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TRANSFORMATION AND MULTIPLICATION OF DERIVATIVES

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ABSTRACT. The author investigates homeomorphisms h of the interval $[0,1]$ onto itself such that the composite function $f \circ h$ belongs to a certain class for each f belonging to another class. These classes consist of derivatives or approximately continuous functions. Then he investigates functions g such that the product fg is a derivative for each derivative f or each Lebesgue function f and shows a connection between families of such functions h and g.

0. INTRODUCTION. Let H be the family of all increasing homeomorphisms of the interval $[0,1]$ onto itself. A. M. Bruckner characterized in [1] the class of all $h \in H$ such that the composite function $f \circ h$ is approximately continuous for each approximately continuous f. In [2] he investigated homeomorphisms $h \in H$ such that $f \circ h \in D$ for each $f \in D$ for which f^2 also belongs to D, where D is the class of all derivatives. M. Laczkovich and G. Petruska characterized in [5] the smaller class of all $h \in H$ such that $f \circ h \in D$ for all $f \in D$. R. J. Fleissner described in [4] the system of all functions g such that $fg \in D$ for each $f \in D$. The present paper contains, among other things, improvements of some of the results obtained in the mentioned articles and also shows connections between these results.

The word "function" means, throughout the paper, a mapping of a subset of $R = (-\infty,\infty)$ to $R^* = R \cup \{-\infty,\infty\}$. A function whose range is in R is called a finite function. Multiplication in R^* is defined in the usual way; in particular, $a\infty = \infty$ for $a > 0$. The word "continuous" refers to the usual topology in R; thus, "continuous function" always means a finite function.

The letter J stands for the interval $[0,1]$. The system of all finite [bounded] derivatives on J is denoted by D [bD]. The

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system of all finite functions that are approximately continuous with respect to J at each point of J is denoted by C_{ap}; the meaning of bC_{ap} is obvious.

Symbols like $\int_a^b f$, $\int_a^b f(t)dt$ denote the corresponding Lebesgue or Perron integral. If $a,b \in R$, $a<b$ and if $K=[a,b]$, we write also $\int_K f$. The meaning of expressions "L-integrable," "P-integrable" is obvious.

The outer Lebesgue measure of a set $S \subset R$ is denoted by $|S|$. Words like "measurable" always refer to the Lebesgue measure in R.

A finite function f on J belongs to D if and only if $f(a) = \lim (x-a)^{-1} \int_a^x f$ $(x \to a, x \in J)$ for each $a \in J$. A finite function f on J is called a Lebesgue function if and only if $(x-a)^{-1} \int_a^x |f-f(a)| \to 0$ $(x \to a, x \in J)$ for each $a \in J$. The system of all Lebesgue functions is denoted by L. It is easy to see that L is a vector space, $bC_{ap} \subset L \subset D$ and that each Lebesgue function is L-integrable. The system of all functions $f \in D$ such that f^2 also belongs to D is denoted by W.

1. DERIVATIVES AND APPROXIMATELY CONTINUOUS FUNCTIONS. This part contains some simple assertions that will be used later. The proofs of 1.1 and 1.2 are left to the reader.

<u>1.1</u>. Let $f \in D$, $f \geq 0$, $c \in J$, $f(c) = 0$. Then f is approximately continuous at c with respect to J.

<u>1.2</u>. Let $f \in L$. Let g be a finite function on R such that $|g(y)-g(x)| \leq |y-x|$ for any $x,y \in R$. Then $g \circ f \in L$.

<u>1.3</u>. $L \subset C_{ap}$.

Proof. Let $c \in J$. By 1.2 we have $|f-f(c)| \in D$. Now we apply 1.1.

<u>1.4</u>. $W \subset L$. (See [2], Theorem 1.)

Proof. Let $w \in W$, $c \in J$, $g = w-w(c)$. It follows from the Cauchy inequality and from the relation $g^2 \in D$ that $((x-c)^{-1} \int_c^x |g|)^2 \leq (x-c)^{-1} \int_c^x g^2 \to 0$ $(x \to c, x \in J)$.

<u>1.5</u>. Let $f \in D$ and let $f \vee a \in D$ for each $a \in R$. Then $f \in L$.

Proof. Let $c \in J$, $a = f(c)$. Obviously $f + a = (f \vee a) + (f \wedge a)$ whence $f \wedge a \in D$. Since $|f-a| = (f \vee a) - (f \wedge a)$, we have $|f-a| \in D$. Thus $(x-c)^{-1} \int_c^x |f-f(c)| \to 0$ $(x \to c, x \in J)$, $f \in L$.

<u>1.6</u>. Let g be a finite function on J such that $g \vee b \in D$ for each $b \in R$. Then $g \in C_{ap}$.

Proof. It follows from 1.5 and 1.3 that $g \vee c \in C_{ap}$ for each $c \in R$. Hence $g \in C_{ap}$.

1.7. Let f be a finite function such that $(f\wedge a)\vee b \in D$ for any $a,b \in R$. Then $f \in C_{ap}$. (See [3], p.50, Theorem 4.1.)

(This follows from 1.6.)

1.8. Let $f,g \in C_{ap}$, $f \in D$ and $|g| \leq f$. Then $g \in L$.

Proof. We may suppose that $g \geq 0$. Let $c \in J$, $a = f(c)$, $f_1 = f\wedge a$, $g_1 = g\wedge a$, $f_2 = f - f_1$, $g_2 = g - g_1$. Then $f_2 = (f\vee a) - a$, $g_2 = (g\vee a) - a$. Since $f_1 \in bC_{ap}$, we have $f_1, f_2 \in D$. Obviously $0 \leq g_2 \leq f_2$, $(x-c)^{-1}\int_c^x g_2 \leq (x-c)^{-1}\int_c^x f_2 \to f_2(c) = 0$ $(x \to c,\ x \in J)$. Since $g_1 \in bC_{ap}$, we have $g_1 \in L$. Thus $g = g_1 + g_2 \in L$.

1.9. W is a vector space. If $f,g \in W$, then $fg \in L$.

Proof. Let $f,g \in W$. It follows from 1.4 and 1.3 that $W \subset C_{ap}$. Since $2|fg| \leq f^2 + g^2$, we have, by 1.8, $fg \in L$. Now it is obvious that $(f+g)^2 \in D$ which shows that W is a vector space.

1.10. Let $f \in L$, $f \geq 0$. Then $f^{1/2} \in W$.

Proof. Obviously $f^{1/2} < 1+f$. Now we apply 1.3 and 1.8.

1.11. Let $f \in L$. Then there are $v,w \in W$ such that $f = vw$.

Proof. By 1.2 and 1.10 there are $w_1, w_2 \in W$ such that $w_1^2 = f\vee 0$, $w_2^2 = (-f)\vee 0$. Set $v = w_1 + w_2$, $w = w_1 - w_2$. By 1.9 we have $v,w \in W$. Obviously $vw = f$.

2. INTEGRATION AND COMPOSITION. This part is connected with the problem of finding conditions under which a composite function is a derivative. Most of the results are of auxiliary character. However, sections 2.3 and 2.4 contain simple estimates of integrals and may be of independent interest.

Let $a,b \in R$, $a < b$ and let φ be a function on the interval $K = [a,b]$. If $\varphi(x) = \pm\infty$ for some $x \in K$, we set $\text{var}(\varphi,K) = \infty$. If $\varphi(K) \subset R$, we define, as usual, $\text{var}(\varphi,K)$ as the least upper bound of the set of all sums $\Sigma_{j=1}^{n}|\varphi(t_j) - \varphi(t_{j-1})|$, where $a = t_0 < t_1 < \ldots < t_n = b$. We write $\text{var}(\varphi,K) = \text{var}(\varphi,a,b) = \text{var}(\varphi,b,a)$. Further we set $\sup(\varphi,a,b) = \sup(\varphi,b,a) = \sup \varphi(K)$ and $\text{var}(\varphi,c,c) = 0$, $\sup(\varphi,c,c) = \varphi(c)$ for each $c \in K$.

