of the simple loop \(P\) only. If we change the orientation, we obtain \(-n\) instead of \(n\) (see Remark at the end of Section 21.2).

The number \(n\) is said to be the degree of the mapping \(f\). If the mapping \(f\) is inessential, then there is a continuous mapping \(\varphi\) of \(P\) into \(E^1\) with \(e^{i\varphi(x)} = f(x)\) for every \(x \in P\).

We may put \(\varphi_1 = \varphi_{P(a, b)}\), \(\varphi_2 = \varphi_{P(b, a)}\), and we obtain in (3) \(n_1 = n_2 = 0\) and consequently \(n = 0\).

On the other hand let \(n = 0\), so that \(n_1 = n_2\) in (3); if \(\varphi_1, \varphi_2\) are the mappings from (2), there is a mapping \(\varphi\) of \(P\) into \(E^1\) such that
\[\begin{align*}
x \in P(a, b) &\text{ implies } \varphi(x) = \varphi_1(x), \\
x \in P(b, a) &\text{ implies } \varphi(x) = \varphi_2(x) - 2n_1\pi.
\end{align*}\]

We have \(e^{i\varphi(x)} = f(x)\) for every \(x \in P\) and the mapping \(f\) is continuous (see ex. 9.5) so that \(f\) is inessential.

The results obtained are stated in the following two theorems:

**24.3.2.** The degree \(n\) of a continuous mapping of an oriented simple loop into \(S^1\) is an integer. If the orientation is changed, \(n\) is replaced by \(-n\).

**24.3.3.** A continuous mapping of an oriented simple loop into \(S^1\) is inessential if and only if its degree is zero.

Moreover, it is easy to prove the following theorem:

**24.3.4.** Let \(f_1, f_2\) be continuous mappings of an oriented simple loop \(P\) into \(S^1\) and let \(n_1, n_2\) be their degrees. Then the degree of the mapping \(f_1f_2\) is equal to \(n_1 + n_2\).

**24.3.5.** Let \(P\) be an oriented simple loop. There are exactly two kinds of homeomorphic mappings of \(P\) onto \(S^1\). The mappings of the first kind have degree one, the mappings of the second kind have degree minus one.

**Proof:** I. Choose \(a \in P\), \(b \in P\), \(a \neq b\). Then \(P(a, b)\) and \(P(b, a)\) are simple arcs with end points \(a, b\), so that there is a homeomorphic mapping \(\varphi_1\) of the interval
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Let \( J = E[0 \leq t \leq 1] \) onto \( P(a, b) \) and a homeomorphic mapping \( \varphi_2 \) of \( J \) onto \( P(b, a) \) such that \( \varphi_1(0) = \varphi_2(0) = a, \varphi_1(1) = \varphi_2(1) = b \). Define \( f_1, f_2 \) by

\[
\begin{align*}
    f_1(x) &= e^{it}, & f_2(x) &= e^{-it} \quad &\text{for} \quad x \in P(a, b), \quad x = \varphi_1(t), \\
    f_1(x) &= e^{-it}, & f_2(x) &= e^{it} \quad &\text{for} \quad x \in P(b, a), \quad x = \varphi_2(t).
\end{align*}
\]

It is easy to prove that \( f_1, f_2 \) are homeomorphic mappings of \( P \) onto \( S_1 \) and that their degrees are \( +1, -1 \).

II. Let \( f \) be a homeomorphic mapping of \( P \) onto \( S_1 \). Put \( a = f^{-1}(1), b = f^{-1}(-1) \). Let \( M_1 \) be the set of all \( e^{it} \) \( (0 \leq t \leq 1) \). Let \( M_2 \) be the set of all \( e^{-it} \) \( (0 \leq t \leq 1) \). Then \( M_1 \cup M_2 = S_1 \), \( M_1 \cap M_2 = (1) \cup (-1) \) and \( M_1, M_2 \) are simple arcs with end points \( +1, -1 \). Thus, \( f^{-1}(M_1) \subset P, f^{-1}(M_2) \subset P \) are two distinct simple arcs with end points \( a, b \). Thus, under a suitable choice of orientation of the simple loop \( P \) we have

\[
P(a, b) = f^{-1}(M_1), \quad P(b, a) = f^{-1}(M_2).
\]

Obviously there is a homeomorphic mapping \( \varphi_1 \) of \( P(a, b) \) onto \( J = E[0 \leq t \leq \pi] \) and a homeomorphic mapping \( \varphi_2 \) of \( P(b, a) \) onto \( J \) such that

\[
\begin{align*}
    f(x) &= e^{i\varphi_1(x)} \quad &\text{for} \quad x \in P(a, b), \\
    f(x) &= e^{-i\varphi_2(x)} \quad &\text{for} \quad x \in P(b, a).
\end{align*}
\]

We have \( \varphi_1(a) = \varphi_2(a) = 0, \varphi_1(b) = \varphi_2(b) = \pi \), so that the degree of \( f \) is equal to \( +1 \). If we change the orientation, the degree of \( f \) is equal to \( -1 \).

24.3.6. Let \( P \) be an oriented simple loop. Let \( n \) be an integer. Then there exists a continuous mapping of \( P \) into \( S_1 \) with degree equal to \( n \).

Proof: By 24.3.5 there is a homeomorphic mapping \( f \) of \( P \) onto \( S_1 \) with degree one. By 24.3.4 (see also 24.3.2) it is easy to prove that the mapping \( f^n \) has degree \( n \).

24.3.7. Let \( P \subset E_1 \). Then every continuous mapping \( f \) of \( P \) into \( S_1 \) is inessential.

Proof: By 24.2.15 there is a set \( G \supset P \) open in \( E_1 \) and a continuous mapping \( g \) of \( G \) into \( S_1 \) such that \( |f(x) - g(x)| < 2 \) for every \( x \in G \). Thus, by 24.2.6 and 24.2.8, it suffices to prove that the mapping \( g \) of \( G \) into \( S_1 \) is inessential.

Let \( g \) be essential. The set \( G \) is separable by 16.1.2 and 16.1.5, locally compact by 17.10.1 (see also ex. 17.20) and locally connected by 22.1.3 and 22.1.8. Thus, by 24.2.18, there is a continuum \( K \subset G \) such that the partial mapping \( g_K \) is essential. This is a contradiction by 19.2.2 and 24.3.1.

24.4. 24.4.1. Let \( Q \subset P \). Let us define \( L(Q) \) in the same manner as in 22.2. Let \( Q \subset M \subset Q \cup L(Q) \). Let \( g \) be a continuous mapping of \( M \) into \( S_1 \). Let the partial mapping \( f_Q \) be inessential. Then \( f \) is inessential.
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**Proof:** I. There is a continuous mapping \( \varphi \) of \( Q \) into \( E_1 \) such that \( e^{i\varphi(x)} = f(x) \) for every \( x \in Q \).

