Chapter VI: Mappings of a space onto the circle

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Chapter VI

MAPPINGS OF A SPACE ONTO THE CIRCLE

§ 24. Inessential mappings onto the circle

24.1. In this and in the following chapter we shall identify couples (x, y) of real numbers with complex numbers $x + iy$, so that $\mathbf{E}_2$ is the set of all the complex numbers nad $\mathbf{S}_1$ (see 17.10) is the set of all complex numbers $x + iy$ with absolute value

$$|x + iy| = +\sqrt{(x^2 + y^2)}$$

equal to one. The set $\mathbf{E}_2$ will be termed the *plane*, the set $\mathbf{S}_1$ will be termed the *circle*. Evidently

$$\varrho(a, b) = |a - b| \quad \text{for} \quad a \in \mathbf{E}_2,\ b \in \mathbf{E}_2\,.$$

As is well known, for any $t \in \mathbf{E}_1$,

$$e^{ti} = \cos t + i \sin t \in \mathbf{S}_1\,.$$

The following two theorems are well known:

24.1.1. *Put* $f(t) = e^{ti}$ *for* $t \in \mathbf{E}_1$. *Then* f *is a continuous mapping of* $\mathbf{E}_1$ *onto* $\mathbf{S}_1$.

24.1.2. *Let* $\alpha \in \mathbf{E}_1$, $J = \underset{t}{\mathrm{E}}[\alpha < t < \alpha + 2\pi]$. *Put* $f(t) = e^{ti}$ *for* $t \in J$. *Then* f *is a homeomorphic mapping of* J *onto* $\mathbf{S}_1 - (e^{i\alpha})$.

24.2. Let P be a metric space. The following two theorems are easy to prove:

24.2.1. *Let* f *and* g *be continuous mappings of* P *into* $\mathbf{S}_1$. *Then* $f \cdot g$ *is a continuous mapping of* P *into* $\mathbf{S}_1$.

24.2.2. *Let* f *be a continuous mapping of* P *into* $\mathbf{S}_1$. *Then* $1/f$ *is a continuous mapping of* P *into* $\mathbf{S}_1$.

It follows easily by 24.1.1:

24.2.3. *Let* φ *be a continuous mapping of* P *into* $\mathbf{E}_1$. *Put* $f(x) = e^{i\varphi(x)}$ *for every* $x \in P$. *Then* f *is a continuous mapping of* P *into* $\mathbf{S}_1$.

Let f be a continuous mapping of P into $\mathbf{S}_1$. We say that f is *inessential*, if there exists a continuous mapping φ of P into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$. A mapping f is said to be *essential*, if it is not inessential.

The following three theorems are evident.

24.2.4. *Let f and g be inessential continuous mappings of P into $\mathbf{S}_1$. Then $f.g$ is an inessential continuous mapping of P into $\mathbf{S}_1$.*

24.2.5. *Let f be an inessential continuous mapping of P into $\mathbf{S}_1$. Then $1/f$ is an inessential continuous mapping of P into $\mathbf{S}_1$.*

24.2.6. *Let $Q \subset P$. Let f be an inessential continuous mapping of P into $\mathbf{S}_1$. Then the partial mapping f_Q is also inessential.*

24.2.7. *Let f be a continuous mapping of P into $\mathbf{S}_1$. If $\mathbf{S}_1 - f(P) \neq \emptyset$, then f is inessential.*

Proof: There is an $\alpha \in \mathbf{E}_1$ with $e^{i\alpha} \in \mathbf{S}_1 - f(P)$. By 24.1.2 there exists a homeomorphic mapping h of $\mathbf{S}_1 - (e^{i\alpha})$ onto the interval $\underset{t}{\mathrm{E}}[\alpha < t < \alpha + 2\pi]$ such that $e^{ih(z)} = z$ for every $z \in \mathbf{S}_1 - (e^{i\alpha})$. Put $\varphi(x) = h[f(x)]$ for $x \in P$. Then φ is a continuous mapping of P into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$.

24.2.8. *Let f and g be continuous mappings of P into $\mathbf{S}_1$. Let f be inessential. Let $|f(x) - g(x)| < 2$ for every $x \in P$. Then g is also inessential.*

Proof: Obviously $g(x)/f(x) \neq -1$ for any $x \in P$. Thus, the mapping $g = f.(g/f)$ is inessential by 24.2.4 and 24.2.7.

24.2.9. *Let $0 < \omega < 2\pi$. Let f be a continuous mapping of P into $\mathbf{S}_1$. Let φ be a mapping of P into $\mathbf{E}_1$. Let $f(x) = e^{i\varphi(x)}$ for every $x \in P$. Let φ not be continuous in a point $a \in P$. Then there is a sequence $\{x_n\}$ in P such that $\lim x_n = a$, $|\varphi(x_n) - \varphi(a)| > \omega$ for every n.*

Proof: Denote by M the set of all $x \in P$ such that $|\varphi(x) - \varphi(a)| > \omega$. By 8.2.1, we have to prove that $a \in \overline{M}$. Let us assume the contrary. Then $U = P - \overline{M}$ is a neighborhood of a such that $x \in U$ implies $|\varphi(x) - \varphi(a)| \leqq \omega$. Evidently there is a neighborhood V of a such that $\mathbf{S}_1 - f(V) \neq \emptyset$. By 24.2.7 there is a continuous mapping ψ of V into $\mathbf{E}_1$ such that, for every $x \in V$

$$e^{i\psi(x)} = f(x) = e^{i\varphi(x)}.$$

In particular $e^{i\psi(a)} = e^{i\varphi(a)}$, so that there is an integer k with $\varphi(a) = \psi(a) + 2\pi k$. Since $\omega < 2\pi$ and since ψ is continuous, there is obviously a neighborhood $W \subset U$ of a such that $x \in W$ implies $|\psi(x) - \psi(a)| < 2\pi - \omega$. For $x \in U \cap W$ we have $|\varphi(x) - \psi(x) - 2k\pi| = |[\varphi(x) - \varphi(a)] - [\psi(x) - \psi(a)]| \leqq |\varphi(x) - \varphi(a)| + |\psi(x) - \psi(a)| < 2\pi$. However, the number

$$\frac{\varphi(x) - \psi(x) - 2k\pi}{2\pi} \tag{1}$$

is an integer, since

$$e^{i\varphi(x)} = e^{i\psi(x)} = e^{i(\psi(x)+2k\pi)}.$$

Thus, (1) is an integer and its absolute value is less than 1, hence $\varphi(x) = \psi(x) + 2k\pi$ for every $x \in U \cap W$. On the other hand, $U \cap W$ is a neighborhood of a and ψ is continuous. Thus, φ is continuous in a. This is a contradiction.

24.2.10. *Let f be a continuous mapping of P into $\mathbf{S}_1$. Let there exist an integer $k \neq 0$ such that the mapping f^k is inessential. Then f is also inessential.*

Proof: There is a continuous mapping φ of P into $\mathbf{E}_1$ with

$$[f(x)]^k = e^{i\varphi(x)}$$

for every $x \in P$. For $x \in P$ put

$$g(x) = \exp\,[i\varphi(x)/k]\,.$$

Then g is an inessential continuous mapping of P into $\mathbf{S}_1$. For every $x \in P$ we have $[f(x)/g(x)]^k = 1$, so that f/g is inessential by 24.2.7. Thus, the mapping

$$f = (f/g) \cdot g$$

is inessential by 24.2.4.

24.2.11. *Let φ_1 and φ_2 be continuous mappings of a connected space P into $\mathbf{E}_1$. Let*

$$e^{i\varphi_1(x)} = e^{i\varphi_2(x)}$$

for every $x \in P$. Then there is an integer k such that

$$\varphi_2(x) = \varphi_1(x) + 2k\pi$$

for every $x \in P$.

Proof: $\varphi = (2\pi)^{-1} \cdot (\varphi_2 - \varphi_1)$ is a continuous mapping of P into $\mathbf{E}_1$ and the set $\varphi(P)$ consists of *integers*, so that $\varphi(P)$ is not an interval. Hence, $\varphi(P)$ is a one-point set by 18.1.10 and 19.2.2.

24.2.12. *Let $K = 1, 2, 3, \ldots$. Let $P = A \cup B$ and let A, B be either both closed or both open. Let $A \cap B$ have at most k components. Let f_λ $(1 \leqq \lambda \leqq k)$ be continuous mappings of P into $\mathbf{S}_1$. Let all the partial mappings*

$$(f_\lambda)_A,\ (f_\lambda)_B \quad (1 \leqq \lambda \leqq k)$$

be inessential. Then there are integers n_λ $(1 \leqq \lambda \leqq k)$ which are not all equal to zero such that the mapping

$$\prod_{\lambda=1}^{k} (f_\lambda)^{n_\lambda}$$

is inessential.

Proof: Let C_μ $(1 \leqq \mu \leqq h)$ be all the components of the set $A \cap B$; thus, $0 \leqq h \leqq k$.

There are continuous mappings φ_λ $(1 \leqq \lambda \leqq k)$ of A into $\mathbf{E}_1$ and continuous mappings ψ_λ $(1 \leqq \lambda \leqq k)$ of B into $\mathbf{E}_1$ such that

$$f_\lambda(x) = e^{i\varphi_\lambda(x)} \quad \text{for} \quad x \in A\,,$$
$$f_\lambda(x) = e^{i\psi_\lambda(x)} \quad \text{for} \quad x \in B\,.$$

By 24.2.11 there are integers $k_{\mu\lambda}$ $(1 \leqq \mu \leqq h,\ 1 \leqq \lambda \leqq k)$ such that

$$\psi_\lambda(x) = \varphi_\lambda(x) + 2\pi k_{\mu\lambda} \quad \text{for} \quad x \in C_\mu\,.$$

Let us determine integers n, n_λ $(1 \leqq \lambda \leqq k)$ satisfying the equations

$$\sum_{\lambda=1}^{k} k_{\mu\lambda} n_\lambda = n. \qquad (1 \leqq \mu \leqq h) \tag{2}$$

Since the number of the equations is less than the number of unknowns and since the coefficients are integers, there exists a solution of (2) such that we do not have $n_1 = \ldots = n_k = 0$.

Put $f = \prod_{\lambda=1}^{k} (f_\lambda)^{n_\lambda}$, so that f is a continuous mapping of P into $\mathbf{S}_1$. We have to prove that f is inessential.

Equations (2) yield that $x \in A \cap B$ implies $\sum_{\lambda=1}^{k} n_\lambda \psi_\lambda(x) = \sum_{\lambda=1}^{k} n_\lambda \varphi_\lambda(x) + 2\pi n$. Thus, we may define a mapping χ of P into $\mathbf{E}_1$ by

$$\chi(x) = \sum_{\lambda=1}^{k} n_\lambda \varphi_\lambda(x) + 2\pi n \quad \text{for} \quad x \in A,$$

$$\chi(x) = \sum_{\lambda=1}^{k} n_\lambda \psi_\lambda(x) \qquad\qquad \text{for} \quad x \in B.$$

Evidently $f(x) = e^{i\chi(x)}$ for every $x \in P$, so that it suffices to prove that χ is continuous. This follows easily from the continuity of the partial mappings χ_A, χ_B (see ex. 9.5).

24.2.13. *Let $P = A \cup B$ and let A, B be either both closed or both open. Let $A \cap B$ be either void or connected. Let f be a continuous mapping of P into $\mathbf{S}_1$. Let both partial mappings f_A, f_B be inessential. Then also f is inessential.*

This follows immediately from 24.2.10 and 24.2.12.*)

24.2.14. *Let $P = \bigcup_{n=1}^{\infty} A_n$. Let $A_n \subset A_{n+1}$ $(n = 1, 2, 3, \ldots)$. Let the sets A_n be connected. For every $x \in P$ let there be an index n such that x is an interior point* (*see* 8.6) *of A_n.*

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*) 24.2.13 is a particular case of theorem 24.2.12. If the proof is carried out for this particular case, we see easily that we do not need theorem 24.2.10.

Let f be a continuous mapping of P into $\mathbf{S}_1$. Let the partial mappings f_{A_n} be inessential ($n = 1, 2, 3, \ldots$). Then f is inessential.