If $\varphi(K) \subset R$, then $U^+\varphi(a)$ $[L^+\varphi(a)]$ is the right upper [lower] derivate of φ at a; the meaning of $U^-\varphi(x)$, $L^-\varphi(x)$ (for $x \in (a,b]$), $U\varphi(x)$, $L\varphi(x)$ (for $x \in (a,b)$) is obvious. If $[a,b]$ is the domain of definition of φ, we write sometimes $U\varphi(a)$, $\varphi'(a)$ instead of $U^+\varphi(a)$, $\varphi'^+(a)$ etc.

In 2.1 and 2.2 we state without proof two well-known results of integration theory. (See, e.g., [6], Chapter VIII.) Symbols a,b,K have the same meaning as before.

2.1. Let f be P-integrable on K and let φ be a finite non-negative nonincreasing function on K. Then $f\varphi$ is P-integrable on K and there is a $\xi \in K$ such that $\int_K f\varphi = \varphi(a)\int_a^\xi f$.

2.2. Let f be a function on K that is P-integrable on $[x,b]$ for each $x \in (a,b)$. If $\lim_{x\to a+}\int_x^b f = \lambda \in R$, then f is P-integrable on K and $\int_K f = \lambda$.

2.3. Let $\sigma \in R$. Let φ be a nonnegative nonincreasing function on $(a,b]$ such that $\int_K \varphi < \infty$. Let f be a function P-integrable on K such that $|\int_a^x f| \le \sigma(x-a)$ for each $x \in K$. Then the function $f\varphi$ is P-integrable on K and $|\int_K f\varphi| \le \sigma\int_K \varphi$.

Proof. Let $\alpha,\beta \in (a,b]$, $\alpha < \beta$. Set $f_1 = f-\alpha$. By 2.1 there is a $\xi \in [\alpha,\beta]$ such that $\int_\alpha^\beta f_1\varphi = \varphi(\alpha)\int_\alpha^\xi f_1 = \varphi(\alpha)(\int_a^\xi f_1 - \int_a^\alpha f_1)$. Since $\int_a^\xi f_1 \le 0$, we have $\int_\alpha^\beta f\varphi \le M$, where $M = \sigma\int_\alpha^\beta \varphi + 2\sigma\varphi(\alpha)(\alpha-a)$. Taking $-f$ instead of f we see that $|\int_\alpha^\beta f\varphi| \le M$. Since $(\alpha-a)\varphi(\alpha) \le \int_a^\alpha \varphi$, there is a finite limit $\lambda = \lim\int_x^b f\varphi$ $(x \to a+)$ and $|\lambda| \le \sigma\int_K \varphi$. By 2.2 we have $\lambda = \int_K f\varphi$.

2.4. Let $\sigma, A \in R$. Let f be a P-integrable function on K such that $|\int_a^x f| \le \sigma(x-a)$ for each $x \in K$. Let α be a function on K such that $\int_K \operatorname{var}(\alpha,t,b)dt = A$. Then $f\alpha$ is P-integrable on K and

$$|\int_K f\alpha| \le \sigma(2A + |\int_K \alpha|).$$

Proof. Suppose first that $\alpha(b) = 0$. Set $\psi(x) = \operatorname{var}(\alpha,x,b)$ $(x \in (a,b])$, $\alpha_1 = (\psi+\alpha)/2$, $\alpha_2 = (\psi-\alpha)/2$. By 2.3 we have $|\int_K f\alpha_j| \le \sigma\int_K \alpha_j$ for $j = 1,2$. Hence

$$|\int_K f\alpha| \le \sigma A. \tag{1}$$

In the general case we have $|K|\alpha(b) = \int_K(\alpha(b)-\alpha) + \int_K \alpha$, $\int_K f\alpha = \int_K f\cdot(\alpha-\alpha(b)) + \alpha(b)\int_K f$ so that $|K||\alpha(b)| \le A + |\int_K \alpha|$ and (see (1)) $|\int_K f\alpha| \le \sigma A + |\alpha(b)|\sigma|K| \le \sigma(2A + |\int_K \alpha|)$.

2.5. Let f be L-integrable on K. Let γ be a nonnegative measurable function on K such that $0 < \int_a^x \gamma < \infty$ for each $x \in (a,b)$. Suppose that $(x-a)^{-1}\int_a^x |f-f(a)| \to 0$ and that

$$\limsup\,(\int_a^x \gamma)^{-1}\int_a^x \sup(\gamma,t,x)dt < \infty \qquad (x \to a+). \tag{2}$$

Then there is a $c \in (a,b)$ such that $f\gamma$ is L-integrable on $[a,c]$ and we have

$$(\int_a^x \gamma)^{-1}\int_a^x f\gamma \to f(a) \qquad (x \to a+).$$

Proof. We may suppose that $f(a) = 0$. For each $x \in (a,b]$ set $\sigma(x) = \sup\{(t-a)^{-1}\int_a^t |f|;\ a < t \le x\}$. Obviously $\sigma(a+) = 0$. There is a $c \in (a,b)$ such that $\int_a^c \sup(\gamma,t,c)dt < \infty$. Choose an $x \in (a,c]$ and set $\varphi(t) = \sup(\gamma,t,x)$ $(a < t \le x)$. By 2.3,

$\int_a^x |f|\varphi \leq \sigma(x)\int_a^x \varphi$. Since $|f\gamma| \leq |f|\varphi$ on $(a,x]$, we have (see (2)) $(\int_a^x \gamma)^{-1}\int_a^x f\gamma \to 0 \quad (x \to a+)$.

2.6. Let α, φ, f be P-integrable functions on K and let $\lambda \in R$. Suppose that $\int_a^x \varphi > 0$ for each $x \in (a,b)$, $(\int_a^x \varphi)^{-1}\int_a^x \alpha \to \lambda$, $(x-a)^{-1}\int_a^x f \to f(a)$ and that

(3) $$\limsup (\int_a^x \varphi)^{-1}\int_a^x \mathrm{var}(\alpha,t,x)dt < \infty \qquad (x \to a+).$$

Then there is a $c \in (a,b)$ such that the function $f\alpha$ is P-integrable on $[a,c]$ and we have

(4) $$(\int_a^x \varphi)^{-1}\int_a^x f\alpha \to \lambda f(a) \qquad (x \to a+).$$

Proof. We may suppose that $f(a) = 0$. For each $x \in (a,b)$ set $\sigma(x) = \sup\{(t-a)^{-1}|\int_a^t f|;\ a < t \leq x\}$. Obviously $\sigma(a+) = 0$. There is a $c \in (a,b)$ such that $\int_a^c \mathrm{var}(\alpha,t,c)dt < \infty$. By 2.4 we have $|\int_a^x f\alpha| \leq \sigma(x)(|\int_a^x \alpha| + 2\int_a^x \mathrm{var}(\alpha,t,x)dt)$ for each $x \in (a,c]$. This easily implies (4).

2.7. Let $b \in (0,\infty)$, $K = [0,b]$. Let g be a continuous increasing function on K, $h = g^{-1}$. Set $\gamma(b) = U^-g(b)$, $\gamma = Ug$ on $(0,b)$. Let M be a number less that $\int_K \sup(\gamma,t,b)dt$. Then there exist an $a \in (0,b)$ and a nonnegative piecewise linear function f on K such that $f = 0$ on $(0,a) \cup \{b\}$, $\int_0^x f \leq x$ for each $x \in K$ and

$$\int_{g(K)} f \circ h > M.$$

Proof. For each $x \in (0,b]$ define $\varphi(x) = \sup(\gamma,x,b)$.