II. Let \( x \in M - Q \). Since \( f \) is continuous, there exists a neighborhood \( V_x \) of \( x \) in the space \( M \) such that \( f(y) = -f(x) \) for \( y \in V_x \). By 8.7.5 there is a neighborhood \( U_x \) of \( x \) in \( P \) such that \( V_x = M \cap U_x \). Since \( M - Q \subset L(Q) \), there is a component \( K_x \) of \( Q \cap U_x = Q \cap V_x \subset M \) such that \( x \) is an interior point of \( K_x \cup (P - Q) \). The partial mapping \( f_{V_x} \) is inessential by 24.2.7, as \( f(V_x) \subset S_1 - [-f(x)]. \) Thus, there exists a continuous mapping \( \chi_x \) of \( V_x \) into \( E_1 \) such that

\[ e^{i\chi_x(y)} = f(y) \quad \text{for} \quad y \in V_x. \]

As \( K_x \) is a connected subset of \( Q \cap V_x \), there is, by 24.2.11, an integer \( k_x \) such that

\[ y \in K_x \Rightarrow \chi_x(y) = \varphi(y) + 2k_x\pi. \]

III. Let us define a mapping \( \psi \) of \( M \) into \( E_1 \) as follows: First, if \( x \in Q \), put \( \psi(x) = \varphi(x) \). Secondly, if \( x \in M - Q \), put \( \psi(x) = \chi_x(x) - 2k_x\pi \). Then we have \( e^{i\psi(x)} = f(x) \) for every \( x \in M \). It remains to prove that \( \psi \) is continuous.

IV. Let \( x \in M \). As \( L(Q) \subset \overline{Q} \), we have \( M \subset \overline{Q} \). Hence (see 8.2.1), there exists a sequence \( \{a_n\} \) such that \( a_n \to x \) and \( a_n \in Q \) for every \( n \). We shall prove that \( \varphi(a_n) \to \psi(x) \).

This is evident for \( x \in Q \). Hence, let \( x \in M - Q \). By II, \( x \) is an interior point of \( K_x \cup (P - Q) \). Thus, there is an index \( p \) such that \( a_n \in K_x \cup (P - Q) \) for \( n > p \). As \( a_n \in Q \), we see that

\[ n > p \Rightarrow a_n \in K_x \Rightarrow \varphi(a_n) = \chi_x(a_n) - 2k_x\pi. \]

On the other hand, \( \chi_x \) is a continuous mapping of the set \( V_x \supset K_x \) into \( E_1 \). Hence,

\[ \varphi(a_n) \to \chi_x(x) - 2k_x\pi = \psi(x). \]

V. Let us choose an \( x \in M \) and prove that \( \psi \) is continuous at the point \( x \). Thus, let \( x_n \in M \), \( x_n \to x \). We have to prove that \( \psi(x_n) \to \psi(x) \). There are sequences \( \{b_{nv}\}_{v=1}^{\infty} \) \( (n = 1, 2, 3, \ldots) \) in \( Q \) such that \( \lim_{v \to \infty} b_{nv} = x_n \). By IV, \( \lim_{v \to \infty} \varphi(b_{nv}) = \psi(x_n) \).

Obviously, for every \( n = 1, 2, 3, \ldots \) there is an index \( v_n \) such that

\[ \varphi(x_n, b_{nv_n}) < n^{-1}, \quad |\psi(x_n) - \varphi(b_{nv_n})| < n^{-1}. \]

As \( x_n \to x \), \( \varphi(x_n, b_{nv_n}) < n^{-1} \), we have \( \lim_{v \to \infty} b_{nv_n} = x \). Moreover, \( b_{nv_n} \in Q \), so that, by IV, \( \lim_{n \to \infty} \varphi(b_{nv_n}) = \psi(x) \). As \( |\psi(x_n) - \varphi(b_{nv_n})| < n^{-1} \), we have also \( \lim_{n \to \infty} \psi(x_n) = \psi(x) \).

24.4.2. Let \( P \) be a topologically complete locally connected space. Let \( f \) be a continuous mapping of \( P \) into \( S_1 \). Let \( f_Q \) be inessential for every simple loop \( Q \subset P \). Then \( f \) is inessential.
24. Inessential mappings onto the circle

Proof: I. Let \( K \) be a component of \( P \). By 24.2.17 it suffices to prove that the partial mapping \( f_K \) is inessential. The space \( K \) is topologically complete by 13.2, 15.5.3 and 18.2.2. Moreover, it is connected and also, by 22.1.6, locally connected.

II. Choose a point \( a \in K \) and a number \( \alpha \in E_1 \) with \( e^{i\alpha} = f(a) \). If \( x \in K \), \( x \neq a \), then by 22.3.1 \( K \) contains at least one simple arc with end points \( a, x \).

Let \( C_1 \subset K, C_2 \subset K \) be simple arcs with end points \( a, x \). By 24.3.1 there is a continuous mapping \( \varphi_1 \) of \( C_1 \) into \( E_1 \) and a continuous mapping \( \varphi_2 \) of \( C_2 \) into \( E_1 \) such that:

1. \( \varphi_1(a) = \varphi_2(a) = \alpha \),
2. \( e^{i\varphi_1(y)} = f(y) \) for every \( y \in C_1 \) and \( e^{i\varphi_2(y)} = f(y) \) for every \( y \in C_2 \). We shall prove that \( \varphi_1(x) = \varphi_2(x) \). Let us assume the contrary.

Let \( C \) be oriented in such a way that \( a \) is the initial point. Define \( M \subset C \) as follows: If \( y \in C \) then \( y \in M \) if and only if \( y \in C_2 \) and \( \varphi_1(y) = \varphi_2(y) \). The set \( M \) is obviously (see 9.5) closed in \( C \). Moreover, \( a \in M \) and hence \( M \neq \emptyset \). By 20.2.7 there exists a last point \( b \) of the set \( M \subset C \). As \( \varphi_1(x) \neq \varphi_2(x) \), we have \( b = x \), so that (see 20.1.8) there exists a simple arc \( C_1(b, x) \subset C \). Evidently

\[ y \in C_2 \cap C_1(b, x), \quad \varphi_1(y) = \varphi_2(y) \Rightarrow y = b. \] (1)

There exists a simple arc \( C_2(b, x) \subset C_2 \). Suppose that it is oriented in such a way that \( b \) is the initial point. We define a set \( M' \subset C_2(b, x) \) as follows: If \( y \in C_2(b, x) \), then \( y \in M' \) if and only if \( y \in C_1(b, x) \) and \( \varphi_1(y) \neq \varphi_2(y) \). As \( e^{i\varphi_1(y)} = e^{i\varphi_2(y)} \), we may write \(| \varphi_1(y) - \varphi_2(y) | \geq 2\pi \) instead of \( \varphi_1(y) \neq \varphi_2(y) \). Thus (see 9.5) the set \( M' \) is closed in \( C_2(b, x) \). Moreover, \( x \in M' \) and hence \( M' \neq \emptyset \). By 20.2.7 there is a first element \( c \) of the set \( M' \subset C_2(b, x) \). By (1), \( c \) is the first point \( y \in C_2(b, x) \) with \( y \in C_1(b, x) - (b) \). There exist simple arcs

\[ C_1(b, c) \subset C_1, \quad C_2(b, c) \subset C_2. \]

Evidently \( C_1(b, c) \cap C_2(b, c) = (b) \cup (c) \), so that \( C_1(b, c) \cup C_2(b, c) = Q \) is a simple loop by 21.1.3. Let \( Q \) be oriented in such a way that

\[ Q(b, c) = C_1(b, c), \quad Q(c, b) = C_2(b, c). \]

Since \( \varphi_1(b) = \varphi_2(b) \), the degree of the mapping \( f_Q \) is equal to

\[ \frac{1}{2\pi} [\varphi_1(c) - \varphi_2(c)] \neq 0, \]

so that the mapping \( f_Q \) is essential by 24.3.3. This is a contradiction.