Proof: Choose an $a \in A_1$, so that $a \in A_n$ for every n. For $n = 1, 2, 3, \ldots$ there is a continuous mapping ψ_n of A_n into $\mathbf{E}_1$ such that $f(x) = e^{i\psi_n(x)}$ for every $x \in A_n$. If $m < n$, then, by 24.2.11, there exists an integer k_{mn} such that $x \in A_m$ implies $\psi_n(x) = \psi_m(x) + 2\pi k_{mn}$. Put $h_n = k_{1n}$. We have

$$\psi_n(a) = \psi_m(a) + 2\pi k_{mn},$$
$$\psi_n(a) = \psi_1(a) + 2\pi h_n,$$
$$\psi_m(a) = \psi_1(a) + 2\pi h_m,$$

hence, $k_{mn} = h_n - h_m$. Thus, we may define a mapping φ of P into $\mathbf{E}_1$ by

$$\varphi(x) = \psi_n(x) - 2\pi h_n \quad \text{for} \quad x \in A_n.$$

Evidently $f(x) = e^{i\varphi(x)}$ for every $x \in P$. Since for every $x \in P$ there is an index n such that x is an interior point of A_n and since the mappings ψ_n are continuous, φ is also continuous. Thus, f is inessential.

24.2.15. *Let $Q \subset P$. Let either $T = \mathbf{E}_1$ or $T = \mathbf{S}_1$. Let $\varepsilon > 0$. Let φ be a continuous mapping of Q into T. Then there is a neighborhood G of Q and a continuous mapping ψ of G into T such that $|\psi(x) - \varphi(x)| \leqq \varepsilon$ for every $x \in Q$.*

Proof: I. First, let $T = \mathbf{E}_1$. We may assume that $Q \neq \emptyset$.

II. Let Γ be the set of all $x \in \bar{Q}$ such that there is a number $\eta_x > 0$ with

$$(a) \cup (b) \subset Q \cap \Omega(x, \eta_x) \Rightarrow |\varphi(a) - \varphi(b)| < \tfrac{1}{2}\varepsilon.$$

As φ is continuous, we have obviously

$$Q \subset \Gamma \subset \bar{Q}.$$

Moreover, it is easy to prove that

$$x \in \Gamma \Rightarrow \bar{Q} \cap \Omega(x, \eta_x) \subset \Gamma$$

so that Γ is open in $\bar{Q}$.

III. For $n = 0, \pm 1, \pm 2, \ldots$ denote by A_n the set of all $x \in Q$ with

$$n\varepsilon \leqq \varphi(x) \leqq (n + 1)\,\varepsilon,$$

so that

$$Q = \bigcup_{n=-\infty}^{\infty} A_n.$$

IV. We have

$$\Gamma \subset \bigcup_{n=-\infty}^{\infty} \bar{A}_n.$$

To prove this, we choose an $x \in \Gamma$. Since $\Gamma \subset \overline{Q}$, we have $0 = \varrho(x, Q) < \eta_x$, so that there is an $a \in Q$ with $\varrho(a, x) < \eta_x$. Choose such an a and determine an integer m with $|\varphi(a) - m\varepsilon| \leqq \frac{1}{2}\varepsilon$. If $0 < \delta \leqq \eta_x$, then $0 = \varrho(x, Q) < \delta$, so that there is a point $b \in Q$ with $\varrho(b, x) < \delta \leqq \eta_x$. By II, $|\varphi(a) - \varphi(b)| < \frac{1}{2}\varepsilon$, so that $|\varphi(b) - m\varepsilon| < \varepsilon$, hence $b \in A_{m-1} \cup A_m$. Thus, $\varrho(x, A_{m-1} \cup A_m) < \delta$ for every $\delta > 0$, $\delta \leqq \eta_x$, so that $\varrho(x, A_{m-1} \cup A_m) = 0$, hence $x \in \overline{A_{m-1} \cup A_m} = \overline{A}_{m-1} \cup \overline{A}_m$.

V. Further, we prove that

$$x \in \Gamma \cap \overline{A}_n,\ y \in \Gamma \cap \overline{A}_m,\ \ \varrho(x, y) < \eta_x \Rightarrow |m - n| \leqq 1\,.$$

(In particular, $x \in \Gamma \cap \overline{A}_n \cap \overline{A}_m \Rightarrow |m - n| \leqq 1$.)

Since $x \in \Gamma \cap \overline{A}_n$, there exists a point $a \in A_n \cap \Omega(x, \eta_x)$. Choose a $\delta > 0$ with $\delta < \eta_y$, $\varrho(x, y) + \delta < \eta_x$. Since $y \in \Gamma \cap \overline{A}_m$, there exists a point $b \in A_m \cap \Omega(y, \delta)$. We have $\varrho(b, x) \leqq \varrho(x, y) + \varrho(b, y) \leqq \varrho(x, y) + \delta < \eta_x$. Hence, $(a) \cup (b) \subset Q \cap \Omega(x, \eta_x)$, so that $|\varphi(a) - \varphi(b)| < \frac{1}{2}\varepsilon$. Since $a \in A_n$, $b \in A_m$, we have $n\varepsilon \leqq \varphi(a) \leqq (n + 1)\varepsilon$, $m\varepsilon \leqq \varphi(b) \leqq (m + 1)\varepsilon$. Since $|\varphi(a) - \varphi(b)| < \varepsilon$, we have $|m - n| \leqq 1$.

VI. Let us define a mapping χ of Γ into $\mathbf{E}_1$ as follows:
If $x \in \Gamma \cap \overline{A}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ then*)

$$\chi(x) = n\varepsilon + \varepsilon \frac{\varrho(x, A_{n-1})}{\varrho(x, A_{n-1}) + \varrho(x, A_{n+1})}$$

(the ratio on the right-hand side is always defined, since $\varrho(x, A_{n-1}) + \varrho(x, A_{n+1}) = 0$ implies $x \in \overline{A}_{n-1} \cap \overline{A}_{n+1}$, which is, for $x \in \Gamma$, impossible by V). By IV, the number $\chi(x)$ is defined for any $x \in \Gamma$ at least in one way. If $x \in \Gamma \cap \overline{A}_m$, $x \in \Gamma \cap \overline{A}_n$, $m \neq n$, then, by V, $m = n \pm 1$. Then we obtain two formally different definitions, which, however, both lead to the same value, namely $\chi(x) = n\varepsilon$ provided $m = n - 1$, $\chi(x) = (n + 1)\varepsilon$ provided $m = n + 1$.

VII. $x \in Q \Rightarrow |\chi(x) - \varphi(x)| \leqq \varepsilon$.

In fact, there is an index n with $x \in A_n \subset \Gamma \cap \overline{A}_n$. By III, $n\varepsilon \leqq \varphi(x) \leqq (n + 1)\varepsilon$, by VI, $n\varepsilon \leqq \chi(x) \leqq (n + 1)\varepsilon$, hence $|\chi(x) - \varphi(x)| \leqq \varepsilon$.

VIII. The mapping χ is continuous. Let $x_r \in \Gamma$ $(r = 1, 2, 3, \ldots)$, $x \in \Gamma$, $\lim x_r = x$. We have to prove that $\lim \chi(x_r) = \chi(x)$. Let us assume the contrary. Then there is a number $\delta > 0$ and a subsequence $\{y_r\}$ of $\{x_r\}$ such that $|\chi(y_r) - \chi(x)| > \delta$ for every r. By IV there is an index n such that $x \in \Gamma \cap \overline{A}_n$. There is an index p such that $r > p$ implies $\varrho(x, y_r) < \eta_x$.

By V, $y_r \in \Gamma \cap (\overline{A}_{n-1} \cup \overline{A}_n \cup \overline{A}_{n+1})$ for every $r > p$. If $y_r \in \Gamma \cap \overline{A}_{n-1}$ for infinitely many indices r, then $(\varrho(x, \overline{A}_{n-1}) \leqq \varrho(x, y_r) \to 0$, hence) $\varrho(x, \overline{A}_{n-1}) = 0$, i.e. $x \in \Gamma \cap A_{n-1}$. Similarly, $x \in \Gamma \cap \overline{A}_{n+1}$ provided there exist infinitely many

*) We arrange to set $\varrho(x, \emptyset) = 1$ for every point x.

indices r with $y_r \in \Gamma \cap \bar{A}_{n+1}$. Thus, there exists an index m ($m = n$ or $m = n - 1$ or $m = n + 1$) such that $x \in \Gamma \cap \bar{A}_m$ and $\{y_r\}$ contains a subsequence $\{z_r\}$ such that $z_r \in \Gamma \cap \bar{A}_m$ for every r. On the other hand, $z_r \to x$ and the partial mapping $\chi_{\Gamma \cap \bar{A}_m}$ is continuous (see ex. 9.10). Hence, $\chi(z_r) \to \chi(x)$. This is a contradiction, since $|\chi(z_r) - \chi(x)| > \delta > 0$ for every r.

IX. The set $\bar{Q} - \Gamma$ is closed by II and 8.7.3, so that the set $G = P - (\bar{Q} - \Gamma)$ is open. Moreover, $\Gamma = \bar{Q} \cap G$, so that Γ is closed in G by 8.7.2. Hence, by VIII and 14.8.3, there exists a continuous mapping ψ of G into $\mathbf{E}_1$ such that $\psi(x) = \chi(x)$ for $x \in \Gamma$. As $Q \subset \Gamma$, $x \in Q$ implies $|\psi(x) - \varphi(x)| \leqq \varepsilon$ by VII.

X. The proof is finished for $T = \mathbf{E}_1$. Now, let us turn to the case of $T = \mathbf{S}_1$. We may assume that $\varepsilon < 1$. For $x \in Q$ put $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$. Then φ_1, φ_2 are continuous mappings of Q into $\mathbf{E}_1$, and, for every $x \in Q$ we have $[\varphi_1(x)]^2 + [\varphi_2(x)]^2 = 1$. Hence, there exist neighborhoods G_1, G_2 of Q, a continuous mapping ψ_1 of G into $\mathbf{E}_1$ and a continuous mapping ψ_2 of G_2 into $\mathbf{E}_1$ such that for every $x \in Q$ we have $|\varphi_1(x) - \psi_1(x)| < \frac{1}{6}\varepsilon$, $|\varphi_2(x) - \psi_2(x)| < \frac{1}{6}\varepsilon$, and hence also

$$||\psi_1(x) + i\psi_2(x)| - 1| = ||\psi_1(x) + i\psi_2(x)| - |\varphi_1(x) + i\varphi_2(x)|| \leqq$$
$$\leqq |[\varphi_1(x) - \psi_1(x)] + i[\varphi_2(x) - \psi_2(x)]| < \frac{1}{3}\varepsilon.$$

Let us denote by G the set of all $x \in G_1 \cap G_2$ with $||\psi_1(x) + i\psi_2(x)| - 1| < \frac{1}{3}\varepsilon$. We see easily that G is a neighborhood of Q, that

$$\psi = \frac{\psi_1 + i\psi_2}{|\psi_1 + i\psi_2|}$$

is a continuous mapping of G into $\mathbf{S}_1$, and that $|\psi(x) - \varphi(x)| \leqq \varepsilon$ for every $x \in Q$.

24.2.16. *Let f be a continuous mapping of P into $\mathbf{S}_1$. Let $Q \subset P$. Let the partial mapping f_Q be inessential. Then there exists a neighborhood G of the set Q such that the partial mapping f_G is inessential.*

Proof: There is a continuous mapping φ of Q into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in Q$. By 24.2.15 there is an open set $G_0 \supset Q$ and a continuous mapping ψ of G_0 into $\mathbf{E}_1$ such that $|\psi(x) - \varphi(x)| < \pi$ for every $x \in Q$. Let G be the set of all $x \in G_0$ with $f(x) \,.\, e^{-i\psi(x)} \neq -1$. Then G is, by 9.2, open in G_0, hence, open in P by 8.7.7. It is easy to prove that $Q \subset G$. If $x \in G$, then $f(x) = f(x) \,.\, e^{-i\psi(x)} \,.\, e^{i\psi(x)}$, $f(x) \,.\, e^{-i\psi(x)} \neq -1$, so that the partial mapping f_G is inessential by 24.2.4 and 24.2.7.

24.2.17. *Let a space P be either compact or locally connected. Let f be a continuous mapping of P into $\mathbf{S}_1$. Let f_K be inessential for every component K of P. Then the mapping f is inessential.*

Proof: We may assume that $P \neq \emptyset$.

I. Let P be locally connected. By 18.2.1 there exists a mapping φ of P into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$, and such that φ_K is continuous for every component K of P. Since the sets K are open (see 22.1.4), we prove easily that φ is continuous.