Suppose first that $\varphi(c) = \infty$ for some $c \in (0,b)$. Set $A = 2|M|/c$. There is a $z \in [c,b]$ such that $\gamma(z) > A$. There are numbers p,q such that either $p=z$ or $q=z$, $q/2 < p < q \leq b$ and $g(q) - g(p) > A(q-p)$. There are numbers α, β such that $p < \alpha < \beta < q$ and $g(\beta) - g(\alpha) > A(q-p)$. Let f be a function such that $f = 0$ on $[0,p]$ and on $[q,b]$ (which means $\{b\}$ for $q=b$), $f = p/(q-p)$ on $[\alpha,\beta]$ and that f is linear on $[p,\alpha]$ and on $[\beta,q]$. Since $\int_0^p f = 0$ and $\int_K f < p$, we have $\int_0^x f \leq x$ for each $x \in K$. Since $2p > q \geq z \geq c$, we have $Ap \geq M$. Obviously $\int_{g(\alpha)}^{g(\beta)} f \circ h = (g(\beta)-g(\alpha))p/(q-p) > Ap$ so that $\int_{g(K)} f \circ h > M$.

Now suppose that $\varphi((0,b)) \subset R$. There are numbers $c \in (0,b)$, $Q \in (1,\infty)$ and $\varepsilon \in (0,\infty)$ such that $\int_c^b \varphi > QM + b\varepsilon$. There are $t_0,\dots,t_n$ such that $c = t_0 < t_1 < \dots < t_n = b$ and that $t_i < Qt_{i-1}$ for $i = 1,\dots,n$. There are integers s and j_k $(s \geq 1)$ such that $\varphi(t_0) = \varphi(t_{j_1}-1) > \varphi(t_{j_1}) = \varphi(t_{j_2-1}) > \varphi(t_{j_2}) = \dots > \varphi(t_{j_{s-1}}) = \varphi(t_{n-1})$. Set $j_0 = 0$, $j_s = n$, $v_k = t_{j_k}$ $(k = 0,\dots,s)$, $A_k = \varphi(t_{j_k-1}) - \varepsilon$

$(k=1,\dots,s)$. For $k=1,\dots,s-1$ there is a $z_k \in [t_{j_k-1}, t_{j_k}]$ such that $\gamma(z_k) > A_k$; there is a $z_s \in [t_{n-1}, t_n]$ such that $\gamma(z_s) > A_s$. Note that $v_k < Qz_k$. There are a_k, b_k such that either $a_k = z_k$ or $b_k = z_k$, $a_1 < b_1 < \dots < a_s < b_s \le b$, $v_k < Qa_k$ and $g(b_k) - g(a_k) > A_k(b_k - a_k)$ for $k=1,\dots,s$. There are numbers α_k, β_k such that $a_k < \alpha_k < \beta_k < b_k$ and that $g(\beta_k) - g(\alpha_k) > A_k(b_k - a_k)$. Let F be a function on $[0,b]$ such that $F=0$ on $[0,a_1] \cup [b_s,b]$ and on $[b_k, a_{k+1}]$ for $k=1,\dots,s-1$, $F = (v_k - v_{k-1})/(b_k - a_k)$ on $[\alpha_k, \beta_k]$ and that F is linear on each of the intervals $[a_k, \alpha_k]$ and $[\beta_k, b_k]$ for $k=1,\dots,s$. Set $a_{s+1} = b$. Obviously $\int_{a_k}^{a_{k+1}} F < v_k - v_{k-1}$. If $a_k < x \le a_{k+1}$, then $\int_0^x F < v_k < Qa_k < Qx$. Further $\int_{g(\alpha_k)}^{g(\beta_k)} f \circ h = (g(\beta_k) - g(\alpha_k))(v_k - v_{k-1})/(b_k - a_k) > A_k(v_k - v_{k-1})$. Since $A_k = \varphi(t_i) - \varepsilon$ for $i = j_{k-1}, \dots, j_k - 1$, we have $A_k(v_k - v_{k-1}) = \Sigma_{i=j_{k-1}+1}^{j_k} \varphi(t_{i-1})(t_i - t_{i-1}) - \varepsilon(v_k - v_{k-1}) \ge \int_{v_{k-1}}^{v_k} \varphi - \varepsilon(v_k - v_{k-1})$. Hence $\int_{g(K)} F \circ h > \int_{v_0}^{v_s} \varphi - \varepsilon b > QM$. Now we set $f = F/Q$.

2.8. Let $a,b \in R$, $a<b$, $K=[a,b]$. Let g be a continuous increasing function on K and let $A,B \in R$. If $Ug \ge A$ on (a,b), then $g(b)-g(a) \ge A|K|$; if $Lg \le B$ on (a,b), then $g(b)-g(a) \le B|K|$.

Proof. If $Ug \ge A$ on (a,b), then, by Theorem 7.4 of Chapter IV of [6], we have $g(b)-g(a) \ge A|K|$.

Now let $Lg \le B$ on (a,b) and let N be the set of all points $x \in (a,b)$ for which $g'(x)$ does not exist. By Theorems 4.4 and 6.5 of Chapter VII of [6] we have $|g(N)| = 0$ and $|g(a,b) \setminus N| \le B|K|$. Therefore $g(b)-g(a) = |g((a,b))| \le B|K|$.

2.9. Let g be a continuous increasing function on $[a,b]$. Then

$$\liminf Ug(x) \le L^+ g(a), \quad U^+ h(a) \le \limsup Lg(x) \qquad (x \to a+).$$

(This follows at once from 2.8.)

2.10. Let $b \in (0,\infty)$. Let ψ be a finite nonincreasing function on $(0,b]$ with $\psi(b) = 0$ and let Q be a number less than $\int_0^b \psi$. Then there are numbers $t_0, \dots, t_n$ such that $0 < t_0 < \dots < t_n < b$, ψ is continuous at t_j for $j=1,\dots,n$ and

$$\Sigma_{j=1}^{n} (t_j - t_{j-1}) \psi(t_j) > Q. \tag{5}$$

Proof. There is a $t_0 \in (0,b)$ and an integer $n > 1$ such that

$$\int_{t_0}^{b} \psi > Q + 2b\psi(t_0)/n. \tag{6}$$

Let $z_j = t_0 + j(b-t_0)/n$ $(j = 0,\ldots,n)$. There are $t_j \in (z_{j-1}, z_j)$ such that ψ is continuous at t_j $(j = 1,\ldots,n)$. Set $t_{n+1} = b$, $S = \Sigma_{j=1}^{n}(t_j - t_{j-1})\psi(t_j)$, $T = \Sigma_{j=1}^{n+1}(t_j - t_{j-1})\psi(t_{j-1})$. Obviously $\int_{t_0}^{b}\psi \le T$, $t_j - t_{j-1} < 2b/n$ so that $\int_{t_0}^{b}\psi - S \le T - S =$ $\Sigma_{j=1}^{n+1}(t_j - t_{j-1})(\psi(t_{j-1}) - \psi(t_j)) \le 2b\psi(t_0)/n$. This and (6) proves (5).

2.11. Let $a,b \in R$, $K = [a,b]$. Let g be a continuous increasing function on K, $h = g^{-1}$. Let γ be a function on K such that $L^+g(a) \le \gamma(a) \le U^+g(a)$, $L^-g(b) \le \gamma(b) \le U^-g(b)$ and that $Lg \le \gamma \le Ug$ on (a,b).

