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ON A SPACE OF FUNCTIONS REPRESENTABLE  
BY DERIVATIVES

The present note was inspired by the original version of [1] where the first three authors proved that

(A) the characteristic function of a closed set of real numbers can be expressed as the product of two derivatives.

This assertion, however, played only an auxiliary role and is not contained in the final version of [1]. Various generalizations of (A) will appear elsewhere; some of the corresponding results have been mentioned in [2], part IV.

This note deals with the space  $\mathcal{A}_0$  of bounded functions on the interval  $[0,1]$  which have only a finite number of discontinuities and with its uniform closure  $\mathcal{A}$ . It is easy to see that  $\mathcal{A}$  is an algebra containing all monotone functions. The main result says, in particular, that for each  $f \in \mathcal{A}$  there are derivatives  $f_j \in \mathcal{A}$  such that  $f = f_1 + f_2 f_3$ .

Notation. 1° The word function means a mapping to  $R = (-\infty, \infty)$ . For each nondegenerate interval  $J \subset R$  let

$$\mathcal{A}(J) = \{f'; f \text{ is a function differentiable on } J\}.$$

2° Let  $f$  be a bounded function on a nondegenerate interval  $J$ . For each  $x \in J$  and each  $h \in (0, \infty)$  set

$$w^*(f, x, h) = \sup f(M), \quad w_*(f, x, h) = \inf f(M) \quad (M = J \cap [x-h, x+h]),$$

$$w^*(f, x) = \lim w^*(f, x, h), \quad w_*(f, x) = \lim w_*(f, x, h) \quad (h \searrow 0),$$

$$w_f(x) = w^*(f, x) - w_*(f, x), \quad \sigma_f(x) = \frac{1}{2}(w^*(f, x) + w_*(f, x)).$$

Lemma 1. Let  $f, g$  be bounded functions on  $R$ . Then  $w_{f+g} \leq w_f + w_g$ . If  $f$  is continuous at a point  $x$ , then  $w_{fg}(x) = |f(x)| \cdot w_g(x)$ .

(The proof is left to the reader.)

Lemma 2. Let  $J = (0, \infty)$  and let  $\delta \in J$ . Let  $f$  be a nonnegative function continuous on  $J$ . Then there are functions  $\gamma, \varphi \in \mathcal{A}([0, \infty))$  which are continuous on  $J$  such that  $\gamma(0) = \varphi(0) = 0$ ,  $\varphi = 0$  on  $(\delta, \infty)$  and  $\gamma \geq 0$ ,  $f = \gamma + \varphi^2$  on  $J$ .

Proof. Choose numbers  $y_n$  such that  $\delta = y_0 > y_1 > \dots$ ,  $y_n \rightarrow 0$ ,  $y_{n-1}/y_n \rightarrow 1$ . Set  $J_n = [y_n, y_{n-1}]$ ,  $M_n = \max\{f(x); x \in J_n\}$ . For  $n = 1, 2, \dots$  find a  $\delta_n \in J$  such that  $\delta_n M_n^{1/2} < y_n/n$  and that  $|f(y) - f(x)| < 1/n$ , whenever  $x, y \in J_n$  and  $|y - x| < \delta_n$ ; further choose an integer  $r_n$  such that  $y_{n-1} - y_n < r_n \delta_n$ . Let

$$S = \{y_n + k(y_{n-1} - y_n)/r_n; n = 1, 2, \dots, k = 1, \dots, r_n\}.$$

We have  $S = \{x_0, x_1, \dots\}$ , where  $\delta = x_0 > x_1 > \dots$ . For  $k = 1, 2, \dots$  set  $L_k = [x_k, x_{k-1}]$ ,  $d_k = x_{k-1} - x_k$ ,  $w_k = \max\{|f(y) - f(x)|; x, y \in L_k\}$ ,  $m_k = \min\{f(x); x \in L_k\}$ ,  $s_k = \frac{1}{2}(x_k + x_{k-1})$ . It is easy to construct a function  $\varphi$  continuous on  $L_k$  such that  $\varphi \geq 0$  on  $[x_k, s_k]$ ,  $\varphi \leq 0$  on  $[s_k, x_{k-1}]$ ,  $\varphi = 0$  on  $\{x_k, x_{k-1}\}$ ,  $\varphi^2 \leq m_k$  on  $L_k$ ,  $\int_{L_k} \varphi = 0$ ,  $\int_{L_k} \varphi^2 > m_k d_k - 2^{-k} x_k$ . In this way

