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ON A THEOREM OF BRUNOVSKÝ FOR PERIODIC OPTIMAL CONTROL

by LAWRENCE MARKUS

CONTROL OF LIMIT CYCLES AND APPLICATIONS TO CARDIOLOGY

In control theory we consider a process or plant or dynamical system described by a differential system

$$\dot{x} = f(x, u)$$

where x is the real state n-vector at time t, and the coefficient f is an n-vector function of the present state x and the control m-vector u. For simplicity we assume the process is autonomous (time-independent) and that f is continuous with continuous first derivatives for all $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$, that is

$$f : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$$

is in class C^1.

We might seek to control $x(t)$ between given initial and final states in some fixed duration $0 \leqq t \leqq T$,

$$x(0) = x_0, \quad x(T) = x_1 ,$$

by choosing a control function $u(t)$ from some admissible function class (say $u \in L_\infty[0, T]$, that is, $u(t)$ is a bounded measurable function on $0 \leqq t \leqq T$). Hence $x(t)$ is a solution of the two-point boundary value problem, with separated end conditions,

$$\dot{x} = f(x, u(t)), \quad x(0) = x_0 , \quad x(T) = x_1 .$$

This constitutes the basic problem of controllability in control theory.

Among all solutions $x(t)$ to this boundary value problem, that is for all admissible control functions, we might try to select and describe the optimal solution $x^*(t)$ for the optimal controller $u^*(t)$ which minimizes some given cost or performance functional $C(u)$. This leads to the central problem of optimal control theory, for which there is a vast literature.

We now turn to a new type of control problem which corresponds to a boundary value problem with periodic conditions rather than separated conditions. That is, we again consider the plant or control dynamics

$$\dot{x} = f(x, u),$$

but the admissible controllers $u(t)$ are now required to be periodic functions, with some period T. Under certain assumptions, each periodic controller $u(t)$ leads to a periodic response $x(t)$ of

$$\dot{x} = f(x, u(t)),$$

where $x(t) \equiv x(t + T)$ for all real t. That is, $x(t)$ is a periodic solution or, in the terminology of oscillation theory, $x(t)$ is a limit cycle. We do not prescribe any initial or final states except for the periodicity boundary condition

$$x(0) = x(T) .$$

We might proceed further to select and describe that limit cycle $x^*(t)$, which minimizes or maximizes some performance functional $C(u)$. In particular, we shall let $C(u)$ be the amplitude of the limit cycle $x(t)$, and find the optimal controller $u^*(t)$ to maximize this amplitude.

From the purely mathematical point of view more general types of boundary conditions could be utilized, or the periodic functions could be generalized to almost periodic functions.

While various extensions of our theory could be pursued, we shall concentrate on the control of a limit cycle. This mathematical problem was motivated by some engineering instrumentation of cardiac assist devices related to heart surgery. For instance, a heart pump must be designed to assist the heart maintain its natural amplitude of systolic and diastolic pressures. Further design improvements would force the controlled heart to maintain a circulatory regime very near to the natural healthy action. Unfortunately the dynamics of the human circulatory system are known too poorly for a useful application of any very sophisticated mathematical or engineering theory. Thus this study can be considered as an introduction to a developing engineering-medical field that could become of great practical significance.

LINEAR DYNAMICS: GEOMETRY OF LIMIT CYCLE CONTROL

We consider a linear control system

$$\dot{x} = Ax + Bu$$

where the state x is a real (column) n-vector, the control u is a real m-vector, and A and B are real constant matrices. For each bounded measurable control $u(t)$, we have the solution $x(t)$ initiating at the state x_0 as prescribed by the Lagrange formula of variations of parameters,

$$x(t) = e^{At}x_0 + e^{At} \int_0^t e^{-As} Bu(s) \, ds .$$

If $u(t)$ is periodic, say $u(t) \equiv u(t + T)$ for almost all real times t, then $x(t)$ has the same period T just in case $x(T) = x(0)$, that is,