Let A be a number less than $\mathrm{var}(\gamma,K)$. Then there is a function f piecewise linear on K such that $f(a) = f(b) = \int_K f = 0$, $|\int_a^x f| \le 1$ for each $x \in K$ and that

$$\int_{g(K)} f \circ h > A .$$

Proof. Suppose first that there is a $c \in K$ such that $\gamma(c) = \infty$. Let, e.g., $U^+g(c) = \infty$. Set $s = (c+b)/2$. Let F be a nonnegative piecewise linear function on $[s,b]$ such that $F(s) = F(b) = 0$ and $\int_s^b F = 1$. Set $B = \int_{g(s)}^{g(b)} F \circ h$. There is a $b_0 \in (c,s)$ such that $\frac{g(b_0)-g(c)}{b_0 - c} > A + B$. There is a $\delta \in (0,\infty)$ such that $c + 2\delta < b_0$ and that $(g(b_0-\delta)-g(c+\delta))/(b_0-c-\delta) > A + B$. Let f be a function on K such that $f(t) = 0$, if $a \le t \le c$, $f = (b_0-c-\delta)^{-1}$ on $[c+\delta, b_0-\delta]$, $f = 0$ on $[b_0,s]$, $f = -F$ on $[s,b]$ and that f is linear on $[c,c+\delta]$ and on $[b_0-\delta,b_0]$. Then $f(a) = f(b) = \int_K f = 0$, $|\int_a^x f| \le 1$ for each $x \in K$ and $\int_{g(K)} f \circ h = \int_{g(c)}^{g(b_0)} f \circ h + \int_{g(s)}^{g(b)} f \circ h >$ $A + B - B = A$.

Now suppose that $\gamma(K) \subset R$. Then there are numbers $t_0,\ldots,t_n$ and η such that $a = t_0 < t_1 < \ldots < t_n = b$, $\eta > 0$ and that

$$\Sigma_{j=1}^{n}|\gamma(t_j) - \gamma(t_{j-1})| > A + 2n\eta . \tag{7}$$

We may suppose that $(\gamma(t_j)-\gamma(t_{j-1}))\cdot(\gamma(t_{j+1})-\gamma(t_j)) < 0$ for $j = 1,\ldots,n-1$. Let, e.g., $\gamma(t_0) > \gamma(t_1) < \gamma(t_2) > \ldots$. Then

$$\Sigma_{j=1}^{n}|\gamma(t_j) - \gamma(t_{j-1})| = \tag{8}$$

$$\gamma(t_0) - 2\gamma(t_1) + \ldots + 2(-1)^{n-1}\gamma(t_{n-1}) + (-1)^n\gamma(t_n) .$$

Choose an $\varepsilon \in (0,\infty)$ such that $2\varepsilon < t_j - t_{j-1}$ for $j = 1,\ldots,n$. There are $z_j \in (a,b)$ such that $(g(z_j)-g(t_j))/(z_j-t_j) > \gamma(t_j) - \eta$ for j even, $(g(z_j)-g(t_j))/(z_j-t_j) < \gamma(t_j) + \eta$ for j odd and $|z_j - t_j| < \varepsilon$ for $j = 0,\ldots,n$.

If $z_j < t_j$, set $a_j = z_j$, $b_j = t_j$; if $z_j > t_j$, set $a_j = t_j$, $b_j = z_j$. Obviously $a = a_0 < b_0 < a_1 < \ldots < b_n = b$. There is a $\delta \in (0,\infty)$ such that $2\delta < b_j - a_j$ for $j = 0,\ldots,n$ and that $(g(b_j-\delta)-g(a_j+\delta))/(b_j-a_j-\delta) > \gamma(t_j)-\eta$ for j even, $(g(b_j)-g(a_j))/(b_j-a_j-\delta) < \gamma(t_j)+\eta$ for j odd. Now let $c_j = (b_j-a_j-\delta)^{-1}$ $(j=0,\ldots,n)$ and let f be a function on K such that $f = c_0$ on $[a_0+\delta, b_0-\delta]$, $f = 2(-1)^j c_j$ on $[a_j+\delta, b_j-\delta]$ $(j=1,\ldots,n-1)$, $f = (-1)^n c_n$ on $[a_n+\delta, b_n-\delta]$, $f = 0$ on $K \setminus \bigcup_{j=0}^{n}(a_j,b_j)$ and f is linear on each of the intervals $[a_j, a_j+\delta]$ and $[b_j-\delta, b_j]$. Set $s_0 = a$, $s_n = b$ and $s_j = (a_j+b_j)/2$ for $j = 1,\ldots,n-1$. It is easy to see that $\int_{a_0}^{b_0} f = 1$, $\int_{a_n}^{b_n} f = (-1)^n$, $\int_{a_j}^{s_j} f = \int_{s_j}^{b_j} f = (-1)^j$ for $j = 1,\ldots,n-1$, $\int_{s_{j-1}}^{s_j} f = 0$ for $j = 1,\ldots,n$ and that $|\int_a^x f| \le 1$ for each $x \in K$. If j is even and $0 < j < n$, then $\int_{g(a_j)}^{g(b_j)} f \circ h > 2c_j(g(b_j-\delta) - g(a_j+\delta)) > 2\gamma(t_j) - 2\eta$; if j is odd and $0 < j < n$, then $\int_{g(a_j)}^{g(b_j)} f \circ h > -2c_j(g(b_j)-g(a_j)) > -2\gamma(t_j) - 2\eta$. Similarly, $\int_{g(a_0)}^{g(b_0)} f \circ h > \gamma(t_0) - \eta$ and $\int_{g(a_n)}^{g(b_n)} f \circ h > (-1)^n \gamma(t_n) - \eta$. Thus (see (7), (8)) $\int_{g(K)} f \circ h > \gamma(t_0) - 2\gamma(t_1) + \ldots + 2(-1)^{n-1}\gamma(t_{n-1}) + (-1)^n \gamma(t_n) - 2n\eta > A$.

<u>2.12</u>. Let $b \in (0,\infty)$, $K = [0,b]$. Let g be a continuous increasing function on K, $h = g^{-1}$. Let γ be a function on $(0,b]$ such that $L^- g(b) \le \gamma(b) \le U^- g(b)$ and $Lg \le \gamma \le Ug$ on $(0,b)$. Let M be a number less than $\int_K \mathrm{var}(\gamma,t,b)dt$. Then there is an $a \in (0,b)$ and a function f piecewise linear on K such that $f = 0$ on $(0,a)$, $f(b) = \int_K f = 0$, $|\int_0^x f| \le x$ for each $x \in K$ and $\int_{g(K)} f \circ h > M$.

Proof. For each $x \in (0,b]$ define $\psi(x) = \mathrm{var}(\gamma,x,b)$.

Suppose first that $\psi(c) = \infty$ for some $c \in (0,b)$. There is an $a \in (0,c)$ such that $g'(a)$ exists. Then, obviously, $\gamma(a) = g'(a) = L^+ g(a) = U^+ g(a)$ and $\mathrm{var}(\gamma,a,b) = \infty$. It follows from 2.11 that there is a piecewise linear function F on $[a,b]$ such that $F(a) = F(b) = \int_a^b F = 0$, $|\int_a^x F| \le 1$ for each $x \in (a,b)$ and that $a \int_{g(a)}^{g(b)} F \circ h > M$. It is easy to see that the function f defined by $f = 0$ on $[0,a]$ and $f = aF$ on $(a,b]$ satisfies our requirements.

