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MULTIPLIERS OF SPACES OF DERIVATIVES

1 Introduction

Let $I = [0,1]$ and let

$$D = \{f : I \to \mathbb{R}; \exists\, F : I \to \mathbb{R},\ F' = f\}.$$

In 1921 Wilkosz showed that there is an $f \in D$ such that $f^2 \notin D$. So it is natural to ask what is the following set.

$$W = \{g \in D; fg \in D\, \forall\, f \in D\}.$$

Using integration by parts it's easy to show $C_1 \subset W$. On the other hand there are differentiable functions that don't belong to W and there are discontinuous functions that do belong to W. In 1977 Fleissner showed that

$$W = \{g \in D; \limsup_{h\to 0} \operatorname{var}(g, y+h, y+2h) < \infty \forall\, y \in I\}.$$

Here the goal is to investigate a more general problem. For X, $Y \subset D$ let

$$M(X,Y) = \{g \in D; \forall\, f \in X,\ fg \in Y\}.$$

For simplicity $M(X,D) = M(X)$. Thus $W = M(D)$. The proof of the first assertion is easy.

Proposition 1. *Let $X_1 \subset X_2 \subset D$ and $Y_1 \subset Y_2 \subset D$. Then $M(X_2,Y_1) \subset M(X_1,Y_2)$.*

*Presenter

2 Subspaces of D

The algebra $C_{ap} = \{g : I \to \mathbb{R}; g \text{ is approximately continuous on } I\}$ isn't a subspace of D, but as is well known, the bounded functions in $C_{ap} = bC_{ap} \subset D$. The subspaces of D of central interest as defined below. For $p \in (0, \infty)$ let

$$S_p = \{g \in D; \lim_{x \to y} \frac{1}{x-y} \int_y^x |f - f(y)|^p = 0, \forall\, y \in I\}.$$

$$T_p = \{g \in D; \limsup_{x \to y} \frac{1}{x-y} \int_y^x |f|^p < \infty, \forall\, y \in I\}.$$

$$S_0 = D \cap C_{ap},\ T_0 = D,\ T_\infty = bD,\ S_\infty = M(T_1).$$

At first sight, the choice of $M(T_1)$ for S_∞ looks strange, but it will seem quite natural in view of Theorem 4 below.

For $p \in (0, \infty]$ let

$$\overline{S}_p = \cap_{q \in (0,p)} S_q \text{ and } \overline{T}_p = \cap_{q \in (0,p)} T_q$$

while for $p \in [0, \infty)$ let

$$\underline{S}_p = \{g \in D;\ \forall\, y \in I,\ \exists\, q > p \lim_{x \to y} \frac{1}{x-y} \int_y^x |f - f(y)|^q = 0\}$$

and

$$\underline{T}_p = \{g \in D;\ \forall\, y \in I,\ \exists\, q > p \limsup_{x \to y} \frac{1}{x-y} \int_y^x |f|^q < \infty\}$$

Proposition 2. *The following containments hold (and are easy to establish). Let $0 < p_1 < p_2 < \infty$. Then*

$$\begin{array}{ccccccccccccccccccccccc}
T_\infty & \subset & \overline{T}_\infty & \subset & \cdots & \subset & \underline{T}_{p_2} & \subset & T_{p_2} & \subset & \overline{T}_{p_2} & \subset & \cdots & \subset & \underline{T}_{p_1} & \subset & T_{p_1} & \subset & \overline{T}_{p_1} & \subset & \cdots & \subset & \underline{T}_0 \subset T_0 \\
 & & \cup & & & & \cup & & \cup & & \cup & & & & \cup & & \cup & & \cup & & & & \cup \quad \cup \\
 & & \overline{S}_\infty & \subset & \cdots & \subset & \underline{S}_{p_2} & \subset & S_{p_2} & \subset & \overline{S}_{p_2} & \subset & \cdots & \subset & \underline{S}_{p_1} & \subset & S_{p_1} & \subset & \overline{S}_{p_1} & \subset & \cdots & \subset & \underline{S}_0 \subset S_0.
\end{array}$$

The two missing containments in the lower left hand corner; namely $S_\infty \subset \overline{S}_\infty$ and $S_\infty \subset T_\infty$, are true as well, but they are not trivial due to the unusual definition of S_∞.

Proposition 3. *Let $p \in (0, \infty]$. Then $\overline{T}_p \cap C_{ap} = \overline{S}_p$. Let $p \in [0, \infty)$. Then $\underline{T}_p \cap C_{ap} = \underline{S}_p$. For $p \in (0, \infty]$, $S_p \subsetneq T_p \cap C_{ap}$.*

3 Multiplier Spaces

The first theorem uses some standard notation. For $p \in [1, \infty]$, p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ where $\frac{1}{\infty} = 0$. In addition S (and later $\widetilde{S}$) is used to denote any of the spaces $\overline{S}_p$, S_p, and $\underline{S}_p$ defined above and similarly for T (and later $\widetilde{T}$).