II. Let P be compact. Let $\mathfrak{K}$ be the system of all components of P. Every $K \in \mathfrak{K}$ has, by 24.2.16, a neighborhood $\Gamma(K)$ such that the partial mapping $f_{\Gamma(K)}$ is inessential. By 19.1.4 (see also 19.1.5) there is a neighborhood $\Delta(K) \subset \Gamma(K)$ of K such that $\Delta(K)$ is both closed and open. Since the sets $\Delta(K)$ are open and since

$$\bigcup_{K \in \mathfrak{K}} \Delta(K) \supset \bigcup_{K \in \mathfrak{K}} K = P,$$

$\mathfrak{K}$ contains by 17.5.4 a finite sequence $\{K_n\}_1^p$ such that $\bigcup_{n=1}^{p} \Delta(K_n) = P$. Put

$$H_1 = \Delta(K_1), \quad H_n = \Delta(K_n) - \bigcup_{s=1}^{n-1} \Delta(K_{s-1}) \quad (2 \leqq n \leqq p).$$

The partial mappings f_{H_n} are inessential by 24.2.6. Moreover, $\bigcup_{n=1}^{p} H_n = P$ with disjoint summands. Thus, there is a mapping φ of P into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in P$ and the partial mappings φ_{H_n} are continuous. Obviously the sets H_n are open. Hence, we see easily that the mapping φ is continuous, so that f is inessential.

24.2.18.*) *Let P be a separable, locally compact and locally connected space. Let f be a continuous mapping of P into $\mathbf{S}_1$. If f is essential, there is a continuum $K \subset P$ such that the partial mapping f_K is essential.*

Proof: By 24.2.17 there exists a component Q of the space P such that the partial mapping f_Q is essential. By 16.1.2, ex. 17.20, 22.1.4 and 22.1.6, Q is a connected, separable, locally compact and locally connected space. Since Q is locally compact, we may associate with every $z \in Q$ a neighborhood $U(z)$ of z in Q such that $\overline{U(z)}$ is compact. Since Q is locally connected, we may find (for every $z \in Q$) a connected neighborhood $V(z)$ of z in Q such that $V(z) \subset U(z)$. The set $\overline{V(z)}$ is connected by 18.1.6 and compact by 17.2.2. By 16.2.2 we may find a sequence $\{z_n\}_1^\infty$ such that $\bigcup_{n=1}^{\infty} V(z_n) = Q$. By 18.4.2 (see also 18.3.1), for every $m = 1, 2, 3, \dots$ there is a finite subsequence $\{u_\lambda^{(m)}\}_{\lambda=0}^{k_m}$ of $\{z_n\}$ such that $u_0^{(m)} = z_1$, $u_{k_m}^{(m)} = z_m$, $V(u_{\lambda-1}^{(m)}) \cap V(u_\lambda^{(m)}) \neq \emptyset$ for $1 \leqq \lambda \leqq k_m$. Put

$$H_m = \bigcup_{\lambda=0}^{k_m} V(u_\lambda^{(m)}), \quad G_n = \bigcup_{m=1}^{n} H_m .$$

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*) This is a particular case of theorem 24.4.2. The proof of the more general theorem is, of course, more complicated.

It is easy to prove that the sets G_n are connected and open in Q. Moreover, $G_n \subset G_{n+1}$, $\bigcup_{n=1}^{\infty} G_n = Q$ and the mapping f_Q is essential. Hence, by 24.2.14, there exists an index n such that f_{G_n} is essential. Hence (see 24.2.6) the mapping f_K is also essential, if $K = \overline{G}_n$. It is easy to prove (see ex. 24.8) that K is a continuum.

24.2.19. *Let Q be a connected dense subset of a space P. Let f be a continuous mapping of P into $\mathbf{S}_1$. Let the partial mapping f_Q be inessential. Then there exists a set $M \subset P$ such that* [1] *M is closed,* [2] *$M \cap Q = \emptyset$,* [3] *if $Q \subset X \subset P$, $M \cap X = \emptyset$, then the partial mapping f_X is inessential,* [4] *if $Q \subset X \subset P$, $M \cap X \neq \emptyset$, then the partial mapping f_X is essential.*

Proof: I. There exists a continuous mapping φ of the set Q into $\mathbf{E}_1$ such that $f(x) = e^{i\varphi(x)}$ for every $x \in Q$. Let G be the set of all $x \in P$ which have the following property: There is a number $\psi(x)$ such that, if $a_n \to x$ and $a_n \in Q$ for every n, then $\varphi(a_n) \to \psi(x)$.

Evidently $Q \subset G$ and

$$\psi(x) = \varphi(x) \quad \text{for} \quad x \in Q .$$

Put $M = P - G$, so that $M \cap Q = \emptyset$. By ex. 12.2, for every $x \in G$ there is a sequence $\{a_n\}$ such that $a_n \in Q$ for every n, $a_n \to x$, so that obviously $f(x) = e^{i\psi(x)}$ for every $x \in G$.

II. ψ is a continuous mapping of G into $\mathbf{E}_1$, so that f_X is inessential whenever $Q \subset X \subset P$, $M \cap X = \emptyset$. Let $x \in G$, $x_n \in G$, $x_n \to x$. We have to prove that $\psi(x_n) \to \psi(x)$. There exist sequences $\{a_{n\nu}\}_{\nu=1}^{\infty}$ such that $a_{n\nu} \in Q$, $\lim_{\nu \to \infty} a_{n\nu} = x_n$. As $x_n \in G$, we have $\lim_{\nu \to \infty} \varphi(a_{n\nu}) = \psi(x_n)$. For every n there is an index ν_n with $\varrho(a_{n,\nu_n}, x_n) < n^{-1}$, $| \varphi(a_{n,\nu_n}) - \psi(x_n) | < n^{-1}$. Thus, $\lim_{n \to \infty} a_{n,\nu_n} = x$, $a_{n,\nu_n} \in Q$, hence $\lim_{n \to \infty} \varphi(a_{n,\nu_n}) = \psi(x)$, so that $\lim_{n \to \infty} \psi(x_n) = \psi(x)$.

III. Let $Q \subset X \subset P$ and let the partial mapping f_X be inessential. We have to prove that $M \cap X = \emptyset$, i.e. that $X \subset G$. There exists a continuous mapping χ of $X \supset Q$ into $\mathbf{E}_1$ such that $f(x) = e^{i\chi(x)}$ for every $x \in X$, so that $e^{i\varphi(x)} = e^{i\chi(x)}$ for every $x \in Q$. By 24.2.11 there exists an integer k with $\varphi(x) = \chi(x) + 2k\pi$ for every $x \in Q$. Choose an $x \in X$. Let $a_n \in Q$, $a_n \to x$ (see ex. 12.2). Then we have $\chi(a_n) \to \chi(x)$, hence $\varphi(a_n) \to \chi(x) + 2k\pi$. Thus, $x \in G$, $\psi(x) = \chi(x) + 2k\pi$, so that in fact $X \subset G$.

IV. It remains to be proved that M is closed, i.e. that G is open. Choose an $a \in G$. By 24.1.2 there is a homeomorphic mapping h of $\mathbf{S}_1 - [-f(a)]$ onto the interval $J = \mathop{\mathrm{E}}_t[\psi(a) - \pi < t < \psi(a) + \pi]$ such that $e^{ih(y)} = y$ for every $y \in \mathbf{S}_1 - [-f(a)]$. Evidently $h[f(a)] = \psi(a)$. There is a neighborhood U of a such that $f(x) \neq -f(a)$ for every $x \in U$. For $x \in U$ put $\Phi(x) = h[f(x)]$. Then Φ is a continuous mapping of U into $\mathbf{E}_1$; we have $\Phi(a) = \psi(a)$, and $f(x) = e^{i\Phi(x)}$ for every $x \in U$. There is a neighborhood $U_1 \subset U$ of a such that $x \in U_1$ implies $|\Phi(x) - \psi(a)| < \frac{1}{2}\pi$. By II there is a neighborhood $U_2 \subset U_1$ of a such that $x \in G \cap U_2$ implies $|\psi(x) - \psi(a)| < \frac{1}{2}\pi$. Thus, $x \in G \cap U_2$ implies $|\Phi(x) - \psi(x)| < \pi$ so that $x \in Q \cap U_2$ implies $|\Phi(x) - \varphi(x)| < \pi$. On the other hand, we have $e^{i\Phi(x)} = f(x) = e^{i\varphi(x)}$ for every $x \in Q \cap U_2$. Hence, $x \in Q \cap U_2$ implies $\Phi(x) = \varphi(x)$. If $x \in U_2$ and if $a_n \in Q$, $a_n \to x$, there exists an index p such that $n > p$ implies $a_n \in U_2$, which implies $\Phi(a_n) = \varphi(a_n)$. We have $\Phi(a_n) \to \Phi(x)$. Thus, $\varphi(a_n) \to \Phi(x)$, i.e. $x \in G$, $\psi(x) = \Phi(x)$. Thus, every $x \in G$ has a neighborhood $U_2 \subset G$ so that the set G is open.

24.3. 24.3.1. *Let P be a simple arc. Then every continuous mapping f of P into $\mathbf{S}_1$ is inessential.*

Proof: By 17.4.4 (see also 9.6.1), there exists an $\varepsilon > 0$ such that

$$x \in P, \quad y \in P, \quad \varrho(x, y) < \varepsilon \quad \text{imply} \quad |f(x) - f(y)| < 2. \tag{1}$$

By 20.1.12 there is a finite point sequence $\{c_i\}_1^{m-1}$ and a finite sequence $\{C_i\}_1^m$ of point sets such that [1] C_i are simple arcs and, hence (see 17.2.2), they are closed sets, [2] $\bigcup_{i=1}^{m} C_i = P$, [3] $C_i \cap C_{i+1} = (c_i)$ $(1 \leqq i \leqq m - 1)$, [4] $C_i \cap C_j = \emptyset$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq m,\ |i - j| \leqq 2)$, [5] $d(C_i) < \varepsilon$ $(1 \leqq i \leqq m)$, so that, by (1), $\mathbf{S}_1 - f(C_i) \neq \emptyset$. Thus, the partial mappings f_{C_i} are inessential by 24.2.7. Put $A_i = \bigcup_{j=1}^{i} C_j$ $(1 \leqq i \leqq m)$. Then $A_1 = C_1$ and for $1 \leqq i \leqq m - 1$ we have $A_{i+1} = A_i \cup C_{i+1}$ with closed summands, $A_i \cap C_{i+1} = (c_i)$. Thus, by 24.2.13, it follows by induction that the partial mappings f_{A_i} $(1 \leqq i \leqq m)$ are inessential. We have $P = A_m$, so that f is inessential.

Now, let P be a simple loop and let f be a continuous mapping of P into $\mathbf{S}_1$. Choose an orientation of P (see 21.2). Choose $a \in P$, $b \in P$, $a \neq b$. By 21.2.2 (see also 21.1.2) we have $P = P(a, b) \cup P(b, a)$, $P(a, b) \cap P(b, a) = (a) \cup (b)$. The sets $P(a, b)$, $P(b, a)$ are simple arcs, so that, by 24.3.1, there exists a continuous mapping φ_1 of $P(a, b)$ into $\mathbf{E}_1$ and a continuous mapping φ_2 of $P(b, a)$ into $\mathbf{E}_1$ such that

$$\begin{aligned} x \in P(a, b) \quad &\text{implies} \quad e^{i\varphi_1(x)} = f(x), \\ x \in P(b, a) \quad &\text{implies} \quad e^{i\varphi_2(x)} = f(x). \end{aligned} \tag{2}$$

We have $e^{i\varphi_1(a)} = e^{i\varphi_2(a)}$, $e^{i\varphi_1(b)} = e^{i\varphi_2(b)}$, so that there are integers n_1, n_2 with

$$\begin{aligned} \varphi_2(a) &= \varphi_1(a) + 2n_1\pi, \\ \varphi_2(b) &= \varphi_1(b) + 2n_2\pi. \end{aligned} \tag{3}$$

Put

$$n = n_1 - n_2,$$

so that n is an integer.