Now suppose that $\psi((0,b]) \subset R$. Choose a number Q with $M < Q < \int_K \psi$, find numbers t_j according to 2.10 and set $t_{n+1} = b$. Then $Q < \Sigma_{j=1}^{n} (t_j - t_{j-1})\psi(t_j) = -t_0 \psi(t_1) + \Sigma_{j=1}^{n} t_j(\psi(t_j) - \psi(t_{j+1})) \le \Sigma_{j=1}^{n} t_j \mathrm{var}(\gamma, t_j, t_{j+1})$. Since ψ is continuous at t_j, γ is continuous at t_j as well. It follows easily from 2.9 that $g'(t_j)$ exists for $j=1,\dots,n$. Let $\varepsilon \in (0,(Q-M)/(nb))$. By 2.11 (note that $\gamma(t_1)=g'(t_1)=L^+g(t_1)$ etc.) there are functions f_j piecewise linear on $[t_j,t_{j+1}]$ such that $f_j(t_j)=f_j(t_{j+1})=\int_{t_j}^{t_{j+1}} f_j=0$, $|\int_{t_j}^{x} f_j|\leq 1$ for each $x\in[t_j,t_{j+1}]$ and that $\int_{g(t_j)}^{g(t_{j+1})} f_j\circ h > \mathrm{var}(\gamma,t_j,t_{j+1})-\varepsilon$. Let f be a function on K such that $f=0$ on $[0,t_1]$ and $f=t_jf_j$ on $[t_j,t_{j+1}]$ $(j=1,\dots,n)$. Then $f(b)=\int_K f=0$ and $\int_{g(K)} f\circ h > \Sigma_{j=1}^{n} t_j(\mathrm{var}(\gamma,t_j,t_{j+1})-\varepsilon) > Q-nb\varepsilon > M$. If $t_j\leq x\leq t_{j+1}$, then $|\int_0^x f|=|\int_{t_j}^x f|\leq t_j\leq x$. This completes the proof.

3. TRANSFORMATIONS VIA INNER HOMEOMORPHISMS. Let AC be the system of all absolutely continuous functions on J. Let H be the system of all increasing homeomorphisms of J onto J and let $\mathcal{A}$ be the system of all functions $h\in H$ such that $f\circ h\in C_{ap}$ for each $f\in C_{ap}$. For each system $S\subset D$ let $\mathrm{Tr}\,S$ be the system of all functions $h\in H$ such that $f\circ h\in D$ for each $f\in S$.

The system $\mathcal{A}$ has been characterized in [1]. We shall see in 3.5 that $\mathcal{A}=\mathrm{Tr}\,b\,C_{ap}$. Thus $\mathrm{Tr}\,S\subset\mathcal{A}$ for each S with $b\,C_{ap}\subset S\subset D$. Theorem 3.6 describes $\mathrm{Tr}\,L$. It is easy to prove that we have even $f\circ h\in L$ for each $f\in L$ and each $h\in\mathrm{Tr}\,L$ (see 3.7). If $h\in H$ and if both functions h and h^{-1} satisfy the Lipschitz condition, then (9) is obviously fulfilled so that $h\in\mathrm{Tr}\,L$. In 3.8 we show that a function $h\in H$ belongs to $\mathrm{Tr}\,L$ if and only if $w\circ h\in W$ for each $w\in W$. In this way we obtain an improvement of Theorem 4 in [2]. Theorem 3.9 gives a characterization (which is simpler than the characterization found in [5]) of the system $\mathrm{Tr}\,D$.

We shall see in 4.7 that the obvious inclusions $\mathrm{Tr}\,D\subset\mathrm{Tr}\,L\subset\mathrm{Tr}\,b\,C_{ap}$ $(=\mathcal{A})$ are proper. If, however, h is a convex or concave element of $\mathcal{A}$, then $h\in\mathrm{Tr}\,D$. This is proved in 3.12 (with the help of 3.11).

First we introduce an auxiliary system.

<u>3.1</u>. Let $\mathcal{B}$ be the system of all functions $g\in H\cap AC$ with the following property: If S is a measurable subset of J and if $x\in J$ is a point of dispersion for S, then $g(x)$ is a point of dispersion for $g(S)$.

Remark. It is easy to see that $g(S)$ is measurable whenever S is measurable, $S \subset J$ and $g \in AC$.

The assertions 3.2 and 3.3 follow easily from Theorems 1 and 2 and Lemma 6 in [1].

3.2. We have $h \in \mathcal{A}$ if and only if $h^{-1} \in \mathcal{B}$.

3.3. Let $h \in \mathcal{A}$, $a \in J$. Then there is a $\delta \in (0,1)$ such that $|h(x)-h(a)|/|x-a|^{\delta} \to 0$ $(x \to a, x \in J)$.

3.4. $\mathcal{A} \subset AC$.

Proof. Let $h \in \mathcal{A}$, $S \subset J$, $|S| = 0$. There is a G_{δ} set T such that $S \subset T \subset J$ and that $|T| = 0$. Suppose that $|h(T)| > 0$. The set $h(T)$ is obviously measurable. Let X be a point of density of $h(T)$. It follows easily from 3.2 that $h^{-1}(x)$ is a point of density of T; this, however, is impossible. Thus $|h(T)| = 0$, $|h(S)| = 0$, $h \in AC$.

3.5. $\mathcal{A} = \operatorname{Tr} b\, C_{ap}$.

Proof. If $h \in \mathcal{A}$ and $\alpha \in b\, C_{ap}$, then $\alpha \circ h \in b\, C_{ap} \subset D$ so that $h \in \operatorname{Tr} b\, C_{ap}$. Now let $h \in \operatorname{Tr} b\, C_{ap}$ and $\alpha \in C_{ap}$. Let $a, b \in R$, $\beta = (\alpha \wedge a) \vee b$. Then $\beta \in b\, C_{ap}$, $((\alpha \circ h) \wedge a) \vee b = \beta \circ h \in D$. By 1.7 we have $\alpha \circ h \in C_{ap}$. Thus $h \in \mathcal{A}$.

3.6. Let $h \in H$, $g = h^{-1}$. Then $h \in \operatorname{Tr} L$ if and only if

$$(9) \quad \limsup (g(x)-g(a))^{-1} \int_a^x \sup(Ug,t,x)\,dt < \infty \qquad (x \to a, x \in J)$$

for each $a \in J$.

Proof. Suppose that (9) holds for each $a \in J$. It is easy to see that there is a finite set $S \subset J$ such that g fulfills the Lipschitz condition on each closed interval contained in $J \setminus S$. This shows that $g \in AC$ so that g is an indefinite integral of Ug. Let $f \in L$. It follows from 2.5 that $f \cdot Ug$ is L-integrable on J. Let $Q(x) = \int_0^x f \cdot Ug$ $(x \in J)$. By 2.5 we have

$$(g(x)-g(a))^{-1}(Q(x)-Q(a)) \to f(a) \qquad (x \to a, x \in J)$$

for each $a \in J$. Hence $(Q \circ h)' = f \circ h$ on J, $h \in \operatorname{Tr} L$.

Now suppose that, e.g., $\limsup (g(x))^{-1} \int_0^x \sup(Ug,t,x)\,dt = \infty$ $(x \to 0+)$. There are $c_n \in (0,1)$ such that $c_n \to 0$ and that $n g(c_n) < T_n$, where $T_n = \int_0^{c_n} \sup(Ug,t,c_n)\,dt$. There are $b_n \in (c_n, 2c_n) \cap (0,1)$ such that $n g(b_n) < T_n$ and that $g'(b_n)$ exists. Then $n g(b_n) < \int_0^{b_n} \sup(Ug,t,b_n)\,dt$ $(n = 1,2,\dots)$. It follows easily from 2.7 that there is a subsequence $\langle a_n \rangle$ of $\langle b_n \rangle$ and nonnegative continuous functions f_n such that $2a_{n+1} < a_n$, $f_n = 0$ on $[0, a_{n+1}] \cup [a_n, 1]$, $\int_0^x f_n \le x$ for $x \in J$ and that

$\int_0^{g(a_n)} f_n \circ h > ng(a_n)$ $(n = 1, 2, \ldots)$. Set $A_n = g(a_n)$, $f = \Sigma n^{-1} f_n$. Let $a_n < x \le a_{n-1}$. Then $\int_0^x f = (n-1)^{-1} \int_0^x f_{n-1} + \Sigma_{k=n}^{\infty} k^{-1} \int_0^{a_k} f_k \le (n-1)^{-1} x + n^{-1} \Sigma_{k=n}^{\infty} a_k < (n-1)^{-1} x + n^{-1} 2a_n < (n-1)^{-1} 3x$. This shows that $f \in D$. It is easy to see that $f \in L$. However, $\int_0^{A_n} f \circ h \ge n^{-1} \int_0^{A_n} f_n \circ h > A_n$ so that $f \circ h \notin D$, $h \notin \mathrm{Tr}\, L$.