Theorem 4. *The spaces of multipliers $M(S)$ and $M(T)$ are displayed in the following two charts. Let $\infty > p > 1 > q > 0$.*

S	S_∞	$\overline{S}_\infty$	$\cdots$	$\underline{S}_p$	S_p	$\overline{S}_p$	$\cdots$	$\underline{S}_1$	S_1	$\overline{S}_1$	$\cdots$	$\underline{S}_q$	S_q	$\overline{S}_q$	$\cdots$	$\underline{S}_0$	S_0
$M(S)$	T_1	$\underline{T}_1$	$\cdots$	$\overline{T}_{p'}$	$T_{p'}$	$\underline{T}_{p'}$	$\cdots$	$\underline{T}_\infty$	T_∞	W	$\cdots$	W	W	W	$\cdots$	W	W

T	T_∞	$\overline{T}_\infty$	$\cdots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\cdots$	$\underline{T}_1$	T_1	$\overline{T}_1$	$\cdots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\cdots$	$\underline{T}_0$	T_0
$M(T)$	S_1	$\underline{S}_1$	$\cdots$	$\overline{S}_{p'}$	$S_{p'}$	$\underline{S}_{p'}$	$\cdots$	$\underline{S}_\infty$	S_∞	W	$\cdots$	W	W	W	$\cdots$	W	W

Now the multiplier spaces $M(X, Y)$ are investigated where X and Y are any of the spaces introduced above. There are four types: $M(T, S)$, $M(S, T)$, $M(T, \widetilde{T})$ and $M(S, \widetilde{S})$. The results are best expressed in charts which are displayed at the end of this report.

Theorem 5. $M(T_\infty, S_0) = \{0\}$.

The $M(T, S)$ can be filled in as a result of Theorem 5 and Proposition 1. Together they assert that for every T space and for every S space, $M(T, S) = \{0\}$. That is, every entry of the $M(T, S)$ chat is $\{0\}$.

Moving on to the $M(S, T)$ chart first note that the last column, the one headed by $T_0 = D$, can be filled in using the bottom row of the second chart of Theorem 4. Similarly the last column of the $M(T, \widetilde{T})$ chart is obtained from the bottom row of the first chart of Theorem 4.

Theorem 6. *For $p \in [0, \infty)$, $M(S_p, \underline{T}_p) = \{0\}$ and for $p \in (0, \infty]$, $M(\overline{S}_p, T_p) = \{0\}$.*

As a consequence of Theorem 6 and Proposition 1 each entry below the main diagonal in the $M(S, T)$ chart and in each of the remaining charts is $\{0\}$. Such entries are denoted by leaving the corresponding places blank.

Theorem 7. *Let $\overline{S}_1 \subset X$. Then $M(X, X) = W = M(D) = M(T_0)$.*

Consequently every entry on the main diagonal below the $\overline{S}_1$ column is W and hence by Proposition 1 the lower right triangle of the $M(S, T)$ chart consists of Ws. The same conclusion holds for each of the remaining charts as well.

Theorem 8. *Let $X \subset T_1 \subset Y$. Then $M(X, Y) = M(X)$.*

This theorem says that the T_1 column = the T_0 column down to the $\overline{S}_1$ row and consequently by Proposition 1 the columns between also = the T_0 one. The corresponding conclusion is also true for the $M(T,\widetilde{T})$ chart. The same conclusion holds for the $M(S,\widetilde{S})$ once the two bounding columns are known.

To complete the remainder of the $M(S,T)$ chart, the following notation is used. Let $1 \le q \le p \le \infty$. Define $r \in [1,\infty]$ by $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

Theorem 9. *For p, $q \in (1,\infty]$ with $q < p$*

$X\backslash Y$	$\underline{T}_q$	T_q	$\overline{T}_q$
$\underline{S}_p$	$\overline{T}_r$	$\overline{T}_r$	$\overline{T}_r$
S_p	$\underline{T}_r$	T_r	$\overline{T}_r$
$\overline{S}_p$	$\underline{T}_r$	$\underline{T}_r$	$\overline{T}_r$

and for $p \in (1,\infty]$,

$X\backslash Y$	$\underline{T}_p$	T_p	$\overline{T}_p$
$\underline{S}_p$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$
S_p		T_∞	$\overline{T}_\infty$
$\overline{S}_p$			$\overline{T}_\infty$

The only cases not covered are the upper part of the column headed by $\underline{T}_1$. Rather than state all of these results, the reader is referred to the $M(S,T)$ chart, Figure 1, page 241.

To determine the $M(T,\widetilde{T})$ chart, we need the analogue of Theorem 9 and some additional notation. For any T space let $\hat{T} = T \cap C_{ap}$.