$$x_0 = e^{AT} x_0 + e^{AT} \int_0^T e^{-As} Bu(s)\, ds .$$

Hence there exists a unique response $x(t)$ with period T, and this initiates at

$$x_0 = (I - e^{AT})^{-1} e^{AT} \int_0^T e^{-As} Bu(s)\, ds ,$$

provided the matrix $(I - e^{AT})$ is invertible. We shall assume, for simplicity, that $T = 1$ and the matrix A has no pure imaginary eigenvalues, $Re\lambda(A) \neq 0$. Then $I - e^{A}$ is invertible, and the initial point $x_0[u]$ for the unique periodic response $x(t)$ to the periodic control $u(t)$ is given by

$$x_0[u] = (I - e^{A})^{-1} e^{A} \int_0^1 e^{-As} Bu(s)\, ds .$$

The problem of optimal control of the amplitude of the limit cycle now depends on finding the admissible controller $u^*(t)$ for which $x_0[u^*]$ leads to the periodic response $x^*(t)$ having the maximal amplitude. We shall restrain the periodic control inputs $u(t)$ by the condition

$$u(t) \in \Omega \quad \text{for all } t,$$

where Ω is a given compact convex subset of the real m-vector space R^m. For instance, Ω could be the cube of unit radius centred at the origin, say $| u^i | \leqq 1$ for $i = 1, \dots, m$. Hence an admissible controller $u(t)$ is a measurable vector in Ω having period of one.

The amplitude of the response $x(t)$ will be taken to be the maximum value of the first component $x^1(t)$ of the vector $x(t)$. That is

$$C(u) = -\max_{0 \leqq t \leqq T} x^1(t)$$

and we seek to minimize the cost $C(u)$. Of course, other costs could be studied, for instance the norm

$$\| x(t) \| = \max_{0 \leqq t \leqq T} [| x^1(t) | + \dots + | x^n(t)|].$$

Definition. Consider the linear control dynamics in R^n

$$\dot{x} = Ax + Bu$$

with compact convex restraint set $\Omega \subset R^m$. For each measurable controller $u(t) \in \Omega$ with period 1, there is a unique initial point $x_0[u]$ for the response of period 1 (assuming A has no pure imaginary eigenvalues). Define the subset of R^n

$$K = \{x_0[u] \mid \text{for all admissible controllers } u(t)\} .$$

The set K corresponds to the attainable set in the usual controllability problem (separated boundary conditions), and the letter K corresponds to the conditions of compactness and convexity.

Theorem 1. *Consider the linear control dynamics in* $\boldsymbol{R}^n$

$$\dot{x} = Ax + Bu \quad (Re\lambda(A) \neq 0)$$

with compact convex restraint set $\Omega \subset R^m$.
Let $K = \{x_0[u] \mid u(t) \equiv u(t+1) \in \Omega$ *almost all* $t \in \boldsymbol{R}^1\}$.
Then

i) *K is a compact convex subset of R^n, and*
ii) *K is the union of all periodic responses $x(t)$.*

Proof. Consider the map

$$u \rightarrow x_0[u] = (I - e^A)^{-1} e^A \int_0^1 e^{-As} Bu(s)\, ds\,.$$

The set of admissible control functions $\mathscr{I}$ is convex, since Ω is convex. Also the map $u \rightarrow x_0[u]$ is linear, and so K is convex.

The function set $\mathscr{I}$ can be embedded in some closed ball B_2 of the Hilbert space $L_2[0, 1]$. We recall that B_2 is weakly compact and any sequence $u_k \in \mathscr{I}$ has a weakly convergent subsequence

$$u_{k_i} \rightharpoonup u^*.$$

Consider points $x_0[u_k] \in K$, and select a subsequence corresponding to controllers u_{ki} converging weakly to u^*. If $u^* \in \mathscr{I}$, then it follows that $x_0[u_{k_i}] \rightarrow x_0[u^*] \in K$.