Remark. It follows from 3.6 that the set $\{x \in J;\ Lh(x) = 0\}$ is finite for each $h \in \mathrm{Tr}\, L$.

3.7. Let $h \in \mathrm{Tr}\, L$ and $f \in L$. Then $f \circ h \in L$.

Proof. Let $a \in R$. By 1.2, $f_{\vee} a \in L$. Thus $f \circ h \in D$, $(f \circ h)_{\vee} a \in D$. By 1.5 we have $f \circ h \in L$.

3.8. Let $h \in H$. Then $h \in \mathrm{Tr}\, L$ if and only if $w \circ h \in W$ for each $w \in W$.

Proof. If $h \in \mathrm{Tr}\, L$ and $w \in W$, then $w, w^2 \in L$ (see 1.9), $w \circ h$, $(w \circ h)^2 \in D$, $w \circ h \in W$.

Now suppose that $w \circ h \in W$ for each $w \in W$. Let $f \in L$. By 1.11 there are $v, w \in W$ such that $f = vw$. Thus $f \circ h = (v \circ h)(w \circ h) \in L$ (see 1.9), $h \in \mathrm{Tr}\, L$.

3.9. Let $h \in H$, $g = h^{-1}$. Let γ be a function such that $Lg \le \gamma \le Ug$ on J. Then the following three conditions are equivalent to each other:

i) There is a function φ such that $g(a) = \int_0^a \varphi$ and that

$$\limsup (g(x) - g(a))^{-1} \int_a^x \mathrm{var}(\varphi, t, x)\, dt < \infty \qquad (x \to a,\ x \in J)$$

for each $a \in J$;

ii) $h \in \mathrm{Tr}\, D$;

iii) the condition

$$\limsup (g(x) - g(a))^{-1} \int_a^x \mathrm{var}(\gamma, t, x)\, dt < \infty \qquad (x \to a,\ x \in J) \tag{10}$$

is fulfilled for each $a \in J$.

Proof. Suppose that i) holds. Let $f \in D$. It follows from 2.6 with $\alpha = \varphi$ that $f\varphi$ is P-integrable on J. Set $Q(x) = \int_0^x f\varphi$ $(x \in J)$. By 2.6 (with $\lambda = 1$) we have $(g(x) - g(a))^{-1}(Q(x) - Q(a)) \to f(a)$ $(x \to a,\ x \in J)$ for each $a \in J$. Hence $(Q \circ h)' = f \circ h$ on J, $h \in \mathrm{Tr}\, D$.

Now let $h \in \mathrm{Tr}\, D$. Suppose that, e.g.,

$$\limsup (g(x))^{-1} \int_0^x \mathrm{var}(\gamma, t, x)\, dt = \infty \qquad (x \to 0+).$$

There are $c_n \in (0,1)$ such that $c_n \to 0$ and that $ng(c_n) < V_n$, where $V_n = \int_0^{c_n} \mathrm{var}(\gamma, t, c_n)\, dt$. There are $b_n \in (c_n, 2c_n) \cap (0,1)$ such that

$ng(b_n) < V_n$ and that $g'(b_n)$ exists. Then $ng(b_n) < \int_0^{b_n} \mathrm{var}(\gamma,t,b_n)dt$ $(n = 1,2,\ldots)$. It follows easily from 2.12 that there is a subsequence $\langle a_n \rangle$ of $\langle b_n \rangle$ and functions f_n continuous on J such that $a_{n+1} < a_n$, $f_n = 0$ on $[0,a_{n+1}] \cup [a_n,1]$, $\int_J f_n = 0$, $|\int_0^x f_n| \leq x$ for each $x \in J$ and that

$$\int_0^{g(a_n)} f_n \circ h > ng(a_n) \qquad (n = 1,2,\ldots). \tag{11}$$

Set $f = \Sigma n^{-1} f_n$, $F = 0$ on $\{0\} \cup (a_1,1]$, $F(x) = n^{-1} \int_{a_{n+1}}^x f_n$ for $x \in (a_{n+1},a_n]$. It is easy to see that $F' = f$ on J. By assumption there is a function G such that $G' = f \circ h$ on J. For $n = 1,2,\ldots$ set $q_n = (G(B) - G(A))/B$, where $A = g(a_{n+1})$, $B = g(a_n)$. Since $G'(0) = 0$, we have $q_n = (G(B)/B) - (A/B)(G(A)/A) \to 0$. However, by (11), $G(B) - G(A) = n^{-1} \int_A^B f_n \circ h > B$ whence $q_n > 1$ for each n. This contradiction proves iii).

Suppose, finally, that iii) holds. It is easy to see that there is a finite set $S \subset J$ such that $\mathrm{var}(\gamma,K) < \infty$ for each closed interval $K \subset J \setminus S$. This shows that $g \in AC$. Thus (i) holds with $\varphi = \gamma$.

Remark 1. Suppose that $h \in H$, $h' > 0$ on J and that $\mathrm{var}(h',J) < \infty$. Then $\mathrm{var}(g',J) < \infty$ and (10) holds for each $a \in J$. Thus $h \in \mathrm{Tr}\, D$. The remark in 3.6 and the example in 3.13 show that the requirement $h' > 0$ is essential.

Remark 2. Let $h \in \mathrm{Tr}\, D$. It follows easily from 3.9 (see the proof of the implication iii) $\to$ i)) that there is a finite set $S \subset J$ such that for each $a \in (0,1) \setminus S$ the unilateral derivatives $h'^{+}(a)$, $h'^{-}(a)$ exist and that there is a countable set $T \subset J$ such that for each $a \in (0,1) \setminus T$ the derivative $h'(a)$ exists; we must, of course, admit also infinite derivatives. The next example shows that h' may be infinite on an uncountable set. (See also [5], p. 195.)

<u>3.10</u>. Example of an $h \in \mathrm{Tr}\, D$ such that $h' = \infty$ on a perfect set.

Let C be the Cantor set. Let G be a function such that $G(0) = 0$ and that $G'(x) = \mathrm{dist}(x,C)$ for each $x \in J$. Then $|G''| = 1$ a.e. whence $\mathrm{var}(G',t,x) = |x-t|$ for all $x,t \in J$. Let $a,b \in J$, $a < b$. Then $\int_a^b \mathrm{var}(G',t,b)dt = \int_a^b (t-b)dt = (b-a)^2/2$. Set $q = 1/3$. There is an integer $n \geq 2$ such that $q^{n-1} < b-a \leq q^{n-2}$. Define $t_k = kq^n$ $(k = 0,\ldots,3^n)$. There are j,k such that $t_{j-1} \leq a < t_j$, $t_k < b \leq t_{k+1}$. It is easy to see that $k \geq j+2$ and that at least one of the intervals (t_j,t_{j+1}), (t_{j+1},t_{j+2}) is contained in $J \setminus C$.