III. Put \( \psi(a) = \alpha \). If \( x \in K - (a) \), we define \( \psi(x) \in E_1 \) as follows: Choose a simple arc \( C \subset K \) with end points \( a, x \) and a continuous mapping \( \varphi \) of \( C \) into \( E_1 \) such that \( \varphi(a) = \alpha \) and that \( e^{i\varphi(y)} = f(y) \) for \( y \in C \).

Then, put \( \psi(x) = \varphi(x) \). By II, \( \psi \) is a uniquely defined mapping of the set \( K \) into \( E_1 \). Evidently \( e^{i\psi(x)} = f(x) \) for every \( x \in K \), so that it suffices to prove that the mapping \( \psi \) is continuous.
IV. Let us choose a point \( x_0 \in K \) and prove that the mapping \( \psi \) is continuous at the point \( x_0 \). As \( f \) is continuous in \( x_0 \), there is a neighborhood \( U \) of the point \( x_0 \) in \( K \) such that \( x \in U \) implies \( f(x) = -f(x_0) \). By 24.2.7 there is a continuous mapping \( \chi \) of \( U \) into \( E_1 \) such that \( e^{ix(x)} = f(x) \) for \( x \in U \) and that \( \chi(x_0) = \psi(x_0) \).

Let \( V \) be the component of \( U \) containing the point \( x_0 \). By 22.1.4, \( V \) is a neighborhood of the point \( x_0 \) in \( K \). \( V \) is a connected space. Moreover, \( V \) is topologically complete by 15.5.3 and locally connected by 22.1.3.

It suffices to prove that \( \chi(x) = \psi(x) \) for \( x \in (x_0) \cup [V - (a)] \). This is evident for \( x = x_0 \). Thus, let \( x \in V \), \( a \neq x \neq x_0 \).

By 22.3.1 there exists a simple arc \( C \subset V \) with end points \( x_0, x \). If \( x_0 = a \), then \( \chi(x) \) is a continuous mapping of \( C \) into \( E_1 \) such that \( e^{ix(y)} = f(y) \) for \( y \in C \) and that \( \chi(x_0) = \chi(x) \). Thus, let \( x_0 \neq a \). Then there exists a simple arc \( C_0 \subset K \) with end points \( a, x_0 \) and a continuous mapping \( \varphi_0 \) of \( C_0 \) into \( E_1 \) such that \( e^{i\varphi_0(y)} = f(y) \) for \( y \in C_0 \) and \( \varphi_0(a) = \chi \). Let \( C_0 \) be oriented in such a way that \( a \) is its initial point. Define a set \( M \subset C_0 \) as follows: If \( y \in C_0 \), then \( y \in M \) if and only if \( y \in C \). It is easy to prove that \( M \) is closed in \( C_0 \). Evidently \( x_0 \in M \), so that \( M \neq \emptyset \). Hence, by 20.2.7 there is a first point \( x_1 \) of the set \( M \subset C_0 \). If \( x_1 = a \), put \( C_1 = (a) \). If \( x_1 \neq a \), put \( C_1 = C_0(a, x_1) \) (see 20.1.8). It is easy to prove that there are simple arcs \( C' \subset C_1 \cup C, C'' \subset C_1 \cup C \) such that \( [1] C' = C_1 \cap (C' \cap C), C'' = C_1 \cap (C'' \cap C), [2] a, x_0 \) are the end points of \( C' \), \( [3] a, x \) are the end points of \( C'' \). As \( e^{i\varphi_0(x_1)} = f(x_1) = e^{i\varphi_0(x_1)} \), there is an integer \( k \) with \( \chi(x_1) = \varphi_0(x_1) + 2k\pi \).

It is easy to prove that there exists a continuous mapping \( \varphi' \) of the set \( C' \) into \( E_1 \) and a continuous mapping \( \varphi'' \) of \( C'' \) into \( E_1 \) such that

\[
y \in C_1 \implies \varphi'(y) = \varphi''(y) = \varphi_0(y),
\]

\[
y \in C' \setminus C_1 \implies \varphi'(y) = \chi(y) - 2k\pi,
\]

\[
y \in C'' \setminus C_1 \implies \varphi''(y) = \chi(y) - 2k\pi.
\]

Evidently: \( e^{i\varphi'(y)} = f(y) \) for \( y \in C' \), \( e^{i\varphi''(y)} = f(y) \) for \( y \in C'' \), \( \varphi'(a) = \varphi''(a) = \chi \).

Thus, we have \( \varphi'(x_0) = \psi(x_0), \varphi''(x) = \psi(x) \). Since \( \varphi'(x_0) = \chi(x_0) - 2k\pi = \psi(x_0) - 2k\pi, \varphi''(x) = \chi(x) - 2k\pi \), we obtain \( k = 0 \) and \( \chi(x) = \psi(x) \).

24.5. 24.5.1. Let \( P \) be a metric space. Let \( Q \) be either a continuum or a connected and locally connected space. Let \( f \) be a continuous mapping of \( P \times Q \) into \( S_1 \). Let, for every \( x \in P \), the partial mapping \( f(x) \times Q \) be inessential. Let there exist a point \( b \in Q \) such that the partial mapping \( f_P \times (b) \) is inessential. Then the mapping \( f \) is inessential.

**Proof:** 1. There exists a continuous mapping \( \chi \) of \( P \) into \( E_1 \) such that \( e^{ix(x)} = f(x, b) \) for every \( x \in P \). For every \( x \in P \) there exists a continuous mapping \( \psi_x \) of \( Q \) into \( E_1 \) such that \( e^{i\psi_x(y)} = f(x, y) \) for every \( y \in Q \). We may assume that \( \psi_x(b) = \chi(x) \) for every \( x \in P \).*

*) Otherwise it suffices to replace the mapping \( \psi_x \) by a mapping \( \psi_x' \) defined by for every \( y \in Q \):

\[
\psi_x'(y) = \psi_x(y) + \chi(x) - \psi_x(b)
\]
For \((x, y) \in P \times Q\) put \(\varphi(x, y) = \psi_x(y)\), so that \(\varphi\) is a mapping of \(P \times Q\) into \(E_1\), such that \(e^{i\varphi(x, y)} = f(x, y)\) for every \((x, y) \in P \times Q\). It remains to prove that the mapping \(\varphi\) is continuous. Let us choose an arbitrary point \(\alpha \in P\) and prove that \(\varphi\) is continuous at the point \((\alpha, y)\) for every \(y \in Q\).

II. Let \(Q\) be a continuum. As \(\chi\) is a continuous mapping of \(P\) into \(E_1\), there is an \(\varepsilon > 0\) such that

\[
x \in P, \quad \varphi(x, x) < \varepsilon \Rightarrow |\chi(x) - \chi(x)| < \pi.
\]

As \(f\) is a continuous mapping of \(P \times Q\) into \(S_1\), we may associate with every \(z \in Q\) a number \(\delta(z) > 0\) such that

\[
x \in P, \quad y \in Q, \quad \varphi(a, x) < \delta(z), \quad \varphi(z, y) < \delta(z) \Rightarrow |f(x, y) - f(a, y)| < 2.
\]

We have

\[
Q = \bigcup_{z \in Q} \Omega[z, \delta(z)]
\]

with open summands. Since \(Q\) is compact, by 17.5.4 there is a finite sequence \(\{z_n\}_1^p\), \(z_n \in Q\), such that

\[
\bigcup_{n=1}^p \Omega[z_n, \delta(z_n)] = Q.
\]

Let \(\eta > 0\) be the least of the \(p + 1\) numbers \(\varepsilon, \delta(z_n)\) \((1 \leq n \leq p)\). Then, first,

\[
x \in P, \quad \varphi(x, x) < \eta \Rightarrow |\chi(x) - \chi(x)| < \pi,
\]

Secondly,

\[
x \in P, \quad \varphi(x, x) < \eta \Rightarrow |f(x, y) - f(x, y)| < 2
\]

for every \(y \in Q\). In fact, for every \(y \in Q\) there is an index \(n\) with \(\varphi(z_n, y) < \delta(z_n)\).