Theorem 10. *For p, $q \in (1,\infty]$ with $q < p$*

$X\backslash Y$	$\underline{T}_q$	T_q	$\overline{T}_q$
$\underline{T}_p$	$\overline{S}_r$	$\overline{S}_r$	$\overline{S}_r$
T_p	$\underline{S}_r$	$\hat{T}_r$	$\overline{S}_r$
$\overline{T}_p$	$\underline{S}_r$	$\underline{S}_r$	$\overline{S}_r$

and for $p \in (1,\infty]$,

$X\backslash Y$	$\underline{T}_p$	T_p	$\overline{T}_p$
$\underline{T}_p$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$
T_p		$\hat{T}_\infty$	$\overline{S}_\infty$
$\overline{T}_p$			$\overline{S}_\infty$

The complete $M(T,\widetilde{T})$ chart is on page 242 where all relevant results can be found.

The body of the $M(S,\widetilde{S})$ chart is similar to that of the $M(T,\widetilde{T})$ chart with some notable exceptions. The top row is identical to the corresponding column headings from S_∞ to S_1. For $p \in [1,\infty]$, $M(S_p, S_1) = M(S_p, S_0) = \hat{T}_{p'}$ thereby determining all corresponding rows from the S_1 column to the S_0 column. The $M(S,\widetilde{S})$ chart is on page 243.

$X\backslash Y$	T_∞	$\overline{T}_\infty$	$\ldots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\ldots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\ldots$	$\underline{T}_1$	T_1	$\overline{T}_1$	$\ldots$	$\underline{T}_u$	T_u	$\overline{T}_u$	$\ldots$	$\underline{T}_0$	T_0
S_∞	T_∞	$\overline{T}_\infty$	$\ldots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\ldots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\ldots$	$\underline{T}_1$	T_1	T_1	$\ldots$	T_1	T_1	T_1	$\ldots$	T_1	T_1
$\overline{S}_\infty$		$\overline{T}_\infty$	$\ldots$	$\underline{T}_p$	$\underline{T}_p$	$\overline{T}_p$	$\ldots$	$\underline{T}_q$	$\underline{T}_q$	$\overline{T}_q$	$\ldots$	$\underline{T}_1$	$\underline{T}_1$	$\underline{T}_1$	$\ldots$	$\underline{T}_1$	$\underline{T}_1$	$\underline{T}_1$	$\ldots$	$\underline{T}_1$	$\underline{T}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\underline{S}_p$				$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\ldots$	$\overline{T}_r$	$\overline{T}_r$	$\overline{T}_r$	$\ldots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\ldots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\overline{T}_{p'}$	$\ldots$	$\overline{T}_{p'}$	$\overline{T}_{p'}$
S_p					T_∞	$\overline{T}_\infty$		$\underline{T}_r$	T_r	$\overline{T}_r$	$\ldots$	$\underline{T}_{p'}$	$T_{p'}$	$T_{p'}$	$\ldots$	$T_{p'}$	$T_{p'}$	$T_{p'}$	$\ldots$	$T_{p'}$	$T_{p'}$
$\overline{S}_p$						$\overline{T}_\infty$	$\ldots$	$\underline{T}_r$	$\underline{T}_r$	$\overline{T}_r$	$\ldots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\ldots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\underline{T}_{p'}$	$\ldots$	$\underline{T}_{p'}$	$\underline{T}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\underline{S}_q$								$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\ldots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\ldots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\overline{T}_{q'}$	$\ldots$	$\overline{T}_{q'}$	$\overline{T}_{q'}$
S_q									T_∞	$\overline{T}_\infty$	$\ldots$	$\underline{T}_{q'}$	$T_{q'}$	$T_{q'}$	$\ldots$	$T_{q'}$	$T_{q'}$	$T_{q'}$	$\ldots$	$T_{q'}$	$T_{q'}$
$\overline{S}_q$										$\overline{T}_\infty$	$\ldots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\ldots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\underline{T}_{q'}$	$\ldots$	$\underline{T}_{q'}$	$\underline{T}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\underline{S}_1$												$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\ldots$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\overline{T}_\infty$	$\ldots$	$\overline{T}_\infty$	$\overline{T}_\infty$
S_1													T_∞	T_∞	$\ldots$	T_∞	T_∞	T_∞	$\ldots$	T_∞	T_∞
$\overline{S}_1$														W	$\ldots$	W	W	W	$\ldots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\underline{S}_u$																W	W	W	$\ldots$	W	W
S_u																	W	W	$\ldots$	W	W
$\overline{S}_u$																		W	$\ldots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{S}_0$																				W	W
S_0																					W

Figure 1: The $M(S,T)$ chart.

