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Multipliers of Classes of Derivatives

The origins of this work go back to the fact, published in 1921 by Wilcosz, that the product of two derivatives need not be a derivative. This suggests the problem of finding all derivatives whose product with each derivative is again a derivative. This problem was solved by Richard Fleissner in 1977. In this paper we solve a similar problem but for several subsets of the class of all derivatives.

Notation. Set $I = [0,1]$ and let $\mathbb{R}$ be the real line. Let

$$D = \{f : I \to \mathbb{R} \ : \text{ for some } F : I \to \mathbb{R} \ \ x \in I \text{ implies } F'(x) = f(x)\}.$$

Let $A \subset D$. Then $M(A) = \{g \in D : fg \in D \text{ for all } f \in A\}$.

To define the classes of derivatives to be consider, we first introduce some additional notation.

Notation. Let $J \subset I$ be a closed and nondegenerate interval, let $f : J \to \mathbb{R}$ be measurable and let $p \in (0, \infty)$. Then $\|f\|_{J,p} = \left(\frac{1}{|J|}\int_J |f|^p\right)^{1/p}$. Also $\|f\|_{J,\infty}$ is the usual L^∞-norm of f on J. If a and b are the endpoints of J, we also write $\|f\|_{a,b,p}$ for $\|f\|_{J,p}$ even if $b < a$.

Note that the norm of the function identically 1 on J is 1.

Proposition 1. *Let f and J be as above and let $0 < p < q \leq \infty$. Then $\|f\|_{J,p} \leq \|f\|_{J,q}$.*

Now we define some of the classes to be investigated.

Definition. Let $p \in (0, \infty)$. Then

$$S_p = \{g \in D : x \in I \text{ implies } \lim_{x \to y, x \in I} \|g - g(y)\|_{y,x,p} = 0\}$$

and

$$T_p = \{g \in D : x \in I \text{ implies } \limsup_{x \to y, x \in I} \|g\|_{y,x,p} < \infty\}.$$

Note that g is continuous at y if and only if $\lim_{x \to y, x \in I} \|g - g(y)\|_{y,x,\infty} = 0$. So we should think of the condition defining S_p as saying that g is continuous at y in the L^p-norm. The meaning of the condition defining T_p is not so clear.

But observe that $\limsup_{x\to y, x\in I} \|g\|_{y,x,\infty} < \infty$ simply means that there is a neighborhood of y on which g is bounded. Thus we may think of $g \in T_p$ as meaning that the derivative g is locally bounded in L^p-norm on I.

Proposition 2. *Let $p, q \in (0,\infty)$ with $p < q$. Then $S_q \subset S_p$, $T_q \subset T_p$ and $S_p \subset T_p$.*

The first two inclusions in the above proposition motivate the following definition of the remaining classes to be studied.

Definition. For $p \in [0,\infty)$ let

$$\underline{S_p} = \{g \in D : y \in I \text{ implies } \lim_{x\to y, x\in I} \|g - g(y)\|_{y,x,q} = 0 \text{ for some } q \in (p,\infty)\}$$

and

$$\underline{T_p} = \{g \in D : y \in I \text{ implies } \limsup_{x\to y, x\in I} \|g\|_{y,x,q} < \infty \text{ for some } q \in (p,\infty)\}.$$

For $p \in (0,\infty]$ let $\overline{S_p} = \cap_{q\in(0,p)} S_q$ and $\overline{T_p} = \cap_{q\in(0,p)} T_q$. Finally let $S_0 = D \cap C_{ap}$ (the approximately continuous functions), let $T_0 = D$, let $S_\infty = M(T_1)$ and let $T_\infty = bD$ (the bounded functions in D).

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Proposition 3. *Let $p, q \in (0,\infty)$ with $p < q$. Then $\underline{S_p} \subset S_p \subset \overline{S_p}$, $\underline{T_p} \subset T_p \subset \overline{T_p}$, $\underline{S_p} \subset \underline{T_p}$, $\overline{S_p} \subset \overline{T_p}$, $\overline{S_q} \subset \underline{S_p}$, and $\overline{T_q} \subset \underline{T_p}$.*

So for $p \in (0,\infty)$ we have

$$\begin{array}{ccccccccccccccccc}
S_0 & \supset & \underline{S_0} & \supset & \cdots & \supset & \overline{S_p} & \supset & S_p & \supset & \underline{S_p} & \supset & \cdots & \supset & \overline{S_\infty} & \supset & S_\infty \\
\cap & & \cap & & & & \cap & & \cap & & \cap & & & & \cap & & \cap \\
T_0 & \supset & \underline{T_0} & \supset & \cdots & \supset & \overline{T_p} & \supset & T_p & \supset & \underline{T_p} & \supset & \cdots & \supset & \overline{T_\infty} & \supset & T_\infty.
\end{array}$$

We are now ready to state the main theorems.

Theorem 1. *Let $S_1 \subset A \subset D$. Then $M(A) = M(D)$.*

For the second theorem we introduce some standard notation.

Notation. Let $p \in (1,\infty)$. Then p' denotes the unique number in $(0,\infty)$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. Also $1' = \infty$ and $\infty' = 1$.

Theorem 2. *Let $p \in [1,\infty]$. Then $M(S_p) = T_{p'}$ and $M(T_p) = S_{p'}$. Let $p \in [1,\infty)$. Then $M(\underline{S_p}) = \overline{T_{p'}}$ and $M(\underline{T_p}) = \overline{S_{p'}}$. Let $p \in (1,\infty]$. Then $M(\overline{S_p}) = \underline{T_{p'}}$ and $M(\overline{T_p}) = \underline{S_{p'}}$.*