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FINITE ELEMENTS AND NUMERICAL STABILITY

by JEAN DESCLOUX

1. INTRODUCTION

We use the following notations. When applied to an element x of $\boldsymbol{R}^N$, $\| \, . \, \|$ is a vector norm; when applied to a real square matrix A of order N, $\| \, . \, \|$ is the matrix norm subordinate to the vector norm $\| \, . \, \|$, i.e. $\| A \| = \sup \| Ax \| / \| x \|$. Here $\| \, . \, \|$ will be either the euclidean norm $\| \, . \, \|_2$ or the uniform norm $\| \, . \, \|_\infty$ defined by $\| x \|_2 = (\sum_{i=1}^{N} x_i^2)^{\frac{1}{2}}$ and $\| x \|_\infty = \max_{i=1,\ldots,N} | x_i |$.

Consider the regular linear system $Ax = b$ and the perturbed system $(A + \delta A)\, x = b + \delta b$ with solutions x_0 and $x_0 + \delta x$. Supposing δA and δb "small" and neglecting terms of "higher order" one gets the approximate relation (see [1]):

$$\frac{\| \delta x \|}{\| x_0 \|} \leqq C(A) \left\{ \frac{\| \delta b \|}{\| b \|} + \frac{\| \delta A \|}{\| A \|} \right\}, \tag{1}$$

where $C(A) = \| A \| \, \| A^{-1} \|$ is the *condition number* of A with respect to $\| \, . \, \|$. Suppose we use a computer to solve numerically the system $Ax = b$, for example, by Gauss elimination. Because of round-off errors the solution x_1 produced by the computer will differ from the exact solution x_0; using the inverse round-off analysis (see [1]), one can show the existence of a "small" matrix δA such that x_1 satisfies the equation $(A + \delta A)\, x_1 = b$. (1) shows the important role of the condition number of A for the discussion of the numerical stability, i.e., the importance of round-off errors, of methods for solving systems of linear equations.

Unfortunately the things are a little bit more complicated. Indeed suppose we solve the system by Gauss elimination without pivoting in binary floating-point arithmetic; it is easy to check that multiplications of the rows and of the columns by powers of 2 will not affect the relative precision of each component of the solution; however by this procedure, for a given norm, the condition number of the matrix can be made as large as one wishes. For this reason Bauer [2] has suggested that the real measure of the numerical stability of a system be defined by the *optimal condition number*:

$$C_{\mathrm{op}}(A) = \inf_{D_1, D_2 \in \mathfrak{D}} C(D_1 A D_2)$$

where $\mathfrak{D}$ is the set of regular diagonal matrices of order N. C_2, $C_{\mathrm{op}2}$, C_∞ and $C_{0p\infty}$ will denote the condition number and the optimal condition number for the euclidean and uniform norms. We recall that for symmetric matrices $C_2(A) \leqq C_\infty(A)$.

When the matrix A is equilibrated, i.e. when the norms of the rows and columns are of the some order, $C(A)$ and $C_{op}(A)$ are not too different; several theorems make this statement precise (see [3]); we stall use the following one [4]: if A is positive definite and possesses Young's property A (in particuliar tridiagonal matrices have this property) and if all diagonal elements are equal, then $C_2(A) = C_{op2}(A)$.

Let L be an elliptic partial differential equation of order $2m$ defined on a domain $G \subset R^p$. Let L_h be the matrix obtained discretizing L by finite differences on a regular net with step h; suppose that stability and consistency are satisfied; they imply respectively the relations: $\| L_h^{-1} \| = O(1)$, $\| L_h \| = O(h^{-2m})$ as $h \to 0$ and consequently $C(L_h) = O(h^{-2m})$. This result is independant of the dimension p of G; since the order N of the matrix L_h is proportionned to h^{-p}, it follows that $C(L_h) = O(N^{2m/p})$. For more precise statements about two-dimensional second order elliptic partial differential equations, see for example [5].

The main purpose of this talk is to discuss the numerical stability of matrices arising from the discretization of elliptic differential operators by the Ritz method. Let $G \subset R^p$ be a bounded domain, V a closed subspace of real Hilbertian Sobolev space $H^m(G)$, A a bilinear form on $V \times V$ of the form $A(u, v) = = \int_G \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) \, dx$; one supposes $A(u, v) = A(v, u)$ and $A(u, v) > 0$ for $u \neq 0$. Let $f_1, f_2, \ldots, f_N$ be independant elements spanning the subspace $U \subset V$. Let H be the positive definite matrix of order N with elements $A(f_i, f_j)$; H is called the *stiffness matrix*. We are interested in the condition number of H.

The following will show the importance of the degenerate case $m = 0$. More precisely, besides H, we introduce the positive definite matrix F of order N with elements $\int_G f_i(x) f_j(x) \, dx$; F is the matrix of the normal equations relative to the problem of least square approximation in the subspace U; F is called the *mass matrix*.

Consider first a classical example. $G = [0, 1]$, $f_i(x) = x^{i-1}$, $i = 1, 2, \ldots, N$; F is then the Hilbert matrix with $F_{ij} = 1/(i + j - 1)$; the Hilbert matrix is well-known for its very large condition number, (see [1]); simple computations give the very optimistic lower bound: $C_\infty(F) > 2^{2N-3}$; in fact for $N = 10$, $C_2(F) = 1.6 \; 10^{13}$. Equilibration cannot improve much the situation. On the other hand, simple examples for the bilinear form A show that one cannot expect a better behaviour for the stiffness matrix. Because of the numerical instability it generates, this set of trial functions is not convenient; besides this, it presents notorious disadvantages: generally this a full matrix; general boundary conditions are difficult to satisfy. The remedy to these difficulties can be found in the method of finite elements.