By 24.1.2 there exists a homeomorphic mapping \(v\) of \(S_1 - (-1)\) onto the interval \(E[-\pi < t < \pi]\) such that \(e^{iv(z)} = z\) for every \(z \in S_1 - (-1)\).

Put \(P_0 = \Omega_p(\alpha, \eta)\). If \((x, y) \in P_0 \times Q\), we have \(\varphi(x, x) < \eta\), hence \(|f(x, y) - f(x, y)| < 2\), hence \(f(x, y)\) is a continuous mapping of \(P_0 \times Q\) into \(E_1\).

Since \(\psi_x(b) = \chi(a)\),

\[
x \in P_0 \Rightarrow |\Phi(x, b) - \chi(x)| < \pi.
\]

Since also \(x \in P_0 \Rightarrow |\chi(x) - \chi(x)| < \pi\),

\[
x \in P_0 \Rightarrow |\Phi(x, b) - \chi(x)| < 2\pi.
\]

On the other hand,

\[
e^{i\Phi(x, b)} = f(x, b) = e^{i\chi(x)},
\]

so that \(\Phi(x, b) = \chi(x)\) for every \(x \in P_0\).
VI. Mappings of a space onto the circle

Choose an \( x \in P_0 \). Let \( \Phi(x, y) = g_x(y) \) for \( y \in Q \), so that \( g_x \) is a continuous mapping of \( Q \) into \( E_1 \). \( \psi_x \) is also a continuous mapping of \( Q \) into \( E_1 \). Moreover, 
\[
e^{i\theta_x(y)} = e^{i\Phi(x, y)} = f(x, y) = e^{i\psi_x(y)} \quad \text{for every} \quad y \in Q.
\]
The space \( Q \) is connected so that, by 24.2.11, there exists an integer \( n_x \) such that 
\[
\Phi(x, y) = g_x(y) = \psi_x(y) + 2n_x\pi \quad \text{for every} \quad y \in Q.
\]
Since \( b \in Q \), \( \Phi(x, b) = \gamma(x) = \psi_x(b) \), we have \( n_x = 0 \). Thus, 
\[
\Phi(x, y) = \psi_x(y) = \varphi(x, y) \quad \text{for} \quad (x, y) \in P_0 \times Q.
\]
Since \( P_0 \times Q \) is open in \( P \times Q \), since \( \Phi \) is a continuous mapping of \( P_0 \times Q \) into \( S_1 \) and since \( x \in P_0 \), the mapping \( \varphi \) is continuous at the point \((x, y)\) for every \( y \in Q \).

III. Let \( Q \) be connected and locally connected. If \( y \in Q \), let \( y \in A \) if \( \varphi \) is continuous at \((x, y)\), \( y \in B \) if \( \varphi \) is not continuous at \((x, y)\). We have to prove that \( B = \emptyset \).

We have \( Q = A \cup B \), \( A \cap B = \emptyset \). We shall prove that the sets \( A, B \) are open in \( Q \), so that \( Q = A \cup B \) with separated summands. Since the space \( Q \) is connected, this will imply that either \( A = \emptyset \) or \( B = \emptyset \). Then the proof will be finished, as soon as we prove that \( b \in A \).

Let \( \beta \in A \), so that \( \varphi \) is continuous at \((x, \beta)\). There exists a neighborhood \( U \) of the point \( x \) in \( P \) and a neighborhood \( V \) of the point \( \beta \) in \( Q \) such that 
\[
x \in U, \quad y \in V \Rightarrow |\varphi(x, y) - \varphi(x, \beta)| < \frac{1}{4}\pi.
\]
If \( y \in V, \,(x_n, y_n) \to (x, y) \), there is an index \( p \) such that for \( n > p \) we have \( x_n \in U \), \( y_n \in V \). Since also \( \alpha \in U, y \in V, n > p \) implies \( |\varphi(x_n, y_n) - \varphi(x, \beta)| < \frac{1}{4}\pi, |\varphi(x, y) - \varphi(x, \beta)| < \frac{1}{4}\pi \), which implies \( |\varphi(x_n, y_n) - \varphi(x, y)| < \pi \), so that, by 24.2.9, the mapping \( \varphi \) is continuous at the point \((x, y)\). Thus, \( V \subset A \). Consequently, \( A \) is open in \( Q \).

Now, let us prove that the set \( B \) is also open in \( Q \). Let \( \beta \in B \) so that \( \varphi \) is not continuous at \((x, \beta)\). We have to prove that there is a neighborhood \( W \) of the point \( \beta \) in \( Q \) such that \( W \subset B \).

By 24.2.9 there exists a sequence \( \{(x_n, y_n)\} \) in \( P \times Q \) such that \( x_n \to x, y_n \to \beta \) and that \( |\varphi(x_n, y_n) - \varphi(x, \beta)| > \pi \) for every \( n \).

Since \( f \) is a continuous mapping of \( P \times Q \) into \( S_1 \), we can find a neighborhood \( U \) of the point \( x \) in \( P \) and a neighborhood \( V_1 \) of the point \( \beta \) in \( Q \) such that 
\[
x \in U, \quad y \in V_1 \Rightarrow |f(x, y) - f(x, \beta)| < 2.
\]
By 24.1.2 there is a homeomorphic mapping \( v \) of \( S_1 \) onto the interval \( E[-\pi < t < \pi] \) such that \( e^{iv(t)} = z \) for every \( z \in S_1 \) such that \( \zeta \).

If \( x \in U, y \in V_1 \), we have \( |f(x, y) - f(x, \beta)| < 2 \) and hence \( f(x, y) \neq -f(x, \beta) \), so that we may put \( \Phi(x, y) = \varphi(x, \beta) + v[f(x, y)/f(x, \beta)] \) for \( x \in U, y \in V_1 \). Then \( \Phi \)
is a continuous mapping of \(U \times V_1\) into \(E_1\) and we have \(e^{i\Phi(x,y)} = f(x,y)\) for every \((x,y) \in U \times V_1\). Moreover,
\[
x \in U, \ y \in V_1 \Rightarrow |\Phi(x,y) - \Phi(x,\beta)| < \pi.
\]
Let \(V_2\) be the component of \(V_1\) containing the point \(\beta\). Then \(V_2 \subseteq V_1\) and, by 22.1.4, \(V_2\) is a neighborhood of the point \(\beta\) in \(Q\).