we define a function  $\varphi$  on  $(0, \delta]$ . Further we set  $\varphi = 0$  on  $\{0\} \cup (\delta, \infty)$ ,  $\gamma = f - \varphi^2$  on  $J$ ,  $\gamma(0) = 0$ .

Then  $\varphi, \gamma$  are continuous on  $J$ . Obviously  $\gamma \geq 0$ ,

$\int_{L_k} \gamma < w_k d_k + 2^{-k} x_k$ . Let  $x \in (0, \delta]$ . There are  $k, n$

such that  $y_n \leq x_k < x \leq x_{k-1} \leq y_{n-1}$ . It follows from

the choice of  $r_n, r_{n+1}, \dots$  that  $w_j < 1/n$

( $j = k, k+1, \dots$ ). Therefore  $\int_0^x \gamma \leq \sum_{j=k}^{\infty} (w_j d_j + 2^{-j} x_j) \leq$

$x_{k-1}/n + 2^{-k+1} x_k < x(y_{n-1}/(ny_n) + 2^{-k+1})$ ,  $0 \leq \int_0^x \varphi =$

$$= \int_{x_k}^x \varphi \leq \frac{1}{2} d_k m_k^{1/2} \leq \delta_n M_n^{1/2} < Y_n/n < x/n. \text{ For } x \rightarrow 0$$

we have  $k \rightarrow \infty$ ,  $n \rightarrow \infty$  so that  $x^{-1} \int_0^x \gamma \rightarrow 0$ ,

$x^{-1} \int_0^x \varphi \rightarrow 0$ . This completes the proof.

Lemma 3. Let  $\delta \in (0, \infty)$ . Let  $f$  be a function bounded on  $R$  and continuous on  $R \setminus \{0\}$ . Then there are functions  $\gamma, \varphi, \psi \in \mathcal{B}(R)$  which are continuous on  $R \setminus \{0\}$  such that  $\varphi = \psi = 0$  on  $R \setminus (-\delta, \delta)$ ,

$$(1) \quad f = \gamma + \varphi\psi, \quad f \wedge f(0) \leq \gamma \leq f \vee f(0), \quad \varphi^2 = \psi^2 \leq |f - f(0)|,$$

$$(2) \quad \omega_\gamma \leq \omega_f, \quad \omega_\varphi \vee \omega_\psi \leq 2 \omega_f^{1/2}.$$

Proof. We may suppose that  $f(0) = 0$ . Let  $f_1 = f \vee 0$ ,  $f_2 = (-f) \vee 0$ . By Lemma 2 there are functions  $\gamma_j, \varphi_j \in \mathcal{B}(R)$  which are continuous on  $R \setminus \{0\}$  such that  $\gamma_j(0) = 0$ ,  $\varphi_j = 0$  on  $R \setminus (-\delta, \delta)$ ,  $\gamma_j \geq 0$  and that  $f_j = \gamma_j + \varphi_j^2$  ( $j = 1, 2$ ). Set  $\gamma = \gamma_1 - \gamma_2$ ,  $\varphi = \varphi_1 + \varphi_2$ ,  $\psi = \varphi_1 - \varphi_2$ ,  $A = \{x \in R; f(x) \geq 0\}$ ,  $B = R \setminus A$ . On  $A$  we have  $\gamma_2 = \varphi_2 = 0$ ,  $0 \leq \gamma = \gamma_1 \leq f_1$ ,  $\varphi = \varphi_1 = \psi$ ,  $\varphi_1^2 \leq f_1 = |f|$ ; on  $B$  we have  $\gamma_1 = \varphi_1 = 0$ ,  $-f_2 \leq -\gamma_2 = \gamma \leq 0$ ,  $\varphi = \varphi_2 = -\psi$ ,  $\varphi_2^2 \leq f_2 = |f|$ . Obviously  $f = \gamma + \varphi\psi$ . This proves (1).