Preserving the points a, b and the chosen orientation of the simple loop P, replace the mappings φ_1, φ_2 by other mappings ψ_1, ψ_2 having the same properties. We obtain integers m_1, m_2 instead of the integers n_1, n_2. By 20.1.1 and 24.2.11 there are integers k_1, k_2 such that

$$x \in P(a, b) \text{ implies } \psi_1(x) = \varphi_1(x) + 2k_1\pi,$$
$$x \in P(b, a) \text{ implies } \psi_2(x) = \varphi_2(x) + 2k_2\pi.$$

Thus,

$$\psi_2(a) = \varphi_2(a) + 2k_2\pi = \varphi_1(a) + 2(n_1 + k_2)\pi = $$
$$= \psi_1(a) + 2(n_1 + k_2 - k_1)\pi,$$

so that $m_1 = n_1 + k_2 - k_1$ and similarly $m_2 = n_2 + k_2 - k_1$. Hence,

$$n = n_1 - n_2 = m_1 - m_2.$$

Thus, the number n does not depend on the choice of φ_1, φ_2. Let us write, more precisely, $n = n(a, b)$. We are going to prove that (with the orientation of P given) the number n does not depend on the choice of a, b. It suffices to prove that the number n remains unchanged whenever we preserve one of the points—say the point a—and replace the point b by another point c; i.e. we prove that $n(a, b) = n(a, c)$ for distinct a, b, c.

For certainty, let $c \in P(a, b)$. It is easy to prove that

$$P(a, c) \cup P(c, b) = P(a, b), \qquad P(a, c) \cap P(c, b) = (c),$$
$$P(c, b) \cup P(b, a) = P(c, a), \qquad P(c, b) \cap P(b, a) = (b).$$

By 24.3.1 there are continuous mappings $\varphi_1, \varphi_2, \varphi_3$ of the simple arcs $P(a, c)$, $P(c, b)$, $P(b, a)$ into $\mathbf{E}_1$ such that

$$x \in P(a, c) \text{ implies } e^{i\varphi_1(x)} = f(x),$$
$$x \in P(c, b) \text{ implies } e^{i\varphi_2(x)} = f(x),$$
$$x \in P(b, a) \text{ implies } e^{i\varphi_3(x)} = f(x).$$

There are integers h_1, h_2, h_3 with

$$\varphi_3(a) = \varphi_1(a) + 2h_1\pi,$$
$$\varphi_3(b) = \varphi_2(b) + 2h_2\pi,$$
$$\varphi_2(c) = \varphi_1(c) + 2h_3\pi.$$

There exist (see ex. 9.5) continuous mappings φ_4, φ_5 of the simple arcs $P(a, b)$, $P(c, a)$ into $\mathbf{E}_1$ such that

$$x \in P(a, c) \Rightarrow \varphi_4(x) = \varphi_1(x), \qquad x \in P(c, b) \Rightarrow \varphi_4(x) = \varphi_2(x) - 2h_3\pi,$$
$$x \in P(c, b) \Rightarrow \varphi_5(x) = \varphi_2(x), \qquad x \in P(b, a) \Rightarrow \varphi_5(x) = \varphi_3(x) - 2h_2\pi.$$

Evidently

$$n(a, b) = n_1 - n_2, \qquad n(a, c) = m_1 - m_2,$$

where

$$\begin{aligned}
2n_1\pi &= \varphi_3(a) - \varphi_4(a) = \varphi_3(a) - \varphi_1(a) = 2h_1\pi \\
2n_2\pi &= \varphi_3(b) - \varphi_4(b) = \varphi_3(b) - [\varphi_2(b) - 2h_3\pi] = 2(h_2 + h_3)\,\pi, \\
2m_1\pi &= \varphi_5(a) - \varphi_1(a) = [\varphi_3(a) - 2h_2\pi] - \varphi_1(a) = 2(h_1 - h_2)\,\pi, \\
2m_2\pi &= \varphi_5(c) - \varphi_1(c) = \varphi_2(c) - \varphi_1(c) = 2h_3\pi,
\end{aligned}$$

so that

$$n_1 - n_2 = h_1 - (h_2 + h_3) = (h_1 - h_2) - h_3 = m_1 - m_2,$$

i.e., $n(a, b) = n(a, c)$.

Thus, the number n—for a given mapping f—depends on the orientation of the simple loop P only. If we change the orientation, we obtain $-n$ instead of n (see Remark at the end of Section 21.2).

The number n is said to be the *degree of the mapping* f. If the mapping f is inessential, then there is a continuous mapping φ of P into $\mathbf{E}_1$ with $e^{i\varphi(x)} = f(x)$ for every $x \in P$. We may put $\varphi_1 = \varphi_{P(a,b)}$, $\varphi_2 = \varphi_{P(b,a)}$, and we obtain in (3) $n_1 = n_2 = 0$ and consequently $n = 0$.

On the other hand let $n = 0$, so that $n_1 = n_2$ in (3); if φ_1, φ_2 are the mappings from (2), there is a mapping φ of P into $\mathbf{E}_1$ such that

$$\begin{aligned}
x \in P(a, b) &\quad \text{implies} \quad \varphi(x) = \varphi_1(x), \\
x \in P(b, a) &\quad \text{implies} \quad \varphi(x) = \varphi_2(x) - 2n_1\pi.
\end{aligned}$$

We have $e^{i\varphi(x)} = f(x)$ for every $x \in P$ and the mapping f is continuous (see ex. 9.5) so that f is inessential.

The results obtained are stated in the following two theorems:

24.3.2. *The degree n of a continuous mapping of an oriented simple loop into $\mathbf{S}_1$ is an integer. If the orientation is changed, n is replaced by $-n$.*

24.3.3. *A continuous mapping of an oriented simple loop into $\mathbf{S}_1$ is inessential if and only if its degree is zero.*

Moreover, it is easy to prove the following theorem:

24.3.4. *Let f_1, f_2 be continuous mappings of an oriented simple loop P into $\mathbf{S}_1$ and let n_1, n_2 be their degrees. Then the degree of the mapping $f_1 f_2$ is equal to $n_1 + n_2$.*

24.3.5. *Let P be an oriented simple loop. There are exactly two kinds of homeomorphic mappings of P onto $\mathbf{S}_1$. The mappings of the first kind have degree one, the mappings of the second kind have degree minus one.*

Proof: I. Choose $a \in P$, $b \in P$, $a \neq b$. Then $P(a, b)$ and $P(b, a)$ are simple arcs with end points a, b, so that there is a homeomorphic mapping φ_1 of the interval

$J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto $P(a, b)$ and a homeomorphic mapping φ_2 of J onto $P(b, a)$ such that $\varphi_1(0) = \varphi_2(0) = a$, $\varphi_1(1) = \varphi_2(1) = b$. Define f_1, f_2 by

$$f_1(x) = \mathrm{e}^{i\pi t}, \quad f_2(x) = \mathrm{e}^{-i\pi t} \quad \text{for} \quad x \in P(a, b), \quad x = \varphi_1(t),$$
$$f_1(x) = \mathrm{e}^{-i\pi t}, \quad f_2(x) = \mathrm{e}^{i\pi t} \quad \text{for} \quad x \in P(b, a), \quad x = \varphi_2(t).$$

It is easy to prove that f_1, f_2 are homeomorphic mappings of P onto $\mathbf{S}_1$ and that their degrees are $+1$, -1.

II. Let f be a homeomorphic mapping of P onto $\mathbf{S}_1$. Put $a = f_{-1}(1)$, $b = f_{-1}(-1)$. Let M_1 be the set of all $\mathrm{e}^{i\pi t}$ $(0 \leqq t \leqq 1)$. Let M_2 be the set of all $\mathrm{e}^{-i\pi t}$ $(0 \leqq t \leqq 1)$. Then $M_1 \cup M_2 = \mathbf{S}_1$, $M_1 \cap M_2 = (1) \cup (-1)$ and M_1, M_2 are simple arcs with end points $+1$, -1. Thus, $f_{-1}(M_1) \subset P$, $f_{-1}(M_2) \subset P$ are two distinct simple arcs with end points a, b. Thus, under a suitable choice of orientation of the simple loop P we have

$$P(a, b) = f_{-1}(M_1), \qquad P(b, a) = f_{-1}(M_2).$$

Obviously there is a homeomorphic mapping φ_1 of $P(a, b)$ onto $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq \pi]$ and a homeomorphic mapping φ_2 of $P(b, a)$ onto J such that

$$f(x) = \mathrm{e}^{i\varphi_1(x)} \quad \text{for} \quad x \in P(a, b),$$
$$f(x) = \mathrm{e}^{-i\varphi_2(x)} \quad \text{for} \quad x \in P(b, a).$$

We have $\varphi_1(a) = \varphi_2(a) = 0$, $\varphi_1(b) = \varphi_2(b) = \pi$, so that the degree of f is equal to $+1$. If we change the orientation, the degree of f is equal to -1.

24.3.6. *Let P be an oriented simple loop. Let n be an integer. Then there exists a continuous mapping of P into $\mathbf{S}_1$ with degree equal to n.*

Proof: By 24.3.5 there is a homeomorphic mapping f of P onto $\mathbf{S}_1$ with degree one. By 24.3.4 (see also 24.3.2) it is easy to prove that the mapping f^n has degree n.

24.3.7. *Let $P \subset \mathbf{E}_1$. Then every continuous mapping f of P into $\mathbf{S}_1$ is inessential.*

Proof: By 24.2.15 there is a set $G \supset P$ open in $\mathbf{E}_1$ and a continuous mapping g of G into $\mathbf{S}_1$ such that $|f(x) - g(x)| < 2$ for every $x \in G$. Thus, by 24.2.6 and 24.2.8, it suffices to prove that the mapping g of G into $\mathbf{S}_1$ is inessential.

Let g be essential. The set G is separable by 16.1.2 and 16.1.5, locally compact by 17.10.1 (see also ex. 17.20) and locally connected by 22.1.3 and 22.1.8. Thus, by 24.2.18, there is a continuum $K \subset G$ such that the partial mapping g_K is essential. This is a contradiction by 19.2.2 and 24.3.1.

24.4. 24.4.1. *Let $Q \subset P$. Let us define $L(Q)$ in the same manner as in 22.2. Let $Q \subset \subset M \subset Q \cup L(Q)$. Let g be a continuous mapping of M into $\mathbf{S}_1$. Let the partial mapping f_Q be inessential. Then f is inessential.*

Proof: I. There is a continuous mapping φ of Q into $\mathbf{E}_1$ such that $e^{i\varphi(x)} = f(x)$ for every $x \in Q$.

II. Let $x \in M - Q$. Since f is continuous, there exists a neighborhood V_x of x in the space M such that $f(y) \neq -f(x)$ for $y \in V_x$. By 8.7.5 there is a neighborhood U_x of x in P such that $V_x = M \cap U_x$. Since $M - Q \subset L(Q)$, there is a component K_x of $Q \cap U_x = Q \cap V_x \subset M$ such that x is an interior point of $K_x \cup (P - Q)$. The partial mapping f_{V_x} is inessential by 24.2.7, as $f(V_x) \subset \mathbf{S}_1 - [-f(x)]$. Thus, there exists a continuous mapping χ_x of V_x into $\mathbf{E}_1$ such that

$$e^{i\chi_x(y)} = f(y) \quad \text{for} \quad y \in V_x.$$

As K_x is a connected subset of $Q \cap V_x$, there is, by 24.2.11, an integer k_x such that

$$y \in K_x \Rightarrow \chi_x(y) = \varphi(y) + 2k_x\pi.$$

III. Let us define a mapping ψ of M into $\mathbf{E}_1$ as follows: First, if $x \in Q$, put $\psi(x) = \varphi(x)$. Secondly, if $x \in M - Q$, put $\psi(x) = \chi_x(x) - 2k_x\pi$. Then we have $e^{i\psi(x)} = f(x)$ for every $x \in M$. It remains to prove that ψ is continuous.

IV. Let $x \in M$. As $L(Q) \subset \overline{Q}$, we have $M \subset \overline{Q}$. Hence (see 8.2.1), there exists a sequence $\{a_n\}$ such that $a_n \to x$ and $a_n \in Q$ for every n. We shall prove that $\varphi(a_n) \to \psi(x)$.