Of course, we can define $u^*(t) \in L_2\ [0, 1]$ to have period one on $\boldsymbol{R}^1$. But we must still verify that the values of $u^*(t)$ lie in Ω for almost all times t. Recall that the weak limit of positive functions is positive (almost everywhere), and use this result to conclude that $u^*(t)$ lies a.e. in any half-space of $\boldsymbol{R}^n$ that contains Ω. But Ω is the intersection of a countable number of closed half-spaces, and hence $u^*(t)$ lies in Ω almost everywhere for $t \in \boldsymbol{R}^1$.

Thus $u^* \in \mathscr{I}$ is an admissible controller, and $x_0[u^*] \in K$. Hence K is compact, and the first conclusion of the theorem has been demonstrated.

The second conclusion follows from the autonomous nature of the control dynamics. For let $u(t) \in \mathscr{I}$ yield the initial state $x_0[u]$ on the periodic response $x(t)$. Then, for each positive number τ, the controller $u(t+\tau)$ yields the periodic response $x(t+\tau)$ with initial state $x(0+\tau)$. Thus the entire set $\{x(t) \mid 0 \leqq t \leqq 1\}$ lies within K, as required. Q.E.D.

We next turn to an examination of the boundary ∂K of the set $K \subset \boldsymbol{R}^n$, and we shall state the appropriate form of the maximal principle for our problem.

Theorem 2. *Consider the linear control dynamics in* R^n

$$\dot{x} = Ax + Bu \quad (Re\lambda(A) \neq 0)$$

with compact convex restraint set $\Omega \subset R^m$.
Then an admissible controller $u^*(t+1) \equiv u^*(t)$ (a.e.) *in* Ω *yields a point* $x_0[u^*] \in \partial K$ *if and only if*:

i) *there exists a nontrivial solution* $\eta^*(t)$ *of* $\dot{\eta} = -\eta A$ *which satisfies the maximal principle*

ii) $\eta^*(t)\, Bu^*(t) = \max_{u \in \Omega} \eta^*(t)\, Bu$, *almost always.*

This theorem was proved by P. BRUNOVSKY and it appears in the doctoral dissertation of Dr. D. SPYKER [2]. In this paper we shall generalise these results to certain nonlinear dynamical systems.

NONLINEAR DYNAMICS: CONTROL OF LIMIT CYCLES

Consider a nonlinear control system in $\boldsymbol{R}^n$

$$\dot{x} = f(x, u)$$

where $u(t) = u(t+1)$ is a periodic control vector lying in a compact set $\Omega \subset \boldsymbol{R}^m$. Suppose there exists a unique periodic response $x(t) = x(t+1)$ lying in some (often) compact constraint set $\Lambda \subset \boldsymbol{R}^n$. We shall seek an optimal controller $u^*(t)$ for which the response $x^*(t)$ assumes the maximal amplitude, in the sense that the first component $x^{*1}(t)$ achieves the maximum possible value at $t = 1$.

Examples from the literature on nonlinear vibrations are illustrative. For instance consider the scalar oscillator

$$\ddot{x} + \dot{x} + x + x^3 = u(t).$$

In the (x, y) phase plane $\boldsymbol{R}^2$ this system becomes

$$\dot{x} = y$$
$$\dot{y} = -x - x^3 - y + u(t),$$

and we seek periodic solutions $(x(t), y(t))$ in a small disc $\Lambda : x^2 + y^2 \leqq \varrho^2$ when the periodic control $u(t)$ is restricted to a small interval $\Omega : |u| \leqq \alpha$. The general theory of perturbations assures us that for a fixed small $\varrho > 0$ there exist suitably small $\alpha > 0$ such that each 1-periodic controller in Ω produces a unique 1-periodic response in Λ.

A more general type of controlled oscillator is described by the vector system in $\boldsymbol{R}^n$

$$\dot{x} = f(x) + Bu(t),$$

for a constant $n \times m$ matrix B. Assume that some basic equation (with $u = u_0(t)$) has a periodic solution $x = \varphi(t)$ with $\varphi(t) = \varphi(t+1)$ giving the shortest period.