Now let  $\theta^*(t) = \omega^*(f, 0, |t|)$ ,  $\theta_*(t) = \omega_*(f, 0, |t|)$  ( $t \neq 0$ ),  $\theta^*(0) = \omega^*(f, 0)$ ,  $\theta_*(0) = \omega_*(f, 0)$ . It follows from (1) that  $\theta_* \leq \gamma \leq \theta^*$ ,  $\varphi^2 = \psi^2 \leq \theta^* - \theta_*$ . This easily implies (2).

Lemma 4. Let  $a, b \in \mathbb{R}$  and let  $A$  be a finite set of real numbers. Let  $f$  be a function on  $\mathbb{R}$  which is continuous on  $\mathbb{R} \setminus A$  such that  $a < f < b$ . Then there are functions  $\gamma, \varphi, \psi \in \mathcal{D}(\mathbb{R})$  such that

$$f = \gamma + \varphi\psi, \quad a < \gamma < b, \quad \varphi^2 = \psi^2 < b - a,$$

$$\omega_\gamma \leq \omega_f, \quad \omega_\varphi \vee \omega_\psi \leq 2 \omega_f^{1/2}.$$

(This follows easily from Lemma 3.)

Lemma 5. Let  $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ . Suppose that  $\varphi, \psi$  are bounded and that at each point of  $\mathbb{R}$  at least one of the functions  $\varphi, \psi$  is continuous. Then  $\varphi\psi \in \mathcal{D}(\mathbb{R})$ .

(The proof is left to the reader.)

Lemma 6. Let  $f$  be a bounded function on the interval  $J = [0, 1]$ ; let  $\eta \in (0, \infty)$  and let  $\omega_f < 2\eta$  on  $J$ . Then there is a function  $h$  continuous on  $J$  such that  $h(0) = \sigma_f(0)$ ,  $h(1) = \sigma_f(1)$  and  $|h - f| < \eta$ .

Proof. Let  $Y$  be the set of all points  $y \in (0, 1]$  with the following property:

(P) There is a function  $g$  continuous on  $[0, y]$  such that  $g(0) = \sigma_f(0)$  and  $|g - f| < \eta$  on  $[0, y]$ .

There is a  $y_1 \in (0, 1]$  such that  $|f - \sigma_f(0)| < \eta$  on  $[0, y_1]$ . Setting  $g = \sigma_f(0)$  on  $[0, y_1]$  we see that  $y_1 \in Y$ . Therefore  $s = \sup Y > 0$ . There are  $s_1, s_2$  such that  $0 < s_1 < s < s_2$  and that  $|f - \sigma_f(s)| < \eta$  on

$J \cap [s_1, s_2]$ . Choose a  $y \in Y \cap (s_1, s]$  and a function  $g$  as in (P). Let  $\lambda$  be a linear function such that  $\lambda(s_1) = 0$ ,  $\lambda(y) = 1$ . Define a function  $h$  on  $[0, s_2]$  setting  $h = g$  on  $[0, s_1]$ ,  $h = g + \lambda \cdot (\sigma_f(s) - g)$  on  $[s_1, y]$ ,  $h = \sigma_f(s)$  on  $[y, s_2]$ . Then  $h$  is continuous on  $[0, s_2]$ ,  $h(0) = \sigma_f(0)$  and  $|h - f| < \eta$  on  $[0, s_2] \cap J$ . This shows that  $s = 1$  and that our assertion holds.

Lemma 7. Let  $f$  be a bounded function on the interval  $J = (0, \infty)$ . Then  $w_f(0+) = 0$  if and only if there is a function  $h$  continuous on  $J$  such that  $(f - h)(0+) = 0$ .

