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SOME REMARKS ON THE ASYMPTOTIC EQUIVALENCE

by MARKO ŠVEC

We shall be dealing with systems

$$x' = Ax + f(t, x), \tag{1}$$

$$y' = Ay, \qquad t \geqq 0, \tag{2}$$

where x, y, f are n-vectors, $A = A(t)$ is continuous nxn matrix for $t \geqq 0$ and $f(t, x)$ is continuous vector function of t and x for $t \geqq 0$, $|x| < \infty$. ($|\ |$ denotes any appropriate vector norm.) Our aim is to make some observations about the asymptotic behavior and asymptotic relationships between the solutions of (1) and (2). Various authors such as Weyl [1], Levinson [2], [3], Wintner [4], [5], Jakubovič [6], Brauer [7], [8], Brauer and Wong [9], [10], Onuchic [11]–[13], Švec [14] and others have dealt with those problems. We can divide them into two main partial problems:

P 1. Find the conditions which guarantee that for each solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that

$$\lim_{t \to \infty} |x(t) - y(t)| = 0. \tag{3}$$

P 2. Find the conditions which guarantee that for each solution $y(t)$ of (2) there exists a solution $x(t)$ of (1) such that (3) holds.

Definition. We will say the two systems (1) and (2) are asymptotically equivalent if and only if both problems **P 1** and **P 2** have solution.

In many papers the problem of asymptotic equivalence is reduced in such a way that the condition (3) is replaced by the condition

$$|x(t) - y(t)| = o(|y(t)|), \qquad t \to \infty, \tag{4}$$

or the asymptotic relationship (3), (4) respectively, is estabilished only between the subsets of solutions (e.g. between the bounded solutions).

The assumptions under which these problems have been studied have concerned the matrix A, or, which is the same, the asymptotic behavior of the solutions of (2) as well as the perturbing term $f(t, x)$. There were studied the cases when A is a constant matrix, all characteristic roots of which have the real parts negative or zero; those with real part zero are simple. This corresponds to the condition that all solutions of (2) are bounded. Another way how to describe the behavior of the solutions of (2), which can omit the condition that A is constant, is to impose some conditions on the fundamental matrix $Y(t)$ of (2), e.g. $|Y(t)| \leqq K$ and $|Y(t)\,Y^{-1}(s)| \leqq K$ for $0 \leqq s \leqq t$; or $|Y(t)| \leqq K\,e^{-at}$, $|Y(t)\,Y^{-1}(s)| \leqq K\,e^{-a(t-s)}$ for $0 \leqq s \leqq t$; or $|Y(t)\,J_1 Y^{-1}(s)| \leqq K$ for $0 \leqq s \leqq t$, $|Y(t)\,J_2 Y^{-1}(s)| \leqq K$ for $0 \leqq t \leqq s$ and $\lim_{t\to\infty} Y(t)\,J_1 = 0$, where J_1 and J_2 are two supplementary projections and K is constant. Only by Jakubovič there is no restriction about the characteristic roots of A (A is assumed to be constant).

The hypotheses about the perturbing term $f(t, x)$ were also various. At the beginning $f(t, x)$ was assumed linear, $f(t, x) = B(t)\,x$, or the norm of $f(t, x)$ was assumed to be majorized by linear function, $|f(t, x)| \leqq \varphi(t)\,|x|$. Later, by Bauer and Wong [9], [10] and Švec [14], there is the assumption that

$$|f(t, x)| \leqq F(t, |x|), \qquad t \geqq 0,\ |x| < \infty, \tag{5}$$

where $F(t, u)$ is a continuous scalar function for $t \geqq 0$, $u \geqq 0$ which is non-decreasing in u for each t.

The most frequently used tools for solving the above problems are those of variation of constants combined with various types of integral inequalities and comparison principle and with the fixed point theorems. In this paper we shall talk about a method which we used mainly in the paper [14] and which gives the general results containing many of the previous as special cases. This method is based on the fact that we have to compare the solutions of two systems, one of which is linear and the other linear perturbed. We start with the systems

$$x' = A(t)\,x + f(t), \tag{1'}$$

$$y' = A(t)\,y. \tag{2'}$$

Then it is very easy to prove

Theorem 1. *The systems* (1') *and* (2') *are asymptotically equivalent if and only if the system* (1') *has a solution* $x_0(t)$ *such that* $\lim_{t\to\infty} x_0(t) = 0$.