If \(x \in U\), put \(g_x(y) = \Phi(x,y), h_x(y) = \psi_x(y)\) for \(y \in V_2\). Then \(g_x\) and \(h_x\) are continuous mappings of the connected \(V_2\) into \(E_1\) and we have \(e^{ig_x(y)} = f(x,y) = e^{ih_x(y)}\) for every \(y \in V_2\). Thus, by 24.2.11 there is an integer \(k_x\) such that \(h_x(y) = g_x(y) + 2k_x\pi\) for \(y \in V_2\). Hence,
\[
x \in U, \ y \in V_2 \Rightarrow \varphi(x,y) = \Phi(x,y) + 2k_x\pi.
\]
Since \(\psi_x\) is a continuous mapping of \(Q\) into \(E_1\), there is a neighborhood \(W \subseteq V_2\) of the point \(\beta\) in \(Q\) such that
\[
y \in W \Rightarrow |\varphi(x,y) - \varphi(x,\beta)| < \frac{1}{2}\pi.
\]
We shall prove that \(W \subseteq B\); then \(B\) will be proved to be open. Since \(x_n \to \alpha, y_n \to \beta\), there is an index \(p\) such that \(n > p\) implies \(x_n \in U, y_n \in W\). If \(n > p\), we have
\[
|\Phi(x_n,y_n) - \varphi(x,\beta)| < \pi, \ |\varphi(x_n,y_n) - \varphi(x,\beta)| > \pi, \ \varphi(x_n,y_n) = \Phi(x_n,y_n) + 2k_n\pi,
\]
\(\Phi(x,\beta) = \varphi(x,\beta), \) hence \(k_2 = 0, k_x \neq 0\). If \(W\) is not contained in \(B\), there is a point \(y \in A \cap W\). We shall obtain a contradiction as follows: Since \(y \in A\), the mapping \(\varphi\) is continuous at the point \((x,y)\). Since \(\Phi\) is also continuous at the point \((x,y)\) and since \(x_n \to \alpha\), we have \(\varphi(x_n,y) \to \varphi(x,y), \ \Phi(x_n,y) \to \Phi(x,y) = \varphi(x,y) + 2k_x\pi = \varphi(x,y), \ \varphi(x_n,y) - \Phi(x_n,y) = 2k_x\pi \to 0\), which is a contradiction, as \(|k_x| \geq 1\).

Since \(k_x \geq 1\) and since \(\Phi(x_n,\beta) \to \Phi(x,\beta) = \varphi(x,\beta), \ \Phi(x_n,\beta) = \varphi(x_n,\beta) - 2k_x\pi, \ \varphi(x_n,\beta)\) cannot converge to \(\varphi(x,\beta)\). On the other hand, evidently \(\varphi(x_n,\beta) \to \varphi(x,\beta)\). Thus, \(\beta = b\) for every \(\beta \in B\), so that \(b \in A\).

**24.5.2. Let \(f\) be a continuous mapping of \(P\) into \(S_1\). Then \(f\) is inessential, if and only if there exists a continuous mapping \(g\) of \(P \times E[0 \leq t \leq 1]\) into \(S_1\) such that**
\[
g(x,0) = f(x), \ g(x,1) = 1 \ \text{for every} \ x \in P.\)

**Proof:** I. Let such a \(g\) exist. Put \(J = E[0 \leq t \leq 1]\). By 24.3.1 the partial mapping
\[
g(x) \times t
\]
is inessential for every \(x \in P\). By 24.2.7 the partial mapping \(g_{P \times (1)}\) is inessential. Hence, by 24.5.1, \(g\) is inessential, so that (see 24.2.6) also the partial mapping \(g_{P \times (0)}\) is inessential. Thus, also the mapping \(f\) is inessential.

*) If \(f_0, f_1\) are mappings of \(X\) into \(Y\) such that there is a continuous mapping \(g\) of \(X \times E[0 \leq t \leq 1]\) into \(Y\) with \(g(x,0) = f_0(x), g(x,1) = f_1(x)\), the mappings \(f_0, f_1\) are said to be homotopic. Thus, the theorem states that a mapping \(f\) of \(P\) into \(S_1\) is inessential if and only if it is homotopic with a constant. (Ed.)
II. Let $f$ be inessential. Then there exists a continuous mapping $\varphi$ of $P$ into $E_1$ such that $e^{i\varphi(x)} = f(x)$ for every $x \in P$. Obviously, it suffices to put $g(x, t) = e^{i(1-t)\varphi(x)}$ for $x \in P$, $0 \leq t \leq 1$.

24.5.3. Let $f$ be a continuous mapping of the euclidean space $E_m$ ($m = 1, 2, 3, \ldots$) into $S_1$. Then $f$ is inessential.

Proof: The statement is true for $m = 1$ by 24.3.7. Since $E_{m+1} = E_m \times E_1$, the general statement may be proved by induction by 24.5.1.

24.5.4. Let $f$ be a continuous mapping of the spherical space $S_m$ ($m = 2, 3, 4, \ldots$) into $S_1$. Then $f$ is inessential.

Proof: If $\hat{a} \in E_m$, it is easy to prove that the set $E_m - \{x\}$ is connected. Consequently, by 17.10.4, $S_m - [(a) \cup (b)]$ is also connected if we choose $a \in S_m$, $b \in S_m$, $a \neq b$. The sets $A = S_m - (a)$, $B = S_m - (b)$ are open in $S_m$ and the partial mappings $f_A, f_B$ are inessential by 17.10.4 and 24.5.3. Moreover, $A \cap B = S_m - [(a) \cup (b)]$ is connected. Thus, $f$ is inessential by 24.2.13.

Exercises

24.1. Let $f$ be a continuous mapping of $E_m$ ($m \geq 2$) onto $S_1$. Let $a \in E_m$, $b \in E_m$, $a \neq b$. Then there exists a point $c \in E_m$ such that either $a = c$, $f(a) = f(c)$ or $b = c$, $f(b) = f(c)$.

24.2. What must we assume about a space $P$ to be allowed to replace $E_m$ in ex. 24.1. by $P$?

24.3. Every continuous mapping of any of the spaces $P_2$, $P_3$, $P_4$, $P_5$, $P_7$ (see exercises to § 19) is inessential. This is not true for the spaces $P_1$, $P_6$.

24.4. We may replace $E_1$ in theorem 24.2.15 by any $E_m$ ($m = 2, 3, 4, \ldots$) or by $U$ (see section 7.3).

Let $m = 1, 2, 3, \ldots$. Let $f$ be a continuous mapping of a space $P$ into $S_m$. We say that $f$ is inessential, if there exists a continuous mapping of $P \times E[0 \leq t \leq 1]$ into $S_m$ such that

$$g(x, 0) = f(x), \quad g(x, 1) = (1, 0, \ldots, 0) \quad \text{for every} \quad x \in P.$$ 

By theorem 24.5.2, this definition is consistent with the definition for $m = 1$ given in the section 24.2.

24.5. In theorems 24.2.6, 24.2.7, 24.2.8, 24.2.16 we may write more generally $S_m$ ($m = 1, 2, 3, \ldots$) instead of $S_1$.

24.6. Let $M \subseteq P$, $a \in M$, $b \in M$, $C \subseteq P$. Let $C$ be a simple arc with end points $a, b$. Let $C \cap \bar{M} = \{a\} \cup \{b\}$. Let $a, b$ belong to distinct quasicomponents of $M$. Let $f$ be a continuous mapping of $M \cup C$ into $S_1$. Let the partial mapping $f_M$ be inessential. Then $f$ is inessential.

24.7. Let $M \subseteq P$, $a \in M$, $b \in M$, $C \subseteq P$. Let $C$ be a simple arc with end points $a, b$. Let $C \cap \bar{M} = \{a\} \cup \{b\}$. Let $a, b$ belong to the same quasicomponent of $M$. Let $g$ be a continuous mapping of $M$ into $S_1$. Then there exists an essential continuous mapping $f$ of $M \cup C$ into $S_1$ such that $f_M = g$.