Proof. If there is such an  $h$ , then, for each  $x \in J$ ,  $w_f(x) = w_{f-h}(x)$  and, obviously,  $w_{f-h}(0+) = 0$ . Now suppose that  $w_f(0+) = 0$ . Let  $x_n \in J$ ,  $x_0 > x_1 > \dots$ ,  $x_n \rightarrow 0$ . Let  $\eta_n = 1/n + \sup\{w_f(x); x_n \leq x \leq x_{n-1}\}$ . It follows from Lemma 6 that there is a function  $h$  continuous on  $J$  such that  $|f - h| < \eta_n$  on  $[x_n, x_{n-1}]$  ( $n = 1, 2, \dots$ ). It is easy to see that  $(f - h)(0+) = 0$ .

Lemma 8. Let  $f$  be a bounded function on  $R$ , let  $\eta \in (0, \infty)$  and let  $A$  be a finite set of real numbers. Suppose that  $w_f(t) \rightarrow 0$  ( $t \rightarrow x$ ) and  $w_f(x) < 2\eta$  for each  $x \in A$ . Then there is a function  $h$  on  $R$  which is continuous on  $R \setminus A$  such that  $|h| < \eta$  and that  $f - h$  is continuous at each point of  $A$ .

Proof. There are disjoint open intervals  $I_x$  such that  $x \in I_x$  and  $|f - \sigma_f(x)| < \eta$  on  $I_x$  ( $x \in A$ ). Let  $x \in A$ . By Lemma 7 there is a function  $h_x$  on  $R$  which is continuous on  $R \setminus \{x\}$  such that  $h_x(x) = f(x)$  and that  $f - h_x$  is continuous at  $x$ . There is a closed interval  $J_x$  containing  $x$  in its interior such that  $J_x \subset I_x$  and that  $|h_x - \sigma_f(x)| < \eta$  on  $J_x$ . Now it is easy to construct a function  $h$  on  $R$  which is continuous on  $R \setminus A$  such that  $|h| < \eta$  and that  $h = h_x - \sigma_f(x)$  on  $J_x$  for each  $x \in A$ . It is obvious that  $f - h$  is continuous at each point of  $A$ .

Notation. In the rest of this note,  $J$  will stand for  $[0,1]$ . Let  $\mathcal{a}_0$  be the system of all bounded functions  $f$  on  $J$  such that  $f$  is continuous on  $J \setminus A$  for some finite set  $A$ . Let  $\mathcal{a}$  be the system of all bounded functions  $f$  on  $J$  such that

$$(3) \quad \omega_f(t) \rightarrow 0 \quad (t \in J, t \rightarrow x) \quad \text{for each } x \in J.$$

Proposition 1. Let  $f$  be a bounded function on  $J$ . Then  $f \in \mathcal{a}$  if and only if the set  $\{x \in J; \omega_f(x) > \eta\}$  is finite for each  $\eta \in (0, \infty)$ .

(The proof is left to the reader.)

Proposition 2. Let  $f$  be a bounded function on  $J$ . Then  $f \in \mathcal{A}$  if and only if for each  $x \in J$  there is a function  $h$  on  $J$  which is continuous on  $J \setminus \{x\}$  such that  $f - h$  is continuous at  $x$ .

(This follows easily from Lemma 7.)

Proposition 3. Let  $f_n \in \mathcal{A}$  ( $n = 1, 2, \dots$ ) and let  $f_n \rightarrow f$  uniformly. Then  $f \in \mathcal{A}$ .

Proof. It is easy to see that  $\omega_{f_n} \rightarrow \omega_f$  uniformly. This implies (3).

Remark. Obviously  $\mathcal{A}_0 \subset \mathcal{A}$ . If  $f$  is a bounded function on  $J$  such that  $f(x+)$  exists for each  $x \in [0, 1)$  and that  $f(x-)$  exists for each  $x \in (0, 1]$ , then  $f \in \mathcal{A}$ . In particular, each function of bounded variation on  $J$  is in  $\mathcal{A}$ .