Also assume that this periodic solution of the free equation is stable, or at least that the Poincaré map

$$x_0 \to x(1, x_0)$$

(where $x(t, x_0)$ is the solution of $\dot{x} = f(x) + Bu_0(t)$ initiating at x_0) does not have 1 as a characteristic multiplier, that is, the matrix $P = \frac{\partial x}{\partial x_0}(1, \varphi(o))$ has no eigenvalue 1.

Then we fix a compact tubular neighborhood Λ of $\{x = \varphi(t)\}$ and thereafter a small closed neighborhood Ω of $\{u = u_0(t)\}$. The general theory of perturbations assures us that each 1-periodic controller in Ω produces a unique 1-periodic response in Λ.

Theorem 3. *Consider the control system in* $\boldsymbol{R}^n$

$$\dot{x} = f(x, u) = f(x) + B(x)\, u$$

where $f(x, u) \in C^1$ *for* (x, u) *in a compact set* $\Lambda \times \Omega \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$ *and* Ω *is convex.*

Assume that each measurable controller

$$u(t) = u(t + 1) \in \Omega \quad \text{(a.e.)}$$

yields some 1*-periodic response* $x_u(t) \in \Lambda$. *Let the cost of* $u(t)$ *be given by*

$$C(u) = g[x_u] + \int_0^1 (f^0(x_u) + G(x_u)\, u)\, \mathrm{d}t,$$

where g is a continuous functional on the function space $C[0, 1]$ *and* $f^0(x, u) = f^0(x) + G(x)\, u$ *is continuous in* $\boldsymbol{R}^{n+m}$.

Then there exists an optimal controller $u^*(t)$ *minimizing (or maximizing) the cost at* $C(u^*)$.

Proof. Since Ω and Λ are compact, and $f^0(x, u)$ and g are continuous, we see that $\inf C(u) = m$ is finite. Take a sequence $u^{(k)}(t)$ of controllers, with corresponding responses $x^{(k)}(t)$, so $C(u^{(k)}) \searrow m$. Using subsequences we can assume (with usual arguments such as those in the text Lee-Markus p. 260 [1])

$$u^{(k)}(t) \rightharpoonup u^*(t) \text{ weakly in } L_2[0, 1].$$
$$x^{(k)}(t) \Rightarrow x^*(t) \text{ uniformly in } C[0, 2]$$

and also, weakly,

$$f(x^{(k)}(t), u^{(k)}(t)) \rightharpoonup f(x^*(t), u^*(t))$$
$$f^0(x^{(k)}(t), u^{(k)}(t)) \rightharpoonup f^0(x^*(t), u^*(t)).$$

Now $x^{(k)}(t)$ is the response to $u^{(k)}(t)$ so

$$x^{(k)}(t) = x^{(k)}(0) + \int_0^t [f(x^{(k)}(s)) + B(x^{(k)}(s))\, u^{(k)}(s)]\, \mathrm{d}s$$

and

$$x^*(t) = x^*(0) + \int_0^t [f(x^*(s)) + B(x^*(s))\, u^*(s)]\, ds.$$

Thus $x^*(t) = x^*(t + 1) \in \Lambda$ is a periodic response to the controller $u^*(t)$. Using the convexity of Ω we can show easily that $u^*(t) \in \Omega$ is an admissible controller.

Further calculations with the cost functional yield

$$C(u^{(k)}) = g[x^{(k)}] + \int_0^1 [f^0(x^{(k)}) + G(x^{(k)})\, u^{(k)}(t)]\, dt.$$

As $k \to \infty$ we compute $C(u^{(k)}) \searrow m$ and

$$\lim_{k \to \infty} C(u^{(k)}) = g[x^*] + \int_0^1 [f^0(x^*) + G(x^*)\, u^*(t)]\, dt.$$

Thus

$$C(u^*) = m,$$

and $u^*(t)$ is an optimal controller, as required. Q.E.D.

Remark. If we define the cost functional

$$C(u) = g[x] = -\max_{0 \leq t \leq 1} x^1(t),$$

then an optimal controller $u^*(t)$ that minimizes $C(u)$ will maximize the amplitude, in the above sense.