24.8.* Complete the proof of theorem 24.2.18.
25. Unicoherence

25.1. A metric space $P$ is said to be unicoherent if [1] $P$ is connected, [2] if $P = A \cup B$ with closed connected summands, then $A \cap B$ is connected.

25.1.1. Let $P \neq \emptyset$ be a locally connected space. $P$ is unicoherent if and only if it has the following property: If $C \subset P$ is closed and connected and if $K$ is a component of $P - C$, then the set $B(K)$ is connected.

25.1.2. Let $P \neq \emptyset$ be a locally connected space. $P$ is unicoherent if and only if it has the following property: If $Q \subset P$ is an irreducible cut of $P$ between points $a, b$, then the set $Q$ is connected.

Proof: I. Let $P \neq \emptyset$ be a locally connected space. Let $U$ designate unicoherence, $V$ the property from theorem 25.1.1 and $W$ the property from theorem 25.1.2. Evidently it suffices to prove the three implications: $U \Rightarrow V, V \Rightarrow W, W \Rightarrow U$.

II. Let $U$ hold. Let $C \subset P$ be closed and connected. Let $K$ be a component of $P - C$. By 22.1.13, $P - K$ is connected. By 18.1.6 the set $K$ is connected. As $P = K \cup (P - K)$ and as $U$ holds, $K \cap (P - K) = K - K$ is also connected, since $P - K$ is closed by 22.1.4. By 10.3.2 and 22.1.4, $K - K = B(K)$. Thus, $V$ holds.

III. Let $V$ hold. Let $Q \subset P$ be an irreducible cut of $P$ between points $a, b$. By 22.1.10 there exist two distinct connected sets $G_1, G_2$ such that

$$a \in G_1, \quad b \in G_2, \quad G_1 \cup G_2 \subset P - Q, \quad B(G_1) = B(G_2) = Q.$$ 

The set $Q$ is closed by 10.3.1 (or by 18.5.4). By 22.1.9, $G_1, G_2$ are components of $P - Q$ so that $G_1 \cap G_2 = \emptyset$. The sets $G_1, G_2$ are open by 22.1.4, so that $\bar{G}_1 \cap G_2 = \emptyset$ by 10.2.6. The set $\bar{G}_1$ is closed and by 18.1.6 connected. The set $G_2$ is connected and $B(G_2) = B(G_1) \subset \bar{G}_1$, while $G_2 \subset P - \bar{G}_1$. Thus, by 22.1.9, $G_2$ is a component of $P - G_1$ so that, by $V$, $B(G_2) = Q$ is connected. Thus, $W$ holds.

IV. Let $W$ hold. If $P$ were not connected, we would have $P = A \cup B$ with non-void separated summands. For $a \in A, b \in B$ the set $\emptyset$ would be an irreducible cut between the points $a$ and $b$. This is impossible, since $W$ holds. Thus, $P$ is connected.

Let $P = A \cup B$ with closed connected summands. We have to prove that the closed set $A \cap B$ is connected. Let us assume the contrary. As $P$ is connected, we have $A \cap B \neq \emptyset$. Hence, $A \cap B = H \cup K$ with non-void separated summands. As $A \cap B$ is closed, $H$ and $K$ are also closed. Moreover, $H \cap K = \emptyset$. Choose $a \in H, b \in K$. Then the set $P - (A \cap B)$ separates the point $a$ from the point $b$ in $P$. By 22.1.12 there is an irreducible cut $S \subset P - (A \cap B)$ of $P$ between the points $a, b$. By $W$ the set $S$ is connected. Since $A, B$ are closed, $A - (A \cap B), B - (A \cap B)$ are evidently separated. On the other hand, $S \subset P - (A \cap B) = [A - (A \cap B)] \cup [B - (A \cap B)]$, so that, by 18.1.2, we have either $A \cap S = \emptyset$ or $B \cap S = \emptyset$. Since $S$ is an irreducible
VI. Mappings of a space onto the circle

Cut of $P$ between the points $a, b, S$ separates $a$ from $b$ in $P$, i.e. the set $P - S$ is not connected between the points $a, b$ so that (see 18.3.3) $M \cap S \neq \emptyset$ for every connected $M \subset P$ containing both the points $a, b$. On the other hand, $a \in H, b \in K, H \cup K = A \cap B$. Thus, each of the connected sets $A, B$ contains both points $a, b$. Hence, $A \cap S \neq \emptyset \neq B \cap S$, which is a contradiction.

25.2. Let $P$ be a connected space. Let every continuous mapping of $P$ into $S_1$ be inessential. Then $P$ is unicoherent.

Proof: Let us assume the contrary. Then there are closed connected sets $A, B$ such that $P = A \cup B$ and $A \cap B$ is not connected. Since $P$ is connected, $A \cap B \neq \emptyset$. Since $A \cap B \neq \emptyset$ is closed and not connected, there are disjoint closed sets $H \neq \emptyset$, $K \neq \emptyset$ with $A \cap B = H \cup K$.

Define a mapping $f$ of $P$ into $S_1$ as follows:*)

$$f(x) = \exp\left(\frac{i\pi \theta(x, H)}{\theta(x, H) + \theta(x, K)}\right) \quad \text{for} \quad x \in A,$$

$$f(x) = \exp\left(-i\pi \frac{\theta(x, H)}{\theta(x, H) + \theta(x, K)}\right) \quad \text{for} \quad x \in B.$$

For $x \in A \cap B = H \cup K$ we have formally two definitions of $f(x)$. Both of them, however, give $f(x) = 1$ for $x \in H$ and $f(x) = -1$ for $x \in K$.

The mapping $f$ is evidently continuous. Thus, $f$ is inessential, i.e., there exists a continuous mapping $\varphi$ of $P$ into $E_1$ such that $e^{i\varphi(x)} = f(x)$ for every $x \in P$. We have

$$\exp\left(\frac{i\pi \theta(x, H)}{\theta(x, H) + \theta(x, K)}\right) = e^{i\varphi(x)} \quad \text{for} \quad x \in A,$$

$$\exp\left(-i\pi \frac{\theta(x, H)}{\theta(x, H) + \theta(x, K)}\right) = e^{i\varphi(x)} \quad \text{for} \quad x \in B,$$

and the sets $A, B$ are connected. Hence, by 24.2.11 there are integers $m, n$ such that

$$\varphi(x) = \pi \frac{\theta(x, H)}{\theta(x, H) + \theta(x, K)} + 2m\pi \quad \text{for} \quad x \in A,$$

$$\varphi(x) = -\pi \frac{\theta(x, H)}{\theta(x, H) + \theta(x, K)} + 2n\pi \quad \text{for} \quad x \in B.$$  

Let us choose $a \in H, b \in K$. We have $a \in A \cap B, b \in A \cap B$, so that

$$\varphi(a) = 2m\pi = 2n\pi,$$

$$\varphi(b) = \pi + 2m\pi = -\pi + 2n\pi,$$

which is a contradiction.

*) Since $H, K$ are closed and since $H \neq \emptyset \neq K, H \cap K = \emptyset$, we have $\theta(x, H) + \theta(x, K) > 0$ for every $x \in P$. 
25.2.2. Let $P$ be a locally compact unicoherent space. Then every continuous mapping $f$ of $P$ into $S_1$ is inessential.