Lemma 9. Let  $f \in \mathcal{A}$ . Let  $\alpha_n \in (0, \infty)$ ,  $2\alpha_0 > \max\{\omega_f(x); x \in J\}$ ,  $\alpha_0 > \alpha_1 > \dots$ ,  $\sum_{n=0}^{\infty} \alpha_n < \infty$ . Let

$$A_n = \{x \in J; \alpha_{n+1} \leq \omega_f(x)/2 < \alpha_n\}.$$

Then there are functions  $g, h_0, h_1, \dots$  on  $J$  such that  $g$  is continuous on  $J$ ,  $h_n$  is continuous on  $J \setminus A_n$ ,  $f - h_n$  is continuous at each point of  $A_n$ ,

$$(4) \quad |h_n| < \alpha_n \quad (n = 0, 1, \dots) \quad \text{and} \quad f = g + \sum_{n=0}^{\infty} h_n.$$

Proof. For  $n = 0, 1, \dots$  let  $h_n$  be a function constructed according to Lemma 8 where we take  $A = A_n$ ,  $\eta = \alpha_n$ . It is easy to see that the function  $f - \sum_{n=0}^{\infty} h_n$  is continuous.

Proposition 4. Let  $f$  be a function on  $J$ . Then  $f \in \mathcal{A}$  if and only if there are  $f_n \in \mathcal{A}_0$  such that  $f_n \rightarrow f$  uniformly.

(This follows from Lemma 9 and Proposition 3.)

Proposition 5. Let  $f, g \in \mathcal{A}$ . Then  $f+g, fg \in \mathcal{A}$ .

Proof. If  $f, g \in \mathcal{A}_0$ , then, obviously,  $f+g, fg \in \mathcal{A}_0$ . Now we apply Proposition 4.

Theorem. Let  $f \in \mathcal{A}$ ,  $\epsilon \in (0, \infty)$ ,  $\epsilon > \max\{\omega_f(x); x \in J\}$ . Then there is a function  $g$  continuous on  $J$  and functions  $\lambda, \varphi, \psi \in \mathcal{A} \cap \mathcal{B}(J)$  such that

$$(5) \quad f = g + \lambda + \varphi\psi, \quad |f - g| < \epsilon/2, \quad |\lambda| < \epsilon/2, \quad \varphi^2 \vee \psi^2 < \epsilon,$$

$$(6) \quad \omega_\lambda \leq 5(\epsilon\omega_f)^{1/2}, \quad \omega_\varphi \vee \omega_\psi \leq 2\omega_f^{1/2}.$$

Proof. Set  $\eta = \epsilon/2$ ,  $\mathcal{B} = \mathcal{B}(J)$ . Choose a  $\beta_0 \in (0, \infty)$  such that  $\max\{\omega_f(x); x \in J\} < 2\beta_0^2 < \epsilon$  (hence  $\beta_0^2 < \eta$ ) and then a  $b \in (0, \frac{1}{2})$  such that

$$(7) \quad \beta_0^2 + b(2 + 4\beta_0) < \eta.$$

Further choose  $\beta_n \in (0, \infty)$  such that  $\beta_0 > \beta_1 > \dots$

and  $\sum_{n=1}^{\infty} \beta_n < b$ . Notice that

$$(8) \quad (\beta_0 + b)^2 < \eta .$$

Set  $\alpha_n = \beta_n^2$  and find sets  $A_n$  and functions  $g, h_n$  according to Lemma 9. It follows from (4) and (7) that

$$|f - g| < \sum_{n=0}^{\infty} \alpha_n < \alpha_0 + b < \eta .$$

By Lemma 4 there are  $\gamma_n, \varphi_n, \psi_n \in \mathcal{D}$  such that

$$(9) \quad h_n = \gamma_n + \varphi_n \psi_n ,$$

$$(10) \quad |\gamma_n| < \alpha_n, \quad |\varphi_n| = |\psi_n| < \beta_n \cdot 2^{1/2} ,$$

$$(11) \quad \omega_{\gamma_n} \leq \omega_{h_n}, \quad \omega_{\varphi_n} \vee \omega_{\psi_n} \leq 2 \omega_{h_n}^{1/2} .$$