We next turn to the maximal principle as a necessary condition for an optimal control of the amplitude of a limit cycle for a nonlinear dynamical system. The concepts and methods are similar to those used in the standard formulation of the maximal principle for the control of a trajectory between given endpoints, see text Lee-Markus pp. 246–256. We use the notations and calculations of this text with no further explanation.

Theorem 4. *Consider the control system in* R^n

$$\dot{x} = f(x, u) \quad \text{in} \quad C^1 \text{ in } \boldsymbol{R}^{n+m}.$$

Use all measurable controllers $u(t) = u(t + 1)$ *lying in the compact restraint set* $\Omega \subset \boldsymbol{R}^m$ (a.e.), *some of which have* 1*-periodic responses* $x(t)$ *in* $\boldsymbol{R}^n$. *Let* $u^*(t)$ *be an optimal controller with response* $x^*(t)$ *maximizing the amplitude* $x^{*1}(t)$ *at* $x^{*1}(1)$.

Assume that $x^*(t)$ *has a Poincaré map such that*

$$P = \left.\frac{\partial x(t, x_0)}{\partial x_0}\right|_{\substack{t=1 \\ x_0 = x^*(0)}} \qquad \text{has no eigenvalue of } 1$$

(*where* $x(t, x_0)$ *is general solution of* $\dot{x} = f(x, u^*(t))$ *initiating at* x_0).

Then there exists a nontrivial row n-vector $\eta^*(t)$ satisfying the adjoint variational equation

i) $$\dot{\eta} = -\eta \frac{\partial f}{\partial x}(x^*(t), u^*(t)),$$

and the maximal principle

ii) $$\eta^*(t) f(x^*(t), u^*(t)) = \max_{u \in \Omega} \eta^*(t) f(x^*(t), u) \quad \text{(a.e.)}$$

with the terminal condition

iii) $$\eta^*(1) = (1, 0, 0, \ldots, 0)(I - P)^{-1}.$$

Remarks. Before discussing the proof of the maximal principle we consider the special case of an autonomous linear system in $\boldsymbol{R}^n$

$$\dot{x} = Ax + Bu.$$

Here $\eta^*(t)$ satisfies

$$\dot{\eta} = -\eta A, \qquad \text{since} \qquad A = \frac{\partial f}{\partial x},$$

and the maximal principle becomes

$$\eta^*(t)[Ax^*(t) + Bu^*(t)] = \max_{u \in \Omega} \eta^*(t)[Ax^*(t) + Bu],$$

or more simply

$$\eta^*(t)\, Bu^*(t) = \max_{u \in \Omega} \eta^*(t)\, Bu.$$

The terminal condition on $\eta^*(t)$ is found by computing $P = e^A$ from the formula

$$x(t, x_0) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu^*(s)\, ds.$$

Clearly P has no eigenvalue 1 just in case A has no eigenvalue that is an integral multiple of the pure imaginary $2\pi i$. In this situation the above theorem on the maximal principle reduces to earlier results for linear systems.

Just as for the earlier linear analysis, the general form of the maximal principle for nonlinear dynamics can often be used to display the bang-bang character of the optimal controller $u^*(t)$ and to guide in the computation of $u^*(t)$.

PROOF OF MAXIMAL PRINCIPLE

We shall only sketch the main ideas in the proof of the theorem, the details following the pattern offered in the text Lee-Markus.

If $x^*(t)$ is an optimal response, with $x^{*1}(1)$ achieving the maximal amplitude, then

$$(1, 0, 0, \ldots, 0)\, x^*(1) = \max\, (1, 0, 0, \ldots, 0)\, x(1),$$

where $x(t)$ is any 1-periodic response to any admissible controller $u(t)$. In other words, the hyperplane $x^1 = x^{*1}(1)$ contains all the possible points $x(1)$ on one side. This is the basic geometric fact which yields the maximal principle as an analytical interpretation.