Let, e.g., $I \cap C = \emptyset$ with $I = (t_j, t_{j+1})$. For $x \in I$ set $\mu(x) = \min(x-t_j, t_{j+1}-x)$. Then $I \subset (a,b)$, $|I| = q^n \geq (b-a)/9$, $G' \geq \mu$ on I and $G(b)-G(a) \geq \int_I \mu = |I|^2/4 \geq (b-a)^2/324$. Hence $(G(b)-G(a))^{-1} \int_a^b \operatorname{var}(G', t, b)dt \leq 162$. Now set $g = 28G$, $h = g^{-1}$. Since $G(1) = \Sigma_{n=1}^{\infty} 2^{n-1}(q^n)^2/4 = 1/28$, we have $h \in H$. By 3.9, $h \in \operatorname{Tr} D$. Obviously $h' = \infty$ on $g(C)$.

3.11. Let $g \in \mathcal{B}$ and let g be convex. Then

$$\limsup xg'^{+}(x)/g(x) < \infty \qquad (x \to 0+).$$

Proof. Suppose that the assertion is false. Then there are $a_n \in (0,1)$ such that $2a_n < a_{n-1}$ and $a_n g'^{+}(a_n) > ng(a_n)$ for $n = 1,2,\ldots$. Set $b_n = (1+n^{-1})a_n$, $A_n = g(a_n)$, $B_n = g(b_n)$, $S = \bigcup(a_n, b_n)$. It is easy to see that $|S \cap (0,x)|/x \to 0$ $(x \to 0+)$. However, $B_n - A_n \geq (b_n - a_n)g'^{+}(a_n) > ng(a_n)(b_n-a_n)/a_n = A_n$ so that $|g(S) \cap (0,B_n)| > B_n - A_n > B_n/2$. Thus 0 is a point of dispersion for S, but not for $g(S)$ which is a contradiction.

3.12. (Cf. [5], Theorem 5.) Let $h \in H$ and let h be concave. Then the following three conditions are equivalent to each other: (i) $h \in \operatorname{Tr} D$; (ii) $h \in \mathcal{A}$; (iii) $\limsup h(x)/(xh'^{+}(x)) < \infty$ $(x \to 0+)$.

Proof. The implication (i) → (ii) follows from 3.5; the implication (ii) → (iii) follows from 3.2 and 3.11. Now suppose that (iii) holds. Set $g = h^{-1}$, $\gamma = Ug$ $(= g'^{+}$ on $[0,1])$. We prove first that (10) holds for $a = 0$. Let $0 < t < x < 1$. Since $\operatorname{var}(\gamma, t, x) \leq \gamma(x)$, we have $(g(x))^{-1} \int_0^x \operatorname{var}(\gamma, t, x)dt \leq x\gamma(x)/g(x)$ and (10) follows from (iii). The reader easily verifies that (10) holds for each $a \in (0,1]$. Now (i) follows from 3.9.

Remark 1. It follows at once from 3.3 that there are concave functions $h \in H \setminus \mathcal{A}$; by 3.12, such an h does not fulfill (iii). It is, however, easy to construct a concave function $h \in H$ violating (iii) such that $h(x) \leq x^{1/2}$ $(x \in J)$.

Remark 2. Condition (iii) is certainly fulfilled (for h in H and concave), if $h'^{+}(0) < \infty$. In particular, each differentiable concave element of H is in $\operatorname{Tr} D$. Similarly, $h \in \operatorname{Tr} D$ for each differentiable convex function $h \in H$. If, however, $h \in H$ and if h is the difference of two differentiable convex functions, we need not have even $h \in \mathcal{A}$ as the following example shows.

3.13. Example of an $h \in H \setminus \mathcal{A}$ such that h'' is continuous on J and $h' > 0$ on $(0,1]$.

Let $a_n = 3^{-n}$ $(n = 0,1,\ldots)$. Set $\varphi(x) = x - \sin x$,

$\lambda_n(x) = 2\pi a_n^{-1}(x-a_n)$, $\alpha_n = 10^{-n}$, $\beta_n = 9\alpha_n$. Note that $\varphi'(0) = \varphi'(2\pi) = \varphi''(0) = \varphi''(2\pi) = 0$ and that φ increases. Let $\psi(0) = 0$, $\psi = \alpha_n + \beta_n(2\pi)^{-1}\varphi\circ\lambda_n$ on $(a_n, 2a_n]$, $\psi = \alpha_{n-1}$ on $(2a_n, 3a_n]$. Obviously $\psi(a_n+) = \alpha_n$, $\psi(2a_n) = \alpha_n + \beta_n = \alpha_{n-1}$ so that ψ is continuous and nondecreasing on J. We have $\psi' = \beta_n 3^n \varphi'\circ\lambda_n$, $\psi'' = 2\pi\beta_n 9^n \varphi''\circ\lambda_n = 18\pi(9/10)^n \sin\circ\lambda_n$ on $(a_n, 2a_n)$. We see that the functions ψ' and ψ'' are continuous on J as well. Set $h(x) = \frac{1}{2}(x^3 + \psi(x))$ $(x \in J)$. Then $h \in H$, $h' > 0$ on $(0,1]$ and h'' is continuous on J. Set $A_n = h(2a_n)$, $B_n = h(3a_n)$ $(n = 1,2,\ldots)$, $S = \bigcup(A_n, B_n)$. Obviously $B_n - A_n = \frac{19}{2}27^{-n}$. If $A_n < x \leq A_{n-1}$, then $|S \cap (0,x)| \leq \sum_{k=n}^{\infty}(B_k - A_k) = \frac{19}{52}27^{1-n}$. Since $A_n > \psi(2a_n)/2 = 10^{1-n}/2$, we have $|S \cap (0,x)|/x \to 0$ $(x \to 0+)$. However, $h^{-1}(S) = \bigcup(2a_n, 3a_n)$. This shows that $h^{-1} \notin \mathcal{B}$. By 3.2 we have $h \in H\setminus\mathcal{A}$.

4. MULTIPLICATION. For each system $S \subset D$ let $\operatorname{Mu} S$ be the family of all functions α such that $f\alpha \in D$ for each $f \in S$. There is a close connection between $\operatorname{Mu} S$ and $\operatorname{Tr} S$. To see this, choose functions $f \in D$ and $h \in H$ such that $0 < h' < \infty$ on J. Let $g = h^{-1}$. It follows from the chain rule that $fg' \in D$ if and only if $f\circ h \in D$. This shows that, for any $S \subset D$, we have $h \in \operatorname{Tr} S$ if and only if $g' \in \operatorname{Mu} S$. This observation helps us to describe $\operatorname{Mu} D$ (see 4.5). We need first the auxiliary assertion 4.1 from which we obtain easily in 4.2 the result $\operatorname{Mu} L = bD$. It is also true that $\operatorname{Mu} bD = L$. This, however, will be proved elsewhere together with the description of systems $\operatorname{Mu} S$ for some other families $S \subset D$.

4.1. Let α be a function on J such that $\limsup \alpha(x) = \infty$ $(x \to 0+)$. Then there is an $f \in D$ such that $f(0) = 0$, f is continuous and nonnegative on $(0,1]$ (in particular, $f \in L$) and $f\alpha \notin D$.

Proof. If α is not a derivative on $(0,1]$, then there is an $a \in (0,1)$ such that α is not a derivative on $(a,1]$. Then the function f such that $f(x) = x$ on $[0,a)$ and $f = a$ on $[a,1]$ fulfills our requirements.