Because of (10) we may define  $\gamma = \sum_{n=0}^{\infty} \gamma_n$ ,  
 $\varphi = \sum_{n=0}^{\infty} \varphi_n$ ,  $\psi = \sum_{n=0}^{\infty} \psi_n$ ,  $\rho = \varphi\psi - \sum_{n=0}^{\infty} \varphi_n \psi_n$ ; further set  
 $\lambda = \gamma - \rho$ . Obviously  $\gamma, \varphi, \psi \in \mathcal{D}$ . Since  $|\varphi_j \psi_k| \leq 2 \beta_j \beta_k$   
and, according to Lemma 5,  $\varphi_j \psi_k \in \mathcal{D}$  whenever  $j \neq k$ ,  
we have  $\rho \in \mathcal{D}$ . Therefore  $\lambda \in \mathcal{D}$ . By (4) and (9),

$$f = g + \gamma + \sum_{n=0}^{\infty} \varphi_n \psi_n = g + \lambda + \varphi\psi .$$

For each nonnegative integer  $p$  let  $S_p$  be the set of all pairs  $(j, k)$  of nonnegative integers such that  $j \neq p \neq k \neq j$  and let  $\sigma_p = \sum \varphi_j \psi_k$  ( $(j, k) \in S_p$ ). Obviously

$$(12) \quad |\sigma_0| < 2b^2 < b .$$

Set  $\gamma_p^* = \gamma - \gamma_p$ ,  $\varphi_p^* = \varphi - \varphi_p$ ,  $\psi_p^* = \psi - \psi_p$ . We have  
(see (10))

$$(13) \quad |\gamma| \vee |\gamma_p^*| < \alpha_0 + b ,$$

$$(14) \quad |\varphi_0^*| \vee |\psi_0^*| < b \cdot 2^{1/2}$$

and, by (8),

$$(15) \quad |\varphi| \vee |\varphi_p^*| \vee |\psi| \vee |\psi_p^*| < (\beta_0 + b) \cdot 2^{1/2} < \epsilon^{1/2} .$$

It is easy to show that

$$(16) \quad \rho = \varphi_p \psi_p^* + \psi_p \varphi_p^* + \sigma_p .$$

Taking  $p = 0$  we get, according to (10), (14) and (12),  
 $|\rho| < 2 \cdot 2 \beta_0 b + b$  so that, by (13) and (7),  $|\lambda| <$   
 $\alpha_0 + b + 4\beta_0 b + b < \eta$ . This proves (5).

Now let  $x \in J$ . If  $f$  is continuous at  $x$ , then  
all the functions  $h_n, \gamma_n, \dots, \lambda, \varphi, \psi$  are continuous at  
 $x$ . Otherwise we choose a  $p$  such that  $x \in A_p$ . Then  
the functions  $\gamma_p^*, \varphi_p^*, \psi_p^*, \sigma_p$  are continuous at  $x$ .  
Since  $f - h_p$  is continuous at  $x$ , we have  $\omega_{h_p}(x) = \omega_f(x)$ .  
Now it follows from (11) that  $\omega_\gamma(x) = \omega_{\gamma_p}(x) \leq \omega_f(x) \leq$   
 $(\epsilon \omega_f(x))^{1/2}$ ,  $\omega_\varphi(x) = \omega_{\varphi_p}(x) \leq 2(\omega_f(x))^{1/2}$ ,  $\omega_\psi(x) = \omega_{\psi_p}(x)$   
 $\leq 2(\omega_f(x))^{1/2}$ . By (16), Lemma 1 and (15) we get

$$\omega_\rho(x) \leq \omega_{\varphi_P}(x) \cdot |\psi_P^*(x)| + \omega_{\psi_P}(x) \cdot |\varphi_P^*(x)| \leq 4(\omega_f(x))^{1/2} \cdot \epsilon^{1/2}.$$

Therefore  $\omega_\lambda(x) \leq 5(\epsilon\omega_f(x))^{1/2}$ . This proves (6) which implies that  $\lambda, \varphi, \psi \in \mathcal{A}$ .