We shall perturb $u^*(t)$ to a new admissible controller $u_\pi(t, \varepsilon)$ with periodic response $x_\pi^*(t, \varepsilon)$ (as defined below) and for $\varepsilon = 0$ the perturbed controller and response reduce to $u^*(t)$ and $x^*(t)$. Then we can assert that

$$(1, 0, 0, \ldots, 0)\, x^*(1) \geqq (1, 0, 0, \ldots, 0)\, x_\pi^*(1, \varepsilon)$$

for every perturbation π.

We approximate $x_\pi^*(1, \varepsilon) - x^*(1)$ by a vector $\frac{\partial x_\pi^*}{\partial \varepsilon}(1, 0)$ based at $x^*(1)$, and write the maximal principle as

$$-(1, 0, 0, \ldots, 0)\frac{\partial x_\pi^*}{\partial \varepsilon}(1, 0) \geqq 0.$$

Our approximation and the resulting conclusions will be valid for suitably small $\varepsilon > 0$.

We shall define a perturbation by data $\pi = \{t_1, l_1, u_1\}$ where t_1 is an instant on $0 \leqq t \leqq 1$, $l_1 \geqq 0$, and u_1 is an arbitrary point in the set Ω. Essentially we shall change $u^*(t)$ to the value u_1, near $t = t_1$, and keep $u^*(t)$ unchanged otherwise. More exactly

$$u(t, \varepsilon, t_1, l_1, u_1) = \begin{cases} u_1 & \text{on} \quad t_1 - l_1\varepsilon \leqq t \leqq t_1 \\ u^*(t) & \text{elsewhere on} \quad 0 \leqq t \leqq 1. \end{cases}$$

We abbreviate this controller by $u_\pi(t, \varepsilon)$ and the corresponding periodic response $x_\pi^*(t, \varepsilon)$.

The significance of this type of perturbation is that, at the time t_1, the solution initiating at x_0^* is jerked or displaced (to first order in ε) by the vector

$$v(t_1) = [f(x^*(t_1), u_1) - f(x^*(t_1), u^*(t_1))]\, l_1.$$

If this vector is then transported along the flow of the system $\dot{x} = f(x, u^*(t))$, we have the final displacement $v(1) = A_{1t_1}v(t_1)$, where A_{tt_1} is the fundamental solution matrix of the variational system

$$\dot{v} = \frac{\partial f}{\partial x}(x^*(t), u^*(t)), \qquad A_{t_1t_1} = I.$$

It is the displacement vector $v(1)$ that enters into the standard formulation of the maximal principle, where the initial point x_0^* is fixed. But in our situation of limit cycle control a more intricate analysis is required.

In order to locate the periodic response to $u_\pi(t, \varepsilon)$ we shall compute the initial point $x_0(\varepsilon, \pi)$ by the implicit function relation

$$x_\pi(1, x_0, \varepsilon) - x_0 = 0.$$

Here $x_\pi(t, x_0, \varepsilon) = x(t, x_0, \varepsilon, t_1, l_1, u_1)$ is the solution of $\dot{x} = f(x, u_\pi(t, \varepsilon))$ initiating at the point x_0. Upon solving this implicit relation for $x_0(\varepsilon, \pi)$, with $x_0(0, \pi) = x_0^*$, we obtain the periodic solution

$$x_\pi^*(t, \varepsilon) = x_\pi(t, x_0(\varepsilon, \pi), \varepsilon).$$

By definition $x_0(\varepsilon, \pi) = x_\pi^*(1, \varepsilon)$ so we can compute the perturbation in the amplitude using either $\frac{\partial x_0}{\partial \varepsilon}(0, \pi)$ or equally well $\frac{\partial x_\pi^*}{\partial \varepsilon}(1, 0)$. If we differentiate the implicit relation for $x_0(\varepsilon, \pi)$ we obtain at $\varepsilon = 0$,

$$\frac{\partial x}{\partial x_0}(1, x_0^*)\frac{\partial x_0}{\partial \varepsilon}(0, \pi) + \frac{\partial x_\pi}{\partial \varepsilon}(1, x_0^*, 0) - \frac{\partial x_0}{\partial \varepsilon}(0, \pi) = 0,$$

or

$$\frac{\partial x_0}{\partial \varepsilon}(0, \pi) = (I - P)^{-1}\frac{\partial x_\pi}{\partial \varepsilon}(1, x_0^*, 0).$$