This is evident for $x \in Q$. Hence, let $x \in M - Q$. By II, x is an interior point of $K_x \cup (P - Q)$. Thus, there is an index p such that $a_n \in K_x \cup (P - Q)$ for $n > p$. As $a_n \in Q$, we see that

$$n > p \Rightarrow a_n \in K_x \Rightarrow \varphi(a_n) = \chi_x(a_n) - 2k_x\pi.$$

On the other hand, χ_x is a continuous mapping of the set $V_x \supset K_x$ into $\mathbf{E}_1$. Hence,

$$\varphi(a_n) \to \chi_x(x) - 2k_x\pi = \psi(x).$$

V. Let us choose an $x \in M$ and prove that ψ is continuous at the point x. Thus, let $x_n \in M$, $x_n \to x$. We have to prove that $\psi(x_n) \to \psi(x)$. There are sequences $\{b_{n\nu}\}_{\nu=1}^{\infty}$ $(n = 1, 2, 3, \ldots)$ in Q such that $\lim_{\nu\to\infty} b_{n\nu} = x_n$. By IV, $\lim_{\nu\to\infty} \varphi(b_{n\nu}) = \psi(x_n)$. Obviously, for every $n = 1, 2, 3, \ldots$ there is an index ν_n such that

$$\varrho(x_n, b_{n\nu_n}) < n^{-1}, \qquad |\psi(x_n) - \varphi(b_{n\nu_n})| < n^{-1}.$$

As $x_n \to x$, $\varrho(x_n, b_{n\nu_n}) < n^{-1}$, we have $\lim_{\nu\to\infty} b_{n\nu_n} = x$. Moreover, $b_{n\nu_n} \in Q$, so that, by IV, $\lim_{n\to\infty} \varphi(b_{n\nu_n}) = \psi(x)$. As $|\psi(x_n) - \varphi(b_{n\nu_n})| < n^{-1}$, we have also $\lim_{n\to\infty} \psi(x_n) = \psi(x)$.

24.4.2. *Let P be a topologically complete locally connected space. Let f be a continuous mapping of P into $\mathbf{S}_1$. Let f_Q be inessential for every simple loop $Q \subset P$. Then f is inessential.*

Proof: I. Let K be a component of P. By 24.2.17 it suffices to prove that the partial mapping f_K is inessential. The space K is topologically complete by 13.2, 15.5.3 and 18.2.2. Moreover, it is connected and also, by 22.1.6, locally connected.

II. Choose a point $a \in K$ and a number $\alpha \in \mathbf{E}_1$ with $e^{i\alpha} = f(a)$. If $x \in K$, $x \neq a$, then by 22.3.1 K contains at least one simple arc with end points a, x.

Let $C_1 \subset K$, $C_2 \subset K$ be simple arcs with end points a, x. By 24.3.1 there is a continuous mapping φ_1 of C_1 into $\mathbf{E}_1$ and a continuous mapping φ_2 of C_2 into $\mathbf{E}_1$ such that: [1] $\varphi_1(a) = \varphi_2(a) = \alpha$, [2] $e^{i\varphi_1(y)} = f(y)$ for every $y \in C_1$ and $e^{i\varphi_2(y)} = f(y)$ for every $y \in C_2$. We shall prove that $\varphi_1(x) = \varphi_2(x)$. Let us assume the contrary. Let C_1 be oriented in such a way that a is the initial point. Define $M \subset C_1$ as follows: If $y \in C_1$ then $y \in M$ if and only if $y \in C_2$ and $\varphi_1(y) = \varphi_2(y)$. The set M is obviously (see 9.5) closed in C_1. Moreover, $a \in M$ and hence $M \neq \emptyset$. By 20.2.7 there exists a last point b of the set $M \subset C_1$. As $\varphi_1(x) \neq \varphi_2(x)$, we have $b \neq x$, so that (see 20.1.8) there exists a simple arc $C_1(b, x) \subset C_1$. Evidently

$$y \in C_2 \cap C_1(b, x), \quad \varphi_1(y) = \varphi_2(y) \Rightarrow y = b. \tag{1}$$

There exists a simple arc $C_2(b, x) \subset C_2$. Suppose that it is oriented in such a way that b is the initial point. We define a set $M' \subset C_2(b, x)$ as follows: If $y \in C_2(b, x)$, then $y \in M'$ if and only if $y \in C_1(b, x)$ and $\varphi_1(y) \neq \varphi_2(y)$. As $e^{i\varphi_1(y)} = e^{i\varphi_2(y)}$, we may write $|\varphi_1(y) - \varphi_2(y)| \geqq 2\pi$ instead of $\varphi_1(y) \neq \varphi_2(y)$. Thus (see 9.5) the set M' is closed in $C_2(b, x)$. Moreover, $x \in M'$ and hence $M' \neq \emptyset$. By 20.2.7 there is a first element c of the set $M' \subset C_2(b, x)$. By (1), c is the first point $y \in C_2(b, x)$ with $y \in C_1(b, x) - (b)$. There exist simple arcs

$$C_1(b, c) \subset C_1, \qquad C_2(b, c) \subset C_2.$$

Evidently $C_1(b, c) \cap C_2(b, c) = (b) \cup (c)$, so that $C_1(b, c) \cup C_2(b, c) = Q$ is a simple loop by 21.1.3. Let Q be oriented in such a way that

$$Q(b, c) = C_1(b, c), \qquad Q(c, b) = C_2(b, c).$$

Since $\varphi_1(b) = \varphi_2(b)$, the degree of the mapping f_Q is equal to

$$\frac{1}{2\pi}[\varphi_1(c) - \varphi_2(c)] \neq 0,$$

so that the mapping f_Q is essential by 24.3.3. This is a contradiction.

III. Put $\psi(a) = \alpha$. If $x \in K - (a)$, we define $\psi(x) \in \mathbf{E}_1$ as follows: Choose a simple arc $C \subset K$ with end points a, x and a continuous mapping φ of C into $\mathbf{E}_1$ such that $\varphi(a) = \alpha$ and that

$$e^{i\varphi(y)} = f(y) \quad \text{for} \quad y \in C.$$

Then, put $\psi(x) = \varphi(x)$. By II, ψ is a uniquely defined mapping of the set K into $\mathbf{E}_1$. Evidently $e^{i\psi(x)} = f(x)$ for every $x \in K$, so that it suffices to prove that the mapping ψ is continuous.

IV. Let us choose a point $x_0 \in K$ and prove that the mapping ψ is continuous at the point x_0. As f is continuous in x_0, there is a neighborhood U of the point x_0 in K such that $x \in U$ implies $f(x) \neq -f(x_0)$. By 24.2.7 there is a continuous mapping χ of U into $\mathbf{E}_1$ such that $e^{i\chi(x)} = f(x)$ for $x \in U$ and that $\chi(x_0) = \psi(x_0)$.

Let V be the component of U containing the point x_0. By 22.1.4, V is a neighborhood of the point x_0 in K. V is a connected space. Moreover, V is topologically complete by 15.5.3 and locally connected by 22.1.3.

It suffices to prove that $\chi(x) = \psi(x)$ for $x \in (x_0) \cup [V - (a)]$. This is evident for $x = x_0$. Thus, let $x \in V$, $a \neq x \neq x_0$.

By 22.3.1 there exists a simple arc $C \subset V$ with end points x_0, x. If $x_0 = a$, then χ_C is a continuous mapping of C into $\mathbf{E}_1$ such that $e^{i\chi(y)} = f(y)$ for $y \in C$ and that $\chi(x_0) = \alpha$, so that $\psi(x) = \chi(x)$. Thus, let $x_0 \neq a$. Then there exists a simple arc $C_0 \subset$ $\subset K$ with end points a, x_0 and a continuous mapping φ_0 of C_0 into $\mathbf{E}_1$ such that $e^{i\varphi_0(y)} = f(y)$ for $y \in C_0$ and $\varphi_0(a) = \alpha$. Let C_0 be oriented in such a way that a is its initial point. Define a set $M \subset C_0$ as follows: If $y \in C_0$, then $y \in M$ if and only if $y \in C$. It is easy to prove that M is closed in C_0. Evidently $x_0 \in M$, so that $M \neq \emptyset$. Hence, by 20.2.7 there is a first point x_1 of the set $M \subset C_0$. If $x_1 = a$, put $C_1 = (a)$. If $x_1 \neq a$, put $C_1 = C_0(a, x_1)$ (see 20.1.8). It is easy to prove that there are simple arcs $C' \subset C_1 \cup C$, $C'' \subset C_1 \cup C$ such that [1] $C' = C_1 \cup (C' \cap C)$, $C'' = C_1 \cup (C'' \cap C)$, [2] a, x_0 are the end points of C', [3] a, x are the end points of C''. As $e^{i\chi(x_1)} = f(x_1) = e^{i\varphi_0(x_1)}$, there is an integer k with $\chi(x_1) = \varphi_0(x_1) + 2k\pi$. It is easy to prove that there exists a continuous mapping φ' of the set C' into $\mathbf{E}_1$ and a continuous mapping φ'' of C'' into $\mathbf{E}_1$ such that

$$y \in C_1 \Rightarrow \varphi'(y) = \varphi''(y) = \varphi_0(y),$$
$$y \in C' - C_1 \Rightarrow \varphi'(y) = \chi(y) - 2k\pi,$$
$$y \in C'' - C_1 \Rightarrow \varphi''(y) = \chi(y) - 2k\pi.$$

Evidently: $e^{i\varphi'(y)} = f(y)$ for $y \in C'$, $e^{i\varphi''(y)} = f(y)$ for $y \in C''$, $\varphi'(a) = \varphi''(a) = \alpha$. Thus, we have $\varphi'(x_0) = \psi(x_0)$, $\varphi''(x) = \psi(x)$. Since $\varphi'(x_0) = \chi(x_0) - 2k\pi = \psi(x_0) - 2k\pi$, $\varphi''(x) = \chi(x) - 2k\pi$, we obtain $k = 0$ and $\chi(x) = \psi(x)$.

24.5. 24.5.1. *Let P be a metric space. Let Q be either a continuum or a connected and locally connected space. Let f be a continuous mapping of $P \times Q$ into $\mathbf{S}_1$. Let, for every $x \in P$, the partial mapping $f_{(x)\times Q}$ be inessential. Let there exist a point $b \in Q$ such that the partial mapping $f_{P\times(b)}$ is inessential. Then the mapping f is inessential.*

Proof: I. There exists a continuous mapping χ of P into $\mathbf{E}_1$ such that $e^{i\chi(x)} = f(x, b)$ for every $x \in P$. For every $x \in P$ there exists a continuous mapping ψ_x of Q into $\mathbf{E}_1$ such that $e^{i\psi_x(y)} = f(x, y)$ for every $y \in Q$. We may assume that $\psi_x(b) = \chi(x)$ for every $x \in P$.*)

*) Otherwise it suffices to replace the mapping ψ_x by a mapping ψ'_x defined by

$$\psi'_x(y) = \psi_x(y) + \chi(x) - \psi_x(b)$$

for every $y \in Q$.

For $(x, y) \in P \times Q$ put $\varphi(x, y) = \psi_x(y)$, so that φ is a mapping of $P \times Q$ into $\mathbf{E}_1$ such that $e^{i\varphi(x,y)} = f(x, y)$ for every $(x, y) \in P \times Q$. It remains to prove that the mapping φ is continuous. Let us choose an arbitrary point $\alpha \in P$ and prove that φ is continuous at the point (α, y) for every $y \in Q$.

II. Let Q be a continuum. As χ is a continuous mapping of P into $\mathbf{E}_1$, there is an $\varepsilon > 0$ such that

$$x \in P, \quad \varrho(\alpha, x) < \varepsilon \;\Rightarrow\; |\chi(x) - \chi(\alpha)| < \pi.$$

As f is a continuous mapping of $P \times Q$ into $\mathbf{S}_1$, we may associate with every $z \in Q$ a number $\delta(z) > 0$ such that

$$x \in P, \quad y \in Q, \quad \varrho(a, x) < \delta(z), \quad \varrho(z, y) < \delta(z) \;\Rightarrow\; |f(x, y) - f(\alpha, y)| < 2.$$

We have

$$Q = \bigcup_{z \in Q} \Omega_Q[z, \delta(z)]$$

with open summands. Since Q is compact, by 17.5.4 there is a finite sequence $\{z_n\}_1^p$, $z_n \in Q$, such that

$$\bigcup_{n=1}^{p} \Omega_Q(z_n, \delta(z_n)] = Q.$$

Let $\eta > 0$ be the least of the $p + 1$ numbers ε, $\delta(z_n)$ $(1 \leqq n \leqq p)$. Then, first,

$$x \in P, \quad \varrho(\alpha, x) < \eta \;\Rightarrow\; |\chi(x) - \chi(\alpha)| < \pi,$$

Secondly,

$$x \in P, \quad \varrho(\alpha, x) < \eta \;\Rightarrow\; |f(x, y) - f(\alpha, y)| < 2$$

for every $y \in Q$. In fact, for every $y \in Q$ there is an index n with $\varrho(z_n, y) < \delta(z_n)$.