$X\backslash Y$	T_∞	$\overline{T}_\infty$	$\dots$	$\underline{T}_p$	T_p	$\overline{T}_p$	$\dots$	$\underline{T}_q$	T_q	$\overline{T}_q$	$\dots$	$\underline{T}_1$	T_1	$\overline{T}_1$	$\dots$	$\underline{T}_u$	T_u	$\overline{T}_u$	$\dots$	$\underline{T}_0$	T_0
T_∞	$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\hat{T}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\hat{T}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	S_1	$\dots$	S_1	S_1	S_1	$\dots$	S_1	S_1
$\overline{T}_\infty$		$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\underline{S}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\underline{S}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_p$				$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_r$	$\overline{S}_r$	$\overline{S}_r$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$
T_p					$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\hat{T}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$S_{p'}$	$S_{p'}$	$\dots$	$S_{p'}$	$S_{p'}$	$S_{p'}$	$\dots$	$S_{p'}$	$S_{p'}$
$\overline{T}_p$						$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\underline{S}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_q$								$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$
T_q									$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$S_{q'}$	$S_{q'}$	$\dots$	$S_{q'}$	$S_{q'}$	$S_{q'}$	$\dots$	$S_{q'}$	$S_{q'}$
$\overline{T}_q$										$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_1$												$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$
T_1													S_∞	S_∞	$\dots$	S_∞	S_∞	S_∞	$\dots$	S_∞	S_∞
$\overline{T}_1$														W	$\dots$	W	W	W	$\dots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{T}_u$																W	W	W	$\dots$	W	W
T_u																	W	W	$\dots$	W	W
$\overline{T}_u$																		W	$\dots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{T}_0$																				W	W
T_0																					W

Figure 2: The $M(T,\widetilde{T})$ Chart.

$X\backslash Y$	S_∞	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	S_p	$\overline{S}_p$	$\dots$	$\underline{S}_q$	S_q	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	$\overline{S}_1$	$\dots$	$\underline{S}_u$	S_u	$\overline{S}_u$	$\dots$	$\underline{S}_0$	S_0
S_∞	S_∞	$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	S_p	$\overline{S}_p$	$\dots$	$\underline{S}_q$	S_q	$\overline{S}_q$	$\dots$	$\underline{S}_1$	S_1	$\hat{T}_1$	$\dots$	$\hat{T}_1$	$\hat{T}_1$	$\hat{T}_1$	$\dots$	$\hat{T}_1$	$\hat{T}_1$
$\overline{S}_\infty$		$\overline{S}_\infty$	$\dots$	$\underline{S}_p$	$\underline{S}_p$	$\overline{S}_p$	$\dots$	$\underline{S}_q$	$\underline{S}_q$	$\overline{S}_q$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$	$\underline{S}_1$	$\dots$	$\underline{S}_1$	$\underline{S}_1$
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_p$				$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_r$	$\overline{S}_r$	$\overline{S}_r$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\overline{S}_{p'}$	$\dots$	$\overline{S}_{p'}$	$\overline{S}_{p'}$
S_p					$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\hat{T}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\dots$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\hat{T}_{p'}$	$\dots$	$\hat{T}_{p'}$	$\hat{T}_{p'}$
$\overline{S}_p$						$\overline{S}_\infty$	$\dots$	$\underline{S}_r$	$\underline{S}_r$	$\overline{S}_r$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\underline{S}_{p'}$	$\dots$	$\underline{S}_{p'}$	$\underline{S}_{p'}$
$\vdots$							$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_q$								$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\overline{S}_{q'}$	$\dots$	$\overline{S}_{q'}$	$\overline{S}_{q'}$
S_q									$\hat{T}_\infty$	$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\dots$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\hat{T}_{q'}$	$\dots$	$\hat{T}_{q'}$	$\hat{T}_{q'}$
$\overline{S}_q$										$\overline{S}_\infty$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\underline{S}_{q'}$	$\dots$	$\underline{S}_{q'}$	$\underline{S}_{q'}$
$\vdots$											$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_1$												$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\overline{S}_\infty$	$\dots$	$\overline{S}_\infty$	$\overline{S}_\infty$
S_1													$\hat{T}_\infty$	$\hat{T}_\infty$	$\dots$	$\hat{T}_\infty$	$\hat{T}_\infty$	$\hat{T}_\infty$	$\dots$	$\hat{T}_\infty$	$\hat{T}_\infty$
$\overline{S}_1$														W	$\dots$	W	W	W	$\dots$	W	W
$\vdots$															$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\underline{S}_u$																W	W	W	$\dots$	W	W
S_u																	W	W	$\dots$	W	W
$\overline{S}_u$																		W	$\dots$	W	W
$\vdots$																			$\ddots$	$\vdots$	$\vdots$
$\underline{S}_0$																				W	W
S_0																					W

Figure 3: The $M(S,\widetilde{S})$ Chart.