The standard discussion of the maximal principle notes that

$$v(1) = \frac{\partial x_\pi}{\partial \varepsilon}(1, x_0^*, 0) = A_{1t_1}[f(x^*(t_1), u_1) - f(x^*(t_1), u^*(t_1))]\, l_1.$$

Thus our present form of the maximal principle is

$$-(1, 0, 0, \ldots, 0)\,(I - P)^{-1}A_{1t_1}[f(x^*(t_1), u_1) - f(x^*(t_1), u^*(t_1))] \geqq 0.$$

If we transport the vector $\eta(1) = (1, 0, 0, \ldots, 0)\,(I - P)^{-1}$ back along the flow according to the adjoint variational oquation

$$\dot{\eta} = -\eta\frac{\partial f}{\partial x}(x^*(t), u^*(t)),$$

then we can conclude that (since $\eta(t)\, v(t) = \text{constant}$)

$$-\eta^*(t_1)\,[f(x^*(t_1), u_1) - f(x^*(t_1),\; u^*(t_1))] \geqq 0,$$

or

$$\eta^*(t_1) f(x^*(t_1), u^*(t_1)) \geqq \eta^*(t_1) f(x^*(t_1), u_1)$$

at each instant t_1 and for each value $u_1 \in \Omega$. This is the required formulation of the maximal principle for our problem.

Finally let us comment on some of the technical difficulties encountered in completing the details of the above proof sketch.

It is easy to see that $x_\pi(t, x_0, \varepsilon) = x(t, x_0, \varepsilon, t_1, l_1, u_1)$ is continuous jointly in all arguments, since

$$\int_0^1 |\, u(t, \varepsilon_1, t_1, l_1, u_1) - u(t, \varepsilon_2, t_2, l_2, u_2)\,|\, \mathrm{d}t \to 0$$

as $(\varepsilon_1, t_1, l_1, u_1) \to (\varepsilon_2, t_2, l_2, u_2)$. It is slightly harder to verify that $\frac{\partial x}{\partial x_0}(t, x_0, \varepsilon, t_1, l_1, u_1)$ is also continuous in $(t, x_0, \varepsilon, t_1, l_1, u_1)$. However this last assertion follows from the observation that $Z(t) = \frac{\partial x}{\partial x_0}(t, x_0, \varepsilon, t_1, l_1, u_1)$ is the fundamental solution matrix of

$$\dot{Z} = \frac{\partial f}{\partial x}(x(t, x_0, \varepsilon, t_1, l_1, u_1), u(t, \varepsilon, t_1, l_1, u_1))\, Z$$

with $Z(0) = I$. Incidentally this calculation validates our use of the implicit function theorem to define $x_0(\varepsilon, t_1, l_1, u_1) = x_0(\varepsilon, \pi)$ which is continuous in all arguments jointly. A more refined study shows that $\frac{\partial x_\pi}{\partial \varepsilon}(1, x_0, 0)$ exists, and so $\frac{\partial x_0}{\partial \varepsilon}(0, \pi)$ is just as computed above.

As in the standard development of the maximal principle we need to take t_1 as a Lebesgue time for $u^*(t)$ and $f(x^*(t), u^*(t))$ so that the perturbation

$$\frac{\partial x_\pi}{\partial \varepsilon}(1, x_0^*, 0) = A_{1t_1}[f(x^*(t_1), u_1) - f(x^*(t_1), u^*(t_1))]\, l_1$$

is correct. Also the use of $\frac{\partial x_\pi^*}{\partial \varepsilon}(1, 0)$ to approximate $x_\pi^*(1, \varepsilon) - x^*(1)$ must be justified as in the standard maximal principle. We omit any further study of the details of the proof.

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