By 24.1.2 there exists a homeomorphic mapping v of $\mathbf{S}_1 - (-1)$ onto the interval $\underset{t}{\mathrm{E}}[-\pi < t < \pi]$ such that $e^{iv(z)} = z$ for every $z \in \mathbf{S}_1 - (-1)$.

Put $P_0 = \Omega_P(\alpha, \eta)$. If $(x, y) \in P_0 \times Q$, we have $\varrho(\alpha, x) < \eta$, hence $|f(x, y) - f(\alpha, y)| < 2$, hence $f(x, y)/f(\alpha, y) \neq -1$; therefore we may put

$$\Phi(x, y) = \psi_x(y) + v[f(x, y)/f(\alpha, y)] \quad \text{for} \quad (x, y) \in P_0 \times Q.$$

Then $e^{i\Phi(x,y)} = f(x, y)$ for every $(x, y) \in P_0 \times Q$ and Φ is a continuous mapping of $P_0 \times Q$ into $\mathbf{E}_1$.

Since $\psi_x(b) = \chi(a)$,

$$x \in P_0 \Rightarrow |\Phi(x, b) - \chi(\alpha)| < \pi.$$

Since also $x \in P_0 \Rightarrow |\chi(x) - \chi(\alpha)| < \pi$,

$$x \in P_0 \Rightarrow |\Phi(x, b) - \chi(x)| < 2\pi.$$

On the other hand,

$$e^{i\Phi(x,b)} = f(x, b) = e^{i\chi(x)},$$

so that $\Phi(x, b) = \chi(x)$ for every $x \in P_0$.

Choose an $x \in P_0$. Let $\Phi(x, y) = g_x(y)$ for $y \in Q$, so that g_x is a continuous mapping of Q into $\mathbf{E}_1$. ψ_x is also a continuous mapping of Q into $\mathbf{E}_1$. Moreover,

$$e^{ig_x(y)} = e^{i\Phi(x,y)} = f(x, y) = e^{i\varphi_x(y)} \quad \text{for every} \quad y \in Q.$$

The space Q is connected so that, by 24.2.11, there exists an integer n_x such that

$$\Phi(x, y) = g_x(y) = \psi_x(y) + 2n_x\pi \quad \text{for every} \quad y \in Q.$$

Since $b \in Q$, $\Phi(x, b) = \chi(x) = \psi_x(b)$, we have $n_x = 0$. Thus,

$$\Phi(x, y) = \psi_x(y) = \varphi(x, y) \quad \text{for} \quad (x, y) \in P_0 \times Q.$$

Since $P_0 \times Q$ is open in $P \times Q$, since Φ is a continuous mapping of $P_0 \times Q$ into $\mathbf{S}_1$ and since $\alpha \in P_0$, the mapping φ is continuous at the point (α, y) for every $y \in Q$.

III. Let Q be connected and locally connected. If $y \in Q$, let $y \in A$ if φ is continuous at (α, y), $y \in B$ if φ is not continuous at (α, y). We have to prove that $B = \emptyset$.

We have $Q = A \cup B$, $A \cap B = \emptyset$. We shall prove that the sets A, B are open in Q, so that $Q = A \cup B$ with separated summands. Since the space Q is connected, this will imply that either $A = \emptyset$ or $B = \emptyset$. Then the proof will be finished, as soon as we prove that $b \in A$.

Let $\beta \in A$, so that φ is continuous at (α, β). There exists a neighborhood U of the point α in P and a neighborhood V of the point β in Q such that

$$x \in U, \quad y \in V \Rightarrow |\varphi(x, y) - \varphi(\alpha, \beta)| < \tfrac{1}{2}\pi.$$

If $y \in V$, $(x_n, y_n) \to (\alpha, y)$, there is an index p such that for $n > p$ we have $x_n \in U$, $y_n \in V$. Since also $\alpha \in U, y \in V, n > p$ implies $|\varphi(x_n, y_n) - \varphi(\alpha, \beta)| < \frac{1}{2}\pi$, $|\varphi(\alpha, y) - \varphi(\alpha, \beta)| < \frac{1}{2}\pi$, which implies $|\varphi(x_n, y_n) - \varphi(\alpha, y)| < \pi$, so that, by 24.2.9, the mapping φ is continuous at the point (α, y). Thus, $V \subset A$. Consequently, A is open in Q.

Now, let us prove that the set B is also open in Q. Let $\beta \in B$ so that φ is not continuous at (α, β). We have to prove that there is a neighborhood W of the point β in Q such that $W \subset B$.

By 24.2.9 there exists a sequence $\{(x_n, y_n)\}$ in $P \times Q$ such that $x_n \to \alpha$, $y_n \to \beta$ and that $|\varphi(x_n, y_n) - \varphi(\alpha, \beta)| > \pi$ for every n.

Since f is a continuous mapping of $P \times Q$ into $\mathbf{S}_1$, we can find a neighborhood U of the point α in P and a neighborhood V_1 of the point β in Q such that

$$x \in U, \quad y \in V_1 \Rightarrow |f(x, y) - f(\alpha, \beta)| < 2.$$

By 24.1.2 there is a homeomorphic mapping v of $\mathbf{S}_1 - (-1)$ onto the interval $\mathrm{E}_t[-\pi < t < \pi]$ such that $e^{iv(z)} = z$ for every $z \in \mathbf{S}_1 - (-1)$.

If $x \in U$, $y \in V_1$, we have $|f(x, y) - f(\alpha, \beta)| < 2$ and hence $f(x, y) \neq -f(\alpha, \beta)$, so that we may put $\Phi(x, y) = \varphi(\alpha, \beta) + v[f(x, y)/f(\alpha, \beta)]$ for $x \in U$, $y \in V_1$. Then Φ

is a continuous mapping of $U \times V_1$ into $\mathbf{E}_1$ and we have $e^{i\Phi(x,y)} = f(x, y)$ for every $(x, y) \in U \times V_1$. Moreover,

$$x \in U, \quad y \in V_1 \quad \Rightarrow \quad | \Phi(x, y) - \Phi(\alpha, \beta) | < \pi.$$

Let V_2 be the component of V_1 containing the point β. Then $V_2 \subset V_1$ and, by 22.1.4, V_2 is a neighborhood of the point β in Q.

If $x \in U$, put $g_x(y) = \Phi(x, y)$, $h_x(y) = \psi_x(y)$ for $y \in V_2$. Then g_x and h_x are continuous mappings of the connected V_2 into $\mathbf{E}_1$ and we have $e^{ig_x(y)} = f(x, y) = e^{ih_x(y)}$ for every $y \in V_2$. Thus, by 24.2.11 there is an integer k_x such that $h_x(y) = g_x(y) + 2k_x\pi$ for $y \in V_2$. Hence,

$$x \in U, \quad y \in V_2 \quad \Rightarrow \quad \varphi(x, y) = \Phi(x, y) + 2k_x\pi.$$

Since ψ_α is a continuous mapping of Q into $\mathbf{E}_1$, there is a neighborhood $W \subset V_2$ of the point β in Q such that

$$y \in W \Rightarrow | \varphi(\alpha, y) - \varphi(\alpha, \beta) | < \tfrac{1}{2}\pi.$$

We shall prove that $W \subset B$; then B will be proved to be open. Since $x_n \to \alpha$, $y_n \to \beta$, there is an index p such that $n > p$ implies $x_n \in U$, $y_n \in W$. If $n > p$, we have $| \Phi(x_n, y_n) - \Phi(\alpha, \beta) | < \pi$, $| \varphi(x_n, y_n) - \varphi(\alpha, \beta) | > \pi$, $\varphi(x_n, y_n) = \Phi(x_n, y_n) + 2k_{x_n}\pi$, $\Phi(\alpha, \beta) = \varphi(\alpha, \beta)$, hence $k_\alpha = 0$, $k_x \neq 0$. If W is not contained in B, there is a point $y \in A \cap W$. We shall obtain a contradiction as follows: Since $y \in A$, the mapping φ is continuous at the point (α, y). Since Φ is also continuous at the point (α, y) and since $x_n \to \alpha$, we have $\varphi(x_n, y) \to \varphi(\alpha, y)$, $\Phi(x_n, y) \to \Phi(\alpha, y) = \varphi(\alpha, y) + 2k_\alpha\pi = \varphi(\alpha, y)$, $\varphi(x_n, y) - \Phi(x_n, y) = 2k_{x_n}\pi \to 0$, which is a contradiction, as $| k_{x_n} | \geqq 1$.

Since $| k_{x_n} | \geqq 1$ and since $\Phi(x_n, \beta) \to \Phi(\alpha, \beta) = \varphi(\alpha, \beta)$, $\Phi(x_n, \beta) = \varphi(x_n, \beta) - 2k_{x_n}\pi$, $\varphi(x_n, \beta)$ cannot converge to $\varphi(\alpha, \beta)$. On the other hand, evidently $\varphi(x_n, b) \to \varphi(\alpha, \beta)$. Thus, $\beta \neq b$ for every $\beta \in B$, so that $b \in A$.

24.5.2. *Let f be a continuous mapping of P into $\mathbf{S}_1$. Then f is inessential, if and only if there exists a continuous mapping g of $P \times \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ into $\mathbf{S}_1$ such that*

$$g(x, 0) = f(x), \quad g(x, 1) = 1 \quad \textit{for every} \quad x \in P.*)$$

Proof: I. Let such a g exist. Put $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$. By 24.3.1 the partial mapping $g_{(x) \times J}$ is inessential for every $x \in P$. By 24.2.7 the partial mapping $g_{P \times (1)}$ is inessential Hence, by 24.5.1, g is inessential, so that (see 24.2.6) also the partial mapping $g_{P \times (0)}$ is inessential. Thus, also the mapping f is inessential.

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*) If f_0, f_1 are mappings of X into Y such that there is a continuous mapping g of $X \times \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ into Y with $g(x, 0) = f_0(x)$, $g(x, 1) = f_1(x)$, the mappings f_0, f_1 are said to be homotopic. Thus, the theorem states that a mapping f of P into $\mathbf{S}_1$ is inessential if and only if it is homotopic with a constant. (Ed.)

II. Let f be inessential. Then there exists a continuous mapping φ of P into $\mathbf{E}_1$ such that $e^{i\varphi(x)} = f(x)$ for every $x \in P$. Obviously, it suffices to put $g(x, t) = e^{i(1-t)\varphi(x)}$ for $x \in P$, $0 \leqq t \leqq 1$.

24.5.3. *Let f be a continuous mapping of the euclidean space $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ into $\mathbf{S}_1$. Then f is inessential.*

Proof: The statement is true for $m = 1$ by 24.3.7. Since $\mathbf{E}_{m+1} = \mathbf{E}_m \times \mathbf{E}_1$, the general statement may be proved by induction by 24.5.1.

24.5.4. *Let f be a continuous mapping of the spherical space $\mathbf{S}_m$ $(m = 2, 3, 4, \ldots)$ into $\mathbf{S}_1$. Then f is inessential.*

Proof: If $\alpha \in \mathbf{E}_m$, it is easy to prove that the set $\mathbf{E}_m - (\alpha)$ is connected. Consequently, by 17.10.4, $\mathbf{S}_m - [(a) \cup (b)]$ is also connected if we choose $a \in \mathbf{S}_m$, $b \in \mathbf{S}_m$, $a \neq b$. The sets $A = \mathbf{S}_m - (a)$, $B = \mathbf{S}_m - (b)$ are open in $\mathbf{S}_m$ and the partial mappings f_A, f_B are inessential by 17.10.4 and 24.5.3. Moreover, $A \cap B = \mathbf{S}_m - [(a) \cup (b)]$ is connected. Thus, f is inessential by 24.2.13.

Exercises

24.1. Let f be a continuous mapping of $\mathbf{E}_m$ $(m \geqq 2)$ onto $\mathbf{S}_1$. Let $a \in \mathbf{E}_m$, $b \in \mathbf{E}_m$, $a \neq b$. Then there exists a point $c \in \mathbf{E}_m$ such that either $a \neq c$, $f(a) = f(c)$ or $b \neq c$, $f(b) = f(c)$.