After this the problem of asymptotic equivalence of (1') and (2') can be reduced to the problem of existence of at least one solution of (1') which converges to zero as $t \to \infty$. The following theorem gives an answer to this problem in the case that A is a constant matrix. We assume, without loss of generality, that A has the Jordan form and that $A = \operatorname{diag}(A_1, A_2)$ such that $\operatorname{Re}\lambda_i(A_1) \leqq -\alpha < 0$ and $\operatorname{Re}\lambda_i(A_2) \geqq 0$. ($\lambda_i(A)$ denotes the characteristic root of A.) Let p be the maximum order of those blocks in A which correspond to characteristic roots with real part zero and let $p = 1$ if there is no characteristic root with real part zero. Let $\lambda = \max \operatorname{Re}\lambda_i(A)$. Let m be the maximum order of those blocks in A which correspond to the characteristic roots $\lambda_i(A)$ such that $\operatorname{Re}\lambda_i(A) = \lambda$.

Theorem 2. *If*

$$\int^{\infty} t^{p-1}\,|f(t)|\,\mathrm{d}t < \infty, \tag{6}$$

then the equation

$$x' = Ax + f(t) \tag{7}$$

has at least one solution $x_0(t)$ converging to zero as $t \to \infty$.

We note that the condition (6) is the best in the sense that there are systems of type (7) with a solution converging to zero as $t \to \infty$ and this fact implies that (6) holds.

Example. The system

$$x_1' = x_2,$$
$$x_2' = f_2(t)$$

has the general solution

$$x_2 = c_2 + \int_0^t f_2(s)\,\mathrm{d}s,$$

$$x_1 = c_1 + c_2 t + \int_0^t (t - s) f_2(s)\,\mathrm{d}s.$$

If $x_1(t) \to 0$, $x_2(t) \to 0$ as $t \to \infty$, then it must be $c_2 = -\int_0^\infty f_2(s)\,\mathrm{d}s$, and $c_1 - t\int_t^\infty f_2(s)\,\mathrm{d}s - \int_0^t s f_2(s)\,\mathrm{d}s \to 0$ as $t \to \infty$. If we assume that $f_2(t) \geqq 0$, then the last condition implies that $\int_0^\infty s f_2(s)\,\mathrm{d}s < \infty$.

Now applying theorems 1 and 2 we can prove easily the two following theorems.

Theorem 3. *Let $F(t, u)$ be a continuous scalar function for $t \geqq 0$, $u \geqq 0$ which is non-decreasing in u for each t. Let (5) be valid and let*

$$\int_0^\infty t^{p-1} F(t, c)\,\mathrm{d}t < \infty \quad \textit{for each} \quad c \leqq 0. \tag{8}$$

Let A be a constant matrix. Then to each bounded solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that (3) holds.

In fact, it is sufficient to consider the system

$$z' = Az + f(t, x(t)). \tag{9}$$

Then

$$\int_0^\infty t^{p-1} \left| f(t, x(t)) \right| \mathrm{d}t \leqq \int_0^\infty t^{p-1} F(t, c)\,\mathrm{d}t < \infty.$$

Thus by theorem 2 the system (9) has a solution $z_0(t)$ converging to zero as $t \to \infty$. Setting $y(t) = x(t) - z_0(t)$ we have that $y(t)$ is a solution of (2) and $x(t) - y(t) = z_0(t) \to 0$ as $t \to \infty$.

We note that theorem 3 is a generalization of theorem 3 of Jakubovič [6].

In the same way can be proved

Theorem 4. *Let A be a constant matrix. Let $x(t)$ be a solution of* (1) *such that*

$$|x(t)| \leqq K e^{\mu t} t^h, \quad t \geqq 0.$$

Let (5) *be valid and let be*

$$\int_0^\infty t^{p-1} F(t, K e^{\mu t} t^h) \, dt < \infty.$$

Then there exists a solution $y(t)$ of (2) *such that* (3) *holds.*

Theorems 2 and 3 are valid also in the case that $A = A(t)$ is a nonconstant matrix such that it can be transformed into a constant matrix B by using the Ljapunov transformation.