Now suppose that α is a derivative on $(0,1]$. There are $a_n \in (0,1)$ such that $2a_n < a_{n-1}$ and $\alpha(a_n) > n$. There are $b_n \in (a_n, 2a_n)$ such that $\int_{a_n}^{b_n}\alpha > n(b_n - a_n)$ $(n = 1,2,\ldots)$. There is a function f continuous and nonnegative on $(0,1]$ such that $f = a_n/(n(b_n - a_n))$ on (a_n, b_n) and that $\int_{a_n}^{a_{n-1}} f < 2a_n/n$. If

$a_n < x \le a_{n-1}$, then $x^{-1}\int_0^x f \le a_n^{-1}\int_0^{a_{n-1}} f < 4/n$. Set $f(0) = 0$. Then $f \in D$. Suppose that there is a Q such that $Q' = f\alpha$ on J. We may suppose that $Q(0) = 0$. Obviously $Q'(0) = 0$ so that $(Q(b_n)-Q(a_n))/b_n = (Q(b_n)/b_n)-(a_n/b_n)Q(a_n)/a_n \to 0$. However, $Q(b_n)-Q(a_n) = (a_n/(n(b_n-a_n)))\int_{a_n}^{b_n}\alpha > a_n > b_n/2$ for each n which is a contradiction.

4.2. $\mathrm{Mu}\, L = bD$.

Proof. Let $f \in L$, $\alpha \in D$, $|\alpha| \le 1$, $c \in J$. Then $|(x-c)^{-1}\int_c^x (f-f(c))\cdot\alpha| \le (x-c)^{-1}\int_c^x |f-f(c)| \to 0$, $(x-c)^{-1}\int_c^x f(c)\cdot\alpha \to f(c)\alpha(c)$ $(x \to c,\ x \in J)$. Thus $f\alpha \in D$, $\alpha \in \mathrm{Mu}\, L$.

Now let $\alpha \in \mathrm{Mu}\, L$. It is obvious that $\alpha \in D$ and it follows easily from 4.1 that α is bounded.

4.3. $\mathrm{Mu}\, D \subset bC_{ap}$. (See [4], Theorems 4 and 8.)

Proof. Obviously $\mathrm{Mu}\, D \subset W \cap \mathrm{Mu}\, L$. Now we apply 1.4, 1.3 and 4.2.

4.4. Let ψ be a finite nonincreasing function on $(0,1)$. Let $A = \limsup(\psi(x)-\psi(2x))$, $B = \limsup x^{-1}\int_0^x(\psi(t)-\psi(x))dt$ $(x \to 0+)$. Then $A \le 2B$, $B \le 2A$.

Proof. If $A < A_1 < \infty$, then there is a $\delta \in (0,1)$ such that $\psi(x/2)-\psi(x) < A_1$, whenever $0 < x < \delta$. Choose such an x. Obviously $\psi(x/2^n)-\psi(x) < nA_1$ for $n = 1,2,\ldots$ so that $\int_0^x(\psi(t)-\psi(x))dt < \Sigma_{n=1}^{\infty} nA_1 x/2^n = 2A_1 x$. Thus $B \le 2A_1$, $B \le 2A$.

If $B < B_1 < \infty$, then there is a $\delta \in (0,1)$ such that $\int_0^{2x}(\psi(t)-\psi(2x))dt < 2B_1 x$, whenever $0 < x < \delta$. Choose such an x. Then $2B_1 x > \int_0^x(\psi(x)-\psi(2x))dt = x(\psi(x)-\psi(2x))$ so that $A \le 2B_1$, $A \le 2B$.

4.5. Let $\alpha \in D$. Then $\alpha \in \mathrm{Mu}\, D$ if and only if

(12) $\limsup \mathrm{var}(\alpha,(a+x)/2,x) < \infty$ $(x \to a,\ x \in J)$ for each $a \in J$.

Proof. If (12) holds, then, by 4.4, $\limsup(x-a)^{-1}\int_a^x \mathrm{var}(\alpha,t,x)dt < \infty$ for each $a \in J$. Let $f \in D$, $a \in J$. It follows from 2.6 with $\varphi = 1$, $\lambda = \alpha(a)$ that $f\alpha$ is P-integrable and that $(x-a)^{-1}\int_a^x f\alpha \to (f\alpha)(a)$ $(x \to a,\ x \in J)$. Thus $f\alpha \in D$, $\alpha \in \mathrm{Mu}\, D$.

Now let $\alpha \in \mathrm{Mu}\, D$. It follows from 4.3 that there is a $c \in R$ such that $\alpha + c > 0$ on J. Let $\gamma = \alpha + c$, $g' = \gamma/\int_J\gamma$, $g(0) = 0$. Since $g \in H$ and $g' \in \mathrm{Mu}\, D$, we have $g^{-1} \in \mathrm{Tr}\, D$ whence, by 3.9,

$$\limsup(g(x)-g(a))^{-1}\int_a^x \mathrm{var}(g',t,x)dt < \infty \text{ for each } a \in J.$$

Therefore $\limsup(x-a)^{-1}\int_a^x \mathrm{var}(\alpha,t,x)dt < \infty$ $(x \to a,\ x \in J)$ for each $a \in J$ and (12) follows from 4.4.

4.6. Example of an $h \in H \setminus \mathrm{Tr}\, D$ such that $\frac{1}{2} < h' < 2$ on J. (Cf. [2], Example 2.)

Let $\alpha \in D \setminus C_{ap}$, $2^{-1/2} < \alpha < 2^{1/2}$. (We may choose, e.g., $\alpha(0) = 1$, $\alpha(x) = 1 + \frac{1}{4} \sin x^{-1}$ for $x \in (0,1]$.) Let $g' = \alpha / \int_J \alpha$, $g(0) = 0$, $h = g^{-1}$. From $a^{\alpha} \in D$ it would follow (see 1.4 and 1.3) that $\alpha \in C_{ap}$; thus $\alpha g' \notin D$, $\alpha \circ h \notin D$, $h \in H \setminus \mathrm{Tr}\, D$. Obviously $\frac{1}{2} < g' < 2$, $\frac{1}{2} < h' < 2$ on J.

4.7. The inclusions $\mathrm{Tr}\, D \subset \mathrm{Tr}\, L \subset \mathrm{Tr}\, b\, C_{ap}$ $(= \mathcal{A})$ are proper.

Proof. It is easy to see that there is a nonnegative function $\varphi \in D$ such that $\varphi^2 \in D$, $\varphi^3 \notin D$. (Such a φ may be continuous on (0,1].) Let $\psi = \varphi + 1$, $g' = \psi / \int_J \psi$, $g(0) = 0$, $h = g^{-1}$. Choose an $\alpha \in b\, C_{ap}$. Since $\alpha, \psi \in W$, we have (see 1.9) $\alpha g' \in L$ so that $\alpha \circ h \in D$, $h \in \mathrm{Tr}\, b\, C_{ap}$. However, $\varphi^2 g' \notin D$ so that $\varphi^2 \circ h \notin D$. Thus (since $\varphi^2 \in L$) $h \notin \mathrm{Tr}\, L$. This shows that the second inclusion is proper. It follows from 4.6 that the first inclusion is proper.

BIBLIOGRAPHY

1. A. M. Bruckner, "Density-preserving homeomorphisms and a theorem of Maximoff," Quart. J. Math. Oxford (2), 21 (1970), 337-347.

2. ________, "On transformations of derivatives," Proc. Amer. Math. Soc. 48 (1975), 101-107.

3. ________, Differentiation of real functions, Lecture Notes in Mathematics, 659, Springer-Verlag, Berlin Heidelberg New York, 1978.

4. R. J. Fleissner, "Distant bounded variation and products of derivatives," Fund. Math. 94 (1977), 1-11.

5. M. Laczkovich and G. Petruska, "On the transformers of derivatives," Fund. Math. (1978), 179-199.

6. S. Saks, Theory of the integral, Monografie Matematyczne 7, Warszawa-Lwów, 1937.

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