24.2. What must we assume about a space P to be allowed to replace $\mathbf{E}_m$ in ex. 24.1. by P?

24.3. Every continuous mapping of any of the spaces P_2, P_3, P_4, P_5, P_7 (see exercises to § 19) is inessential. This is not true for the spaces P_1, P_6.

24.4. We may replace $\mathbf{E}_1$ in theorem 24.2.15 by any $\mathbf{E}_m$ $(m = 2, 3, 4, \ldots)$ or by $\mathbf{U}$ (see section 7.3).

Let $m = 1, 2, 3, \ldots$. Let f be a continuous mapping of a space P into $\mathbf{S}_m$. We say that f is *inessential*, if there exists a continuous mapping of $P \times \underset{t}{E}[0 \leqq t \leqq 1]$ into $\mathbf{S}_m$ such that

$$g(x, 0) = f(x)\,, \quad g(x, 1) = (1, 0, \ldots, 0) \quad \text{for every} \quad x \in P\,.$$

By theorem 24.5.2, this definition is consistent with the definition for $m = 1$ given in the section 24.2.

24.5. In theorems 24.2.6, 24.2.7, 24.2.8, 24.2.16 we may write more generally $\mathbf{S}_m$ $(m = 1, 2, 3, \ldots)$ instead of $\mathbf{S}_1$.

24.6. Let $M \subset P$, $a \in M$, $b \in M$, $C \subset P$. Let C be a simple arc with end points a, b. Let $C \cap \overline{M} = (a) \cup (b)$. Let a, b belong to distinct quasicomponents of M. Let f be a continuous mapping of $M \cup C$ into $\mathbf{S}_1$. Let the partial mapping f_M be inessential. Then f is inessential.

24.7. Let $M \subset P$, $a \in M$, $b \in M$, $C \subset P$. Let C be a simple arc with end points a, b. Let $C \cap \overline{M} = (a) \cup (b)$. Let a, b belong to the same quasicomponent of M. Let g be a continuous mapping of M into $\mathbf{S}_1$. Then there exists an essential continuous mapping f of $M \cup C$ into $\mathbf{S}_1$ such that $f_M = g$.

24.8.* Complete the proof of theorem 24.2.18.

§ 25. Unicoherence

25.1. A metric space P is said to be *unicoherent* if [1] P is connected, [2] if $P = A \cup B$ with closed connected summands, then $A \cap B$ is connected.

25.1.1. *Let $P \neq \emptyset$ be a locally connected space. P is unicoherent if and only if it has the following property: If $C \subset P$ is closed and connected and if K is a component of $P - C$, then the set $B(K)$ is connected.*

25.1.2. *Let $P \neq \emptyset$ be a locally connected space. P is unicoherent if and only if it has the following property: If $Q \subset P$ is an irreducible cut of P between points a, b, then the set Q is connected.*

Proof: I. Let $P \neq \emptyset$ be a locally connected space. Let $\mathbf{U}$ designate unicoherence, $\mathbf{V}$ the property from theorem 25.1.1 and $\mathbf{W}$ the property from theorem 25.1.2. Evidently it suffices to prove the three implications: $\mathbf{U} \Rightarrow \mathbf{V}$, $\mathbf{V} \Rightarrow \mathbf{W}$, $\mathbf{W} \Rightarrow \mathbf{U}$.

II. Let $\mathbf{U}$ hold. Let $C \subset P$ be closed and connected. Let K be a component of $P - C$. By 22.1.13, $P - K$ is connected. By 18.1.6 the set $\overline{K}$ is connected. As $P = \overline{K} \cup (P - K)$ and as $\mathbf{U}$ holds, $\overline{K} \cap (P - K) = \overline{K} - K$ is also connected, since $P - K$ is closed by 22.1.4. By 10.3.2 and 22.1.4, $\overline{K} - K = B(K)$. Thus, $\mathbf{V}$ holds.

III. Let $\mathbf{V}$ hold. Let $Q \subset P$ be an irreducible cut of P between points a, b. By 22.1.10 there exist two distinct connected sets G_1, G_2 such that

$$a \in G_1, \qquad b \in G_2, \qquad G_1 \cup G_2 \subset P - Q, \qquad B(G_1) = B(G_2) = Q.$$

The set Q is closed by 10.3.1 (or by 18.5.4). By 22.1.9, G_1, G_2 are components of $P - Q$ so that $G_1 \cap G_2 = \emptyset$. The sets G_1, G_2 are open by 22.1.4, so that $\overline{G}_1 \cap G_2 = \emptyset$ by 10.2.6. The set $\overline{G}_1$ is closed and by 18.1.6 connected. The set G_2 is connected and $B(G_2) = B(G_1) \subset \overline{G}_1$, while $G_2 \subset P - \overline{G}_1$. Thus, by 22.1.9, G_2 is a component of $P - \overline{G}_1$ so that, by $\mathbf{V}$, $B(G_2) = Q$ is connected. Thus, $\mathbf{W}$ holds.

IV. Let $\mathbf{W}$ hold. If P were not connected, we would have $P = A \cup B$ with non-void separated summands. For $a \in A$, $b \in B$ the set $\emptyset$ would be an irreducible cut between the points a and b. This is impossible, since $\mathbf{W}$ holds. Thus, P is connected.

Let $P = A \cup B$ with closed connected summands. We have to prove that the closed set $A \cap B$ is connected. Let us assume the contrary. As P is connected, we have $A \cap B \neq \emptyset$. Hence, $A \cap B = H \cup K$ with non-void separated summands. As $A \cap B$ is closed, H and K are also closed. Moreover, $H \cap K = \emptyset$. Choose $a \in H$, $b \in K$. Then the set $P - (A \cap B)$ separates the point a from the point b in P. By 22.1.12 there is an irreducible cut $S \subset P - (A \cap B)$ of P between the points a, b. By $\mathbf{W}$ the set S is connected. Since A, B are closed, $A - (A \cap B)$, $B - (A \cap B)$ are evidently separated. On the other hand, $S \subset P - (A \cap B) = [A - (A \cap B)] \cup [B - (A \cap B)]$, so that, by 18.1.2, we have either $A \cap S = \emptyset$ or $B \cap S = \emptyset$. Since S is an irreducible cut of P between the points a, b, S separates a from b in P, i.e. the set $P - S$ is not connected between the points a, b so that (see 18.3.3) $M \cap S \neq \emptyset$ for every connected $M \subset P$ containing both the points a, b. On the other hand, $a \in H$, $b \in K$, $H \cup K = A \cap B$. Thus, each of the connected sets A, B contains both points a, b. Hence, $A \cap S \neq \emptyset \neq B \cap S$, which is a contradiction.

25.2. 25.2.1. *Let P be a connected space. Let every continuous mapping of P into $\mathbf{S}_1$ be inessential. Then P is unicoherent.*

Proof: Let us assume the contrary. Then there are closed connected sets A, B such that $P = A \cup B$ and $A \cap B$ is not connected. Since P is connected, $A \cap B \neq \emptyset$. Since $A \cap B \neq \emptyset$ is closed and not connected, there are disjoint closed sets $H \neq \emptyset$, $K \neq \emptyset$ with $A \cap B = H \cup K$.

Define a mapping f of P into $\mathbf{S}_1$ as follows:*)

$$f(x) = \exp\left(i\pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)}\right) \quad \text{for} \quad x \in A,$$

$$f(x) = \exp\left(-i\pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)}\right) \quad \text{for} \quad x \in B.$$

For $x \in A \cap B = H \cup K$ we have formally two definitions of $f(x)$. Both of them, however, give $f(x) = 1$ for $x \in H$ and $f(x) = -1$ for $x \in K$.

The mapping f is evidently continuous. Thus, f is inessential, i.e., there exists a continuous mapping φ of P into $\mathbf{E}_1$ such that $e^{i\varphi(x)} = f(x)$ for every $x \in P$. We have

$$\exp\left(i\pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)}\right) = e^{i\varphi(x)} \quad \text{for} \quad x \in A,$$

$$\exp\left(-i\pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)}\right) = e^{i\varphi(x)} \quad \text{for} \quad x \in B,$$

and the sets A, B are connected. Hence, by 24.2.11 there are integers m, n such that

$$\varphi(x) = \pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)} + 2m\pi \quad \text{for} \quad x \in A,$$

$$\varphi(x) = -\pi \frac{\varrho(x, H)}{\varrho(x, H) + \varrho(x, K)} + 2n\pi \quad \text{for} \quad x \in B.$$

Let us choose $a \in H$, $b \in K$. We have $a \in A \cap B$, $b \in A \cap B$, so that

$$\varphi(a) = 2m\pi = 2n\pi,$$
$$\varphi(b) = \pi + 2m\pi = -\pi + 2n\pi,$$

which is a contradiction.

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*) Since H, K are closed and since $H \neq \emptyset \neq K$, $H \cap K = \emptyset$, we have $\varrho(x, H) + \varrho(x, K) > 0$ for every $x \in P$.

25.2.2. *Let P be a locally compact unicoherent space. Then every continuous mapping f of P into $\mathbf{S}_1$ is inessential.*

Proof: I. Put

$$\text{Real}\,(a + bi) = a, \qquad \text{Im}\,(a + bi) = b.$$

Define point sets Q_1, Q_2, Q_3, Q_4 as follows. If $x \in P$, then

$$x \in Q_1 \Leftrightarrow \text{Real}\, f(x) > 0, \qquad x \in Q_2 \Leftrightarrow \text{Real}\, f(x) < 0,$$
$$x \in Q_3 \Leftrightarrow \text{Im}\, f(x) > 0, \qquad x \in Q_4 \Leftrightarrow \text{Im}\, f(x) < 0.$$

We have $P = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ and, by 9.2, Q_λ $(\lambda = 1, 2, 3, 4)$ are open sets.

II. For $1 \leqq \lambda \leqq 4$ choose $M_\lambda \subset Q_\lambda$ such that M_λ contains exactly one point of every component of Q_λ. It is easy to prove (see ex. 25.5) that (with the exception of the trivial case with a one-point P) we may assume that the sets M_λ $(\lambda = 1, 2, 3, 4)$ are disjoint. For every $x \in M_\lambda$ let $V(x)$ be the component of Q_λ containing the point x. The sets $V(x)$ are connected and, by 22.1.4, open. Moreover

$$\bigcup_{x \in M_\lambda} V(x) = Q_\lambda$$

with disjoint summands.

Put $M = M_1 \cup M_2 \cup M_3 \cup M_4$.

III. Let $x' \in M$, $x'' \in M$, $x' \neq x''$, $V(x') \cap V(x'') \neq \emptyset$. Evidently $x' \in M_\lambda$, $x'' \in M_\mu$ where the couple (λ, μ) is one of the following eight ones

$$(1, 3),\ (3, 1),\ (1, 4),\ (4, 1),\ (2, 3),\ (3, 2),\ (2, 4),\ (4, 2).$$

IV. Let $\{x_r\}_1^m$ be a finite sequence such that [1] $x_r \in M$ for $1 \leqq r \leqq m$, [2] if $1 \leqq r < s \leqq m$, then $V(x_r) \cap V(x_s) \neq \emptyset$ if and only if either $s = r + 1$ or $r = 1$, $s = m$. Then there is an index λ $(1 \leqq \lambda \leqq 4)$ such that $x_r \in M_\lambda$ for no r $(1 \leqq r \leqq m)$.

Let us assume the contrary, so that $m \geqq 4$. Put $x_0 = x_m$, $x_{m+1} = x_1$. It follows easily by III that there exists an index s $(1 \leqq s \leqq m)$ such that

$$x_{s-1} \in M_\lambda, \qquad x_s \in M_\mu, \qquad x_{s+1} \in M_\nu,$$

where the triple (λ, μ, ν) is one of the following

$$(3, 1, 4),\ (4, 1, 3),\ (3, 2, 4),\ (4, 2, 3).$$

All four cases lead to a contradiction in the same way. Hence, it suffices to treat, one of them. E.g. let

$$x_{s-1} \in M_3, \qquad x_s \in M_1, \qquad x_{s+1} \in M_4.$$

By the assumption there is an index t $(1 \leqq t \leqq m)$ such that $x_t \in M_2$.