Now suppose that (1) and (2) are asymptoticaly equivalent. Then necessarily the asymptotic behavior of the solution $x(t)$ of (1) must be the same as that of the corresponding solution $y(t)$ of (2). We still assume that A is constant. Let $Y(t)$, $Y(0) = E$, be the fundamental matrix of (2). Then

$$|Y(t)| \leqq c_0 e^{\lambda t} \chi_m(t), \quad t \leqq 0 \tag{10}$$

and

$$|y(t)| = |Y(t) y(0)| \leqq c_0 |y(0)| e^{\lambda t} \chi_m(t), \quad t \leqq 0, \tag{11}$$

where

$$\chi_m(t) = \begin{cases} t^{m-1}, & t \geqq 1, \\ 1 & 0 \leqq t \leqq 1. \end{cases} \tag{12}$$

Therefore, if (1) and (2) are asymptotically equivalent, the solutions $x(t)$ of (1) satisfies the estimate

$$|x(t)| \leqq D e^{\lambda t} \chi_m(t) + o(1), \quad \text{for} \quad t \geqq t_0,$$

where D and t_0 are suitable constants.

Now it is desirable for our considerations to establish the conditions which guarantee that for the solutions $x(t)$ of (1) the estimates

$$|x(t)| \leqq D e^{\lambda t} \chi_m(t), \quad t \geqq t_0 \tag{13}$$

are valid. The following theorem deals with this.

Theorem 5. *Let be satisfied* (5), *let be*

$$\int_0^\infty e^{-\lambda t} F(t, c e^{\lambda t} \chi_m(t)) \, dt < \infty \quad \textit{for each} \quad c \geqq 0 \tag{14}$$

and let t_0 be such that

$$\sup_{[1,\infty)} \frac{1}{c} \int_{t_0}^\infty e^{-\lambda t} F(t, c e^{\lambda t} \chi_m(t)) \, dt = S < \frac{1}{c_0}. \tag{15}$$

Then for the solutions $x(t)$ of (1) *the estimate* (13) *holds.*

It is evident that the condition (15) can be substituted by the condition

$$\sup_{[1,\infty)} \frac{1}{c} \int_0^\infty e^{-\lambda t} F(t, c\, e^{\lambda t} \chi_m(t))\, dt < \infty. \tag{16}$$

We note that the hypotheses of theorem 5 guarantee that each solution $x(t)$ of (1) which exists in some right neighborhood of t_0 can be continued to the whole interval $[t_0, \infty)$.

Theorem 5 gives us the main tool for proving

Theorem 6. *Let be satisfied* (5). *Let be*

$$\int_0^\infty t^{p-1} F(t, c\, e^{\lambda t} \chi_m(t))\, dt < \infty \quad \textit{for each} \quad c \geqq 0 \tag{17}$$

if $\lambda \geqq 0$ *and*

$$\int_0^\infty e^{-\lambda t} F(t, c\, e^{\lambda t} \chi_m(t))\, dt < \infty \quad \textit{for each} \quad c \geqq 0 \tag{18}$$

if $\lambda < 0$. *Let be satisfied* (16). *Let A be constant. Then the systems* (1) *and* (2) *are asymptotically equivalent.*

The proof of the first part of this theorem, that is, of the part that to each solution $x(t)$ of (1) there is a solution $y(t)$ of (2) such that (3) holds, consists in application of the theorem 5 and then of the theorems 3 and 2. The proof of the second part is more difficult. If $y(t)$, $y(t_0) = y_0$, is a solution of (2), then the solution of the integral equation

$$x(t) = y(t) + Y_1(t) \int_0^t Y_1^{-1}(s) f(s, x(s))\, ds - Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s, x(s))\, ds \tag{19}$$

is the solution of (1) and it satisfies (3). Here

$$Y_1(t) = \operatorname{diag}(e^{tA_1}, 0), \qquad Y_2(t) = \operatorname{diag}(0, e^{tA_2}), \qquad Y(t) = Y_1(t) + Y_2(t).$$

The existence of the solution $x(t)$ of (19) can be proved via the Schauder fixed point theorem.

Theorem 6 contains as special cases the theorem 2 of Jakubovič [6], the theorm of Levinson and its extension due to Brauer and Wong [10] and others.

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