Remark. It is natural to ask whether each function  $f \in \mathcal{A}$  can be expressed as  $\gamma + \varphi\psi$ , where  $\gamma, \varphi, \psi \in \mathcal{D} = \mathcal{D}(J)$  and where at least one of these three functions is continuous. The answer to this question is contained in Proposition 6. We need four lemmas.

Lemma 10. Let  $a, b, c \in \mathbb{R}$ ,  $0 \leq a < b \leq 1$ ,  $c > 0$ . Let  $\varphi, \psi \in \mathcal{D}$  and let  $\varphi\psi \geq c$  on  $(a, b)$ . Then  $\varphi(a)\psi(a) \geq c$ .

Proof. For each  $x \in (a, b)$  we have  $(x-a)^2 \cdot c \leq (\int_L (\varphi\psi)^{1/2})^2 \leq \int_L |\varphi| \cdot \int_L |\psi| = \int_L \varphi \cdot \int_L \psi$  ( $L = (a, x)$ ).

Lemma 11. Let  $a \in (0, 1)$ . Let  $\gamma$  be a function on  $J$  which is continuous at  $a$  and let  $\varphi, \psi \in \mathcal{D}$ . Define  $f = \gamma + \varphi\psi$ ,

$$(17) \quad \Lambda_+ = \limsup f(x), \quad \lambda_+ = \liminf f(x) \quad (x \searrow a),$$

$$(18) \quad \Lambda_- = \limsup f(x), \quad \lambda_- = \liminf f(x) \quad (x \nearrow a),$$

$$(19) \quad M = \max(\lambda_+, \lambda_-), \quad m = \min(\Lambda_+, \Lambda_-).$$

1° If  $\gamma(a) < M$ , then  $f(a) \geq M$ .

2° If  $\gamma(a) > m$ , then  $f(a) \leq m$ .

Proof. Suppose that  $\gamma(a) < M$ . Let, e.g.,  $M = \lambda_+$ . Choose numbers  $p, q$  such that  $\gamma(a) < p < q < M$ . There is a  $b \in (a, 1)$  such that  $\gamma < p, q < f$  on  $(a, b)$ . Then  $\varphi\psi = f - \gamma > q - p$  on  $(a, b)$ . By Lemma 10 we have  $(\varphi\psi)(a) \geq q - p$  whence  $(\varphi\psi)(a) \geq M - \gamma(a)$ ,  $f(a) \geq M$ . This proves 1° which easily implies 2°.

Lemma 12. Let  $a \in (0, 1)$  and let  $f$  be a function on  $J$ . Define  $m, M$  by (17)-(19). If  $m < f(a) < M$ , then  $f$  cannot be expressed as  $\gamma + \varphi\psi$ , where  $\gamma$  is a function continuous at  $a$  and  $\varphi, \psi \in \mathcal{D}$ .

(This follows at once from Lemma 11.)

Lemma 13. Let  $\psi \in \mathcal{D}$  and let  $\varphi$  be a function continuous on  $J$ . Then there is an interval  $L \subset J$  such that the restriction of the function  $\varphi\psi$  to  $L$  is in  $\mathcal{A}(L)$ .

(This follows from [1], Theorem 4.)

Proposition 6. Let  $f$  be an increasing function on  $J$ . Suppose that the set  $\{x \in (0, 1); f(x-) < f(x+)\}$  is dense in  $J$  and that  $f(x-) < f(x) < f(x+)$  for some  $x \in (0, 1)$ . Let  $f = \gamma + \varphi\psi$ , where  $\gamma, \varphi, \psi \in \mathcal{D}$ . Then none of the functions  $\gamma, \varphi, \psi$  is continuous on  $J$ .

(This follows from Lemmas 12 and 13.)

## REFERENCES

- [1] S.J. Agronsky, R. Biskner, A.M. Bruckner and J. Mařík, Representations of functions by derivatives, *Trans. Amer. Math. Soc.* 263 (1981), 493-500.
- [2] A.M. Bruckner, Current trends in differentiation theory, *Real Analysis Exchange*, Vol. 5, No. 1 (1979-80), 9-60.

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