We have $x_s \in V(x_s)$. Since $x_s \in M_1$, $y \in V(x_s)$ implies $\text{Real} f(y) > 0$, so that $y \in \overline{V(x_s)}$ implies $\text{Real} f(y) \geqq 0$, while $x_t \in M_2$, so that $\text{Real} f(x_t) < 0$. Thus, $x_t \in$

$\in P - \overline{V(x_s)}$, so that, by 18.5.3, $B[V(x_s)]$ separates the point x_s from the point x_t in P. By 22.1.12 and 25.1.2 there exists a connected set $S \subset B[V(x_s)]$, which separates the point x_s from the point x_t in P. Put

$$W_1 = \bigcup_r V(x_r) \qquad (1 \leqq r \leqq m, \quad s - 1 \neq r \neq m + s - 1),$$

$$W_2 = \bigcup_r V(x_r) \qquad (1 \leqq r \leqq m, \quad s + 1 \neq r \neq s + 1 - m).$$

Among the summands of the first union are the sets $V(x_s)$, $V(x_{s+1})$; for every other summand $V(x_r)$ of this union we have $V(x_r) \cap V(x_s) = \emptyset$ and hence (see 10.2.6) $V(x_r) \cap \overline{V(x_s)} = \emptyset$. On the other hand, $S \subset B[V(x_s)] = \overline{V(x_s)} - V(x_s)$ (see 10.3.2). Thus, $S \cap W_1 = S \cap V(x_{s+1})$, and we may deduce similarly that $S \cap W_2 = S \cap \cap V(x_{s-1})$. By 18.1.4 we see easily that the sets W_1, W_2 are connected; moreover, $x_s \in W_1 \cap W_2$, $x_t \in W_1 \cap W_2$. As S separates the point x_s from the point x_t in P, the set $P - S$ is not connected between the points x_s, x_t, so that, by 18.3.3, $S \cap W_1 \neq \neq \emptyset \neq S \cap W_2$, i.e.

$$S \cap V(x_{s-1}) \neq \emptyset \neq S \cap V(x_{s+1}). \tag{1}$$

Since $S \subset \overline{V(x_s)}$, we have $\text{Real} f(y) \geqq 0$ for $y \in S$. By 22.1.9, however, $S \subset B[V(x_s)] \subset \subset P - Q_1$, i.e., $\text{Real} f(y) \leqq 0$ for $y \in S$. Hence, $\text{Real} f(y) = 0$ for $y \in S$, i.e. $f(y) = = \pm i$ for $y \in S$. As $x_{s-1} \in M_3$, $x_{s+1} \in M_4$, we have $\text{Im} f(y) > 0$ for $y \in V(x_{s-1})$, $\text{Im} f(y) < 0$ for $y \in V(x_{s+1})$. Thus [see (1)], $f(S) = (i) + (-i)$, so that $f(S)$ is not connected. This is a contradiction (see 18.1.10).

V. By 24.1.2 there exists a homeomorphic mapping v of $\mathbf{S}_1 - (-1)$ onto the interval $\underset{t}{\mathrm{E}}[-\pi < t < \pi]$ such that $e^{iv(z)} = z$ for every $z \in \mathbf{S}_1 - (-1)$. Evidently, $v(z^{-1}) = -v(z)$ for every $z \in \mathbf{S}_1 - (-1)$.

If $x \in M$, $y' \in V(x)$, $y'' \in V(x)$, we have obviously $f(y') + f(y'') \neq 0$, so that there exists a number

$$v\left(\frac{f(y'')}{f(y')}\right).$$

VI. Let $\{x_r\}_1^m$, $\{y_r\}_1^m$ be finite sequences ($m \geqq 2$) such that [1] $x_r \in M$ for $1 \leqq \leqq r \leqq m$, [2] $y_r \in V(x_r)$ for $1 \leqq r \leqq m$, $y_{r+1} \in V(x_r)$ for $1 \leqq r \leqq m - 1$, $y_1 \in V(x_m)$. Then we have

$$\sum_{r=1}^{m-1} v\left(\frac{f(y_{r+1})}{f(y_r)}\right) = v\left(\frac{f(y_m)}{f(y_1)}\right). \tag{1}$$

This statement is evident for $m = 2$. Hence, let $m \geqq 3$. It suffices to prove it under the assumption (denote it by **H**) that equations analogous to (1) in which m is replaced by a number less than m, are valid. Consider two cases.

First case. There exist indices h, k such that $V(x_h) \cap V(x_k) \neq \emptyset$, $1 \leqq h < k \leqq m$, and neither $k = h + 1$ nor $(h, k) = (1, m)$. Obviously $m \geqq 4$. Choose a point $z \in$

$\in V(x_h) \cap V(x_k)$. Then we obtain, by assumption **H**, the following four equations

$$\sum_{r=1}^{h-1} v\left(\frac{f(y_{r+1})}{f(y_r)}\right) + v\left(\frac{f(z)}{f(y_h)}\right) + v\left(\frac{f(y_{k+1})}{f(z)}\right) + \sum_{r=k+1}^{m-1} v\left(\frac{f(y_{r+1})}{f(y_r)}\right) = v\left(\frac{f(y_m)}{f(y_1)}\right),$$

$$\sum_{r=k+1}^{k-1} v\left(\frac{f(y_{r+1})}{f(y_r)}\right) + v\left(\frac{f(z)}{f(y_k)}\right) = v\left(\frac{f(z)}{f(y_{h+1})}\right),$$

$$v\left(\frac{f(y_h)}{f(z)}\right) + v\left(\frac{f(y_{h+1})}{f(y_h)}\right) = v\left(\frac{f(y_{h+1})}{f(z)}\right),$$

$$v\left(\frac{f(y_k)}{f(z)}\right) + v\left(\frac{f(y_{k+1})}{f(y_k)}\right) = v\left(\frac{f(y_{k+1})}{f(z)}\right).$$

We obtain (1) by adding them, since $v(u^{-1}) = -v(u)$ for every $u \in \mathbf{S}_1$.

Second case. If $1 \leqq r < s \leqq m$, $V(x_r) \cap V(x_s) \neq \emptyset$, we have either $s = r + 1$, or $(r, s) = (1, m)$. By IV there is an index λ $(1 \leqq \lambda \leqq 4)$ such that $x_r \in M_\lambda$ for no r $(1 \leqq r \leqq m)$. Obviously

$$\mathbf{S}_1 - f[\bigcup_{r=1}^{m} V(x_r)] \neq \emptyset,$$

so that by 24.2.7 there exists a continuous mapping φ of $W = \bigcup_{r=1}^{m} V(x_r)$ into $\mathbf{E}_1$ such that $e^{i\varphi(y)} = f(y)$ for every $y \in W$. If $e^{i\beta_r} = f(y_r)$ $(1 \leqq r \leqq m)$, then

$$e^{i\varphi(y)} = \exp\left\{i\left[\beta_r + v\left(\frac{f(y)}{f(y_r)}\right)\right]\right\} \quad \text{for} \quad y \in V(x_r),$$

so that, by 24.2.11, there are integers k_r $(1 \leqq r \leqq m)$ such that

$$\varphi(y) = \beta_r + v\left(\frac{f(y)}{f(y_r)}\right) + 2k_r\pi \quad \text{for} \quad y \in V(x_r).$$

Hence

$$v\left(\frac{f(y_{r+1})}{f(y_r)}\right) = \varphi(y_{r+1}) - \varphi(y_r) \qquad (1 \leqq r \leqq m - 1),$$

$$v\left(\frac{f(y_m)}{f(y_1)}\right) = \varphi(y_m) - \varphi(y_1),$$

which yields (1).

VII. Choose a fixed $a \in P$ and $\alpha \in \mathbf{E}_1$ such that $e^{i\alpha} = f(a)$. For every $y \in P$ there are, by 18.4.2, finite sequences $\{x_r\}_1^m$, $\{y_r\}_0^m$ such that [1] $y_0 = a$, $y_m = y$, [2] $x_r \in M$ for $1 \leqq r \leqq m$, [3] $y_{r-1} \in V(x_r)$, $y_r \in V(x_r)$ for $1 \leqq r \leqq m$. Put (see V)

$$\psi(y) = \alpha + \sum_{r=1}^{m} v\left(\frac{f(y_r)}{f(y_{r-1})}\right), \tag{2}$$

We shall show later that the number $\psi(y)$ is uniquely determined for every $y \in P$. Thus, ψ is a mapping of P into $\mathbf{E}_1$. Evidently $e^{i\psi(y)} = f(y)$ for every $y \in P$. We have to prove that the mapping ψ is continuous. For a given y and given sequences $\{x_r\}_1^m$, $\{y_r\}_0^m$, $V(x_m)$ is a neighborhood of y. Replacing the point y by a point $y' \in V(x_m)$, we may preserve the points x_r $(1 \leqq r \leqq m)$, y_r $(0 \leqq r \leqq m-1)$ and take $y_m = y'$ instead of $y_m = y$. Formula (2) yields

$$\psi(y') - \psi(y) = v\left(\frac{f(y')}{f(y_{m-1})}\right) - v\left(\frac{f(y)}{f(y_{m-1})}\right) \quad \text{for} \quad y' \in V(x_m).$$

As $V(x_m)$ is a neighborhood of the point y, ψ is continuous at the point y.

It remains to prove that the number $\psi(y)$ is, for a given $y \in P$, uniquely determined. Replace the sequences $\{x_r\}_1^m$, $\{y_r\}_0^m$ by other similar sequences $\{x'_r\}_1^n$, $\{y'_r\}_0^n$. We have to prove that

$$\sum_{r=1}^{m} v\left(\frac{f(y_r)}{f(y_{r-1})}\right) = \sum_{r=1}^{n} v\left(\frac{f(y'_r)}{f(y'_{r-1})}\right) = -\sum_{r=1}^{n} v\left(\frac{f(y'_{r-1})}{f(y'_r)}\right).$$

Put $x_{m+r} = x'_{n-r+1}$ for $1 \leqq r \leqq n$, $y_{m+r} = y'_{n-r}$ for $1 \leqq r \leqq n$. We have then [1] $y_0 = y_{m+n} = a$; [2] $x_r \in M$ for $1 \leqq r \leqq m+n$, [3] $y_{r-1} \in V(x_r)$, $y_r \in V(x_r)$ for $1 \leqq r \leqq m+n$ and we have to prove that

$$\sum_{r=1}^{m+n} v\left(\frac{f(y_r)}{f(y_{r-1})}\right) = 0 = v\left(\frac{f(a)}{f(a)}\right) = v\left(\frac{f(y_{m+n})}{f(y_0)}\right).$$

This follows by VI.

25.2.3. *The euclidean space* $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ *is unicoherent.*

This follows by 19.2.4, 24.5.3 and 25.2.1.

25.2.4. *The spherical spaces* $\mathbf{S}_0$, $\mathbf{S}_1$ *are not unicoherent. The spherical spaces* $\mathbf{S}_m$ $(m = 2, 3, 4, \ldots)$ *are unicoherent.*

Proof: I. $\mathbf{S}_0$ is not connected, hence, it is not unicoherent. $\mathbf{S}_1$ is a simple loop, hence (see 20.1.1 and 21.1.2), $\mathbf{S}_1$ is a union of two continua, whose intersection is not connected, so that $\mathbf{S}_1$ is not unicoherent.

II. Let $m \geqq 2$. The space $\mathbf{S}_m$ is connected by 19.2.5. Thus, $\mathbf{S}_m$ is unicoherent by 24.5.4 and 25.2.1.

25.2.5. *Let P, Q be locally connected unicoherent spaces. Then the space $P \times Q$ is unicoherent.*

Proof: The spaces P, Q are connected, so that $P \times Q$ is connected by 18.1.13. Hence, by 25.2.1, it suffices to prove that every continuous mapping of $P \times Q$ into $\mathbf{S}_1$ is inessential. This follows by 24.4.2 and 25.2.2.

Exercises

The spaces $P_1, P_2, \ldots, P_9$ were defined in exercises to § 19.

25.1. The spaces P_2, P_3, P_4, P_5, P_7, P_8 are unicoherent.

25.2. The spaces P_1, P_6, P_9 are not unicoherent.

25.3. Let $P \subset \mathbf{E}_2$ be the space consisting of all (x, y) such that $x^2 + y^2 = 1$ and of all (x, y) of the form $x = (1 + t^{-1}) \cos t$, $y = (1 + t^{-1}) \sin t$, $t > 1$. Then P is a unicoherent space.

25.4. We cannot omit in theorem 25.2.2 the assumption that P is a locally connected space.

25.5.* Prove that the sets M_λ $(\lambda = 1, 2, 3, 4)$ in part II of the proof of theorem 25.2.2 may be found disjoint.