**Proof:** I. Put

\[
\text{Real}\,(a + bi) = a, \quad \text{Im}\,(a + bi) = b.
\]

Define point sets $Q_1, Q_2, Q_3, Q_4$ as follows. If $x \in P$, then

\[
\begin{align*}
x & \in Q_1 \iff \text{Real}\,f(x) > 0, \\
x & \in Q_2 \iff \text{Real}\,f(x) < 0, \\
x & \in Q_3 \iff \text{Im}\,f(x) > 0, \\
x & \in Q_4 \iff \text{Im}\,f(x) < 0.
\end{align*}
\]

We have $P = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ and, by 9.2, $Q_\lambda (\lambda = 1, 2, 3, 4)$ are open sets.

II. For $1 \leq \lambda \leq 4$ choose $M_\lambda \subset Q_\lambda$ such that $M_\lambda$ contains exactly one point of every component of $Q_\lambda$. It is easy to prove (see ex. 25.5) that (with the exception of the trivial case with a one-point $P$) we may assume that the sets $M_\lambda (\lambda = 1, 2, 3, 4)$ are disjoint. For every $x \in M_\lambda$ let $V(x)$ be the component of $Q_\lambda$ containing the point $x$. The sets $V(x)$ are connected and, by 22.1.4, open. Moreover

\[
\bigcup_{x \in M_\lambda} V(x) = Q_\lambda
\]

with disjoint summands.

Put $M = M_1 \cup M_2 \cup M_3 \cup M_4$.

III. Let $x' \in M, x'' \in M, x' \neq x''$, $V(x') \cap V(x'') \neq \emptyset$. Evidently $x' \in M_\lambda, x'' \in M_\mu$ where the couple $(\lambda, \mu)$ is one of the following eight ones

\[
(1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2).
\]

IV. Let $\{x_r\}_{r=1}^m$ be a finite sequence such that [1] $x_r \in M$ for $1 \leq r \leq m$, [2] if $1 \leq r < s \leq m$, then $V(x_r) \cap V(x_s) \neq \emptyset$ if and only if either $s = r + 1$ or $r = 1, s = m$. Then there is an index $\lambda (1 \leq \lambda \leq 4)$ such that $x_r \in M_\lambda$ for no $r (1 \leq r \leq m)$.

Let us assume the contrary, so that $m \geq 4$. Put $x_0 = x_m, x_{m+1} = x_1$. It follows easily by III that there exists an index $s (1 \leq s \leq m)$ such that

\[
x_{s-1} \in M_\lambda, \quad x_s \in M_\mu, \quad x_{s+1} \in M_\nu,
\]

where the triple $(\lambda, \mu, \nu)$ is one of the following

\[
(3, 1, 4), (4, 1, 3), (3, 2, 4), (4, 2, 3).
\]

All four cases lead to a contradiction in the same way. Hence, it suffices to treat, one of them. E.g. let

\[
x_{s-1} \in M_3, \quad x_s \in M_1, \quad x_{s+1} \in M_4.
\]

By the assumption there is an index $t (1 \leq t \leq m)$ such that $x_t \in M_2$.

We have $x_s \in V(x_s)$. Since $x_t \in M_1, y \in V(x_s)$ implies $\text{Real}\,f(y) > 0$, so that $y \in \overline{V(x_s)}$ implies $\text{Real}\,f(y) \geq 0$, while $x_t \in M_2$, so that $\text{Real}\,f(x_t) < 0$. Thus, $x_t \in \overline{V(x_s)}$.
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\[ W_1 = \bigcup_r V(x_r) \quad (1 \leq r \leq m, \quad s - 1 - r \neq m + s - 1), \]

\[ W_2 = \bigcup_r V(x_r) \quad (1 \leq r \leq m, \quad s + 1 + r \neq s + 1 - m). \]

Among the summands of the first union are the sets \( V(x_s), V(x_{s+1}) \); for every other summand \( V(x_r) \) of this union we have \( V(x_r) \cap V(x_s) = 0 \) and hence (see 10.2.6) \( V(x_s) \cap \overline{V(x_s)} = 0 \). On the other hand, \( S \subset B[V(x_s)] \subset P - Q_1 \), i.e., \( \text{Real} f(y) \geq 0 \) for \( y \in S \). Hence, \( \text{Real} f(y) = 0 \) for \( y \in S \), i.e. \( f(y) = \pm i \) for \( y \in S \). As \( x_{s-1} \in M_3, x_{s+1} \in M_4 \), we have \( \text{Im} f(y) > 0 \) for \( y \in V(x_{s-1}) \), \( \text{Im} f(y) < 0 \) for \( y \in V(x_{s+1}) \). Thus [see (1)], \( f(S) = (i) + (-i) \), so that \( f(S) \) is not connected. This is a contradiction (see 18.1.10).

V. By 24.1.2 there exists a homeomorphic mapping \( v \) of \( S_1 - (-1) \) onto the interval \( E[\pi < t < \pi] \) such that \( e^{iv(z)} = z \) for every \( z \in S_1 - (-1) \). Evidently, \( v(z^{-1}) = -v(z) \) for every \( z \in S_1 - (-1) \).

If \( x \in M, y' \in V(x), y'' \in V(x) \), we have obviously \( f(y') + f(y'') \neq 0 \), so that there exists a number

\[ v \left( \frac{f(y'')}{f(y')} \right). \]

VI. Let \( \{x_r\}_1^m, \{y_r\}_1^m \) be finite sequences \( (m \geq 2) \) such that \( [1] x_r \in M \) for \( 1 \leq r \leq m \), \( [2] y_r \in V(x_r) \) for \( 1 \leq r \leq m \), \( y_{r+1} \in V(x_r) \) for \( 1 \leq r \leq m - 1 \), \( y_1 \in V(x_m) \).

Then we have

\[ \sum_{r=1}^{m-1} v \left( \frac{f(y_{r+1})}{f(y_r)} \right) = v \left( \frac{f(y_m)}{f(y_1)} \right). \]

This statement is evident for \( m = 2 \). Hence, let \( m \geq 3 \). It suffices to prove it under the assumption (denote it by \( H \)) that equations analogous to (1) in which \( m \) is replaced by a number less than \( m \), are valid. Consider two cases.

First case. There exist indices \( h, k \) such that \( V(x_h) \cap V(x_k) \neq 0 \), \( 1 \leq h < k \leq m \), and neither \( k = h + 1 \) nor \( (h, k) = (1, m) \). Obviously \( m \geq 4 \). Choose a point \( z \in \)
25. Unicoherence

\[ e \in V(x_k) \cap V(x_i). \text{ Then we obtain, by assumption } H, \text{ the following four equations} \]

\[
\sum_{r=1}^{h-1} v \left( \frac{f(y_{r+1})}{f(y_r)} \right) + v \left( \frac{f(z)}{f(y_h)} \right) + v \left( \frac{f(y_{k+1})}{f(z)} \right) + \sum_{r=k+1}^{m-1} v \left( \frac{f(y_{r+1})}{f(y_r)} \right) = v \left( \frac{f(y_m)}{f(y_{k+1})} \right),
\]

\[
\sum_{r=k+1}^{k-1} v \left( \frac{f(y_{r+1})}{f(y_r)} \right) + v \left( \frac{f(z)}{f(y_h)} \right) + v \left( \frac{f(y_{k+1})}{f(z)} \right) = v \left( \frac{f(y_{k+1})}{f(y_{h+1})} \right),
\]

\[
v \left( \frac{f(y_h)}{f(z)} \right) + v \left( \frac{f(y_{h+1})}{f(y_h)} \right) = v \left( \frac{f(y_{h+1})}{f(z)} \right),
\]

\[
v \left( \frac{f(y_k)}{f(z)} \right) + v \left( \frac{f(y_{k+1})}{f(y_k)} \right) = v \left( \frac{f(y_{k+1})}{f(z)} \right).
\]

We obtain (1) by adding them, since \( v(u^{-1}) = -v(u) \) for every \( u \in S_1 \).

**Second case.** If \( 1 \leq r < s \leq m \), \( V(x_r) \cap V(x_s) \neq \emptyset \), we have either \( s = r + 1 \), or \((r, s) = (1, m)\). By IV there is an index \( \lambda \) \((1 \leq \lambda \leq 4)\) such that \( x_r \in M_\lambda \) for no \( r \) \((1 \leq r \leq m)\). Obviously

\[ S_1 = f(\bigcup_{r=1}^{m} V(x_r)) \neq \emptyset, \]

so that by 24.2.7 there exists a continuous mapping \( \varphi \) of \( W = \bigcup_{r=1}^{m} V(x_r) \) into \( E_1 \) such that \( e^{i\varphi(y)} = f(y) \) for every \( y \in W \). If \( e^{ip_r} = f(y_r) \) \((1 \leq r \leq m)\), then

\[ e^{i\varphi(y)} = \exp \left\{ i \left[ \beta_r + v \left( \frac{f(y)}{f(y_r)} \right) \right] \right\} \text{ for } y \in V(x_r), \]

so that, by 24.2.11, there are integers \( k_r \) \((1 \leq r \leq m)\) such that

\[ \varphi(y) = \beta_r + v \left( \frac{f(y)}{f(y_r)} \right) + 2k_r \pi \text{ for } y \in V(x_r). \]

Hence

\[ v \left( \frac{f(y_{r+1})}{f(y_r)} \right) = \varphi(y_{r+1}) - \varphi(y_r) \quad (1 \leq r \leq m - 1), \]

\[ v \left( \frac{f(y_m)}{f(y_1)} \right) = \varphi(y_m) - \varphi(y_1), \]

which yields (1).

VII. Choose a fixed \( a \in P \) and \( \alpha \in E_1 \) such that \( e^{i\alpha} = f(a) \). For every \( y \in P \) there are, by 18.4.2, finite sequences \( \{x_r\}^m_1, \{y_r\}^m_1 \) such that [1] \( y_0 = a, y_m = y, [2] x_r \in M \) for \( 1 \leq r \leq m, [3] y_{r-1} \in V(x_r), y_r \in V(x_r) \) for \( 1 \leq r \leq m \). Put (see V)

\[ \psi(y) = \alpha + \sum_{r=1}^{m} v \left( \frac{f(y_r)}{f(y_{r-1})} \right), \quad (2) \]

We shall show later that the number \( \psi(y) \) is uniquely determined for every \( y \in P \). Thus, \( \psi \) is a mapping of \( P \) into \( E_1 \). Evidently \( e^{i\psi(y)} = f(y) \) for every \( y \in P \). We have
to prove that the mapping $\psi$ is continuous. For a given $y$ and given sequences $\{x_r\}_m^n$, $\{y_r\}_0^n$, $V(x_m)$ is a neighborhood of $y$. Replacing the point $y$ by a point $y' \in V(x_m)$, we may preserve the points $x_r$ ($1 \leq r \leq m$), $y_r$ ($0 \leq r \leq m - 1$) and take $y_m = y'$ instead of $y_m = y$. Formula (2) yields

$$\psi(y') - \psi(y) = v\left(\frac{f(y')}{f(y_m-1)}\right) - v\left(\frac{f(y)}{f(y_m-1)}\right) \quad \text{for} \quad y' \in V(x_m).$$

As $V(x_m)$ is a neighborhood of the point $y$, $\psi$ is continuous at the point $y$.

It remains to prove that the number $\psi(y)$ is, for a given $y \in P$, uniquely determined. Replace the sequences $\{x_r\}_m^n$, $\{y_r\}_0^n$ by other similar sequences $\{x'_r\}_m^n$, $\{y'_r\}_0^n$. We have to prove that

$$\sum_{r=1}^{m} v\left(\frac{f(y_r)}{f(y_{r-1})}\right) = \sum_{r=1}^{n} v\left(\frac{f(y'_r)}{f(y'_{r-1})}\right) = -\sum_{r=1}^{n} v\left(\frac{f(y'_{r-1})}{f(y'_r)}\right).$$

Put $x_{m+r} = x'_{n-r+1}$ for $1 \leq r \leq n$, $y_{m+r} = y'_{n-r}$, for $1 \leq r \leq n$. We have then [1] $y_0 = y_{m+n} = a$; [2] $x_r \in M$ for $1 \leq r \leq m + n$, [3] $y_{r-1} \in V(x_r)$, $y_r \in V(x_r)$ for $1 \leq r \leq m + n$ and we have to prove that

$$\sum_{r=1}^{m+n} v\left(\frac{f(y_r)}{f(y_{r-1})}\right) = 0 = v\left(\frac{f(a)}{f(a)}\right) = v\left(\frac{f(y_{m+n})}{f(y_0)}\right).$$

This follows by VI.

25.2.3. The euclidean space $\mathbb{E}_m$ ($m = 1, 2, 3, \ldots$) is unicoherent.

This follows by 19.2.4, 24.5.3 and 25.2.1.

25.2.4. The spherical spaces $\mathbf{S}_0$, $\mathbf{S}_1$ are not unicoherent. The spherical spaces $\mathbf{S}_m$ ($m = 2, 3, 4, \ldots$) are unicoherent.

Proof: I. $\mathbf{S}_0$ is not connected, hence, it is not unicoherent. $\mathbf{S}_1$ is a simple loop, hence (see 20.1.1 and 21.1.2), $\mathbf{S}_1$ is a union of two continua, whose intersection is not connected, so that $\mathbf{S}_1$ is not unicoherent.

II. Let $m \geq 2$. The space $\mathbf{S}_m$ is connected by 19.2.5. Thus, $\mathbf{S}_m$ is unicoherent by 24.5.4 and 25.2.1.

25.2.5. Let $P$, $Q$ be locally connected unicoherent spaces. Then the space $P \times Q$ is unicoherent.

Proof: The spaces $P$, $Q$ are connected, so that $P \times Q$ is connected by 18.1.13. Hence, by 25.2.1, it suffices to prove that every continuous mapping of $P \times Q$ into $\mathbf{S}_1$ is inessential. This follows by 24.4.2 and 25.2.2.
25. Unicoherence

Exercises

The spaces $P_1, P_2, \ldots, P_9$ were defined in exercises to § 19.

25.1. The spaces $P_2, P_3, P_4, P_5, P_7, P_8$ are unicoherent.

25.2. The spaces $P_1, P_6, P_9$ are not unicoherent.

25.3. Let $P \subseteq \mathbf{E}_2$ be the space consisting of all $(x, y)$ such that $x^2 + y^2 = 1$ and of all $(x, y)$ of the form $x = (1 + t^{-1}) \cos t, \ y = (1 + t^{-1}) \sin t, \ t > 1$. Then $P$ is a unicoherent space.

25.4. We cannot omit in theorem 25.2.2 the assumption that $P$ is a locally connected space.

25.5.* Prove that the sets $M_\lambda (\lambda = 1, 2, 3, 4)$ in part II of the proof of theorem 25.2.2 may be found disjoint.