We say that the set of functions $f_1, f_2, \ldots, f_N$ spanning $U \subset V$ is of finite element type if the following situation is present. Let C_i be the support of f_i. One supposes the C_i are small and one supposes the existence of sets $e_1, e_2, \ldots, e_E$ called *elements* with the following properties:

1) $G = \bigcup_{k=1}^{E} e_k$; measure $(e_k) > 0$; measure $(e_i \cap e_j) = 0 \; i \neq j$;

2) any C_i is the union of a small number of elements; any element is covered by at least one C_i;

We consider two simple examples for the bilinear form $A(u, v) = \int_0^1 u'(x)\, v'(x)\, dx$. *First* let $V = \{u \in H^1[0, 1];\ u(0) = u(1) = 0\}$; one divides $[0, 1]$ in $N + 1$ elements $e_k = [x_{k-1}, x_k]$; f_i, $i = 1, 2, \ldots, N$ is the hat function of figure 1. *Second* let $V = \{u \in H^1[0, 1];\ u(0) = 0\}$; one divides $[0, 1]$ in N elements $e_k = [x_{k-1}, x_k]$; for $i = 1, 2, \ldots, N - 1$ f_i is the hat function of figure 1; f_N is given by figure 2. More general examples can be found in a very rich literature (see for example [6], [7]).

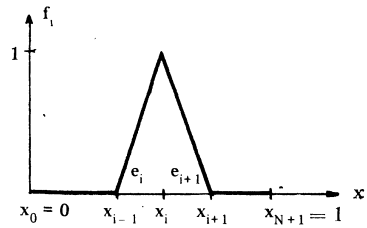

Figure 1

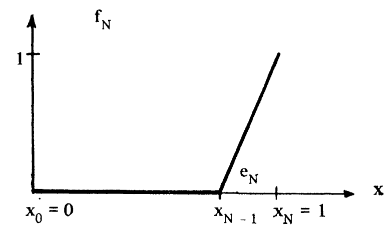

Figure 2

For the element e_k let I_k be the set of indices j for which $C_j \supset e_k$; for $x \in R^N$ let x^k be the subvector of x corresponding to the set of indices I_k. One can write

$$x^t F x = \int_G \Big(\sum_{i=1}^{N} x_i f_i\Big)^2 = \sum_{k=1}^{E} \sum_{i,j \in I_k} \Big(\int_{e_k} f_i f_j\Big) x_i x_j = \sum_{k=1}^{E} x^{kt} F_k x^k; \tag{2}$$

$$x^t H x = A\Big(\sum_{i=1}^{N} x_i f_i, \sum_{j=1}^{N} x_j f_j\Big) = \sum_{k=1}^{E} \sum_{i,j \in I_k} \Big(\int_{e_k} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta} D^\alpha f_i D^\beta f_j\Big) x_i x_j = \sum_{k=1}^{E} x^{kt} H_k x^k; \tag{3}$$

F_k and H_k are symmetric matrices both of order equal to the number of elements of I_k; they are clearly defined by (2) and (3); they are called the *mass and stiffness matrices of the element* e_k. We suppose F_k to be positive definite and H_k to be semi positive definite. Let α_k and w_k be the smallest and the largest eigenvalues of F_k and let ϑ_k be the largest eigenvalue of H_k. Finally let $\alpha = \min_{k=1,\ldots,N} \alpha_k$, $w = \max_{k=1,\ldots,N} w_k$, $\vartheta = \max_{k=1,\ldots,N} \vartheta^k$, $\mu = w/\alpha$.

For the first example considered above F_k and H_k are given by (4) and (5) for $k = 2, 3, \ldots, N - 1$ and by (6) for $k = 1$ and $k = N$ (matrices of order 1); for the second example (4) and (5) are valid for $k = 2, 3, \ldots, N$ and (6) is valid for $k = 1$; in the following relations, we set $h_k = x_k - x_{k-1}$:

$$F_k = \frac{h_k}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \alpha_k = \frac{h_k}{6}, \qquad w_k = \frac{h_k}{2}; \tag{4}$$

$$H_k = \frac{1}{h_k}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \vartheta_k = \frac{2}{h_k}; \tag{5}$$

$$F_k = h_k/3, \quad \alpha_k = w_k = h_k/3; \quad H_k = 1/h_k, \quad \vartheta_k = 1/h_k. \tag{6}$$

2. CONDITION NUMBER FOR THE EUCLIDEAN NORM

The results of this section are due to ISAAC FRIED [8], [9]. Let P be the maximum number of elements contained in any of the supports C_i of f_i and let

$$\lambda = \inf_{u \in V} \frac{(u, u)}{\int_D u^2} > 0.$$

Theorem 1: $\| F \|_2 \leqq Pw, \quad \| F^{-1} \|_2 \leqq 1/\alpha, \quad C_2(F) \leqq Pw/\alpha;$ (7)

$$\| H \|_2 \leqq P\vartheta, \quad \| H^{-1} \|_2 \leqq 1/\lambda\alpha, \quad C_2(F) \leqq P\vartheta/\lambda\alpha. \tag{8}$$

Proof: From (2) one has for any $x \in R^N$:

$$\alpha x^t x \leqq \sum_{k=1}^{E} \alpha_k x^{kt} x^k \leqq \sum_{k=1}^{E} x^{kt} F_k x^k = x^t F_x \leqq \sum_{k=1}^{E} w_k x^{kt} x^k \leqq wPx^t x,$$

which proves (7); the first inequality of (8) is obtained in the same way from (3); the second one is a consequence of the definition of λ; indeed for $u = \sum_{i=1}^{N} x_i f_i$ one has

$$\lambda\alpha x^t x \leqq \lambda x^t F x = \lambda \int_G u^2 \leqq A(u, u) = x^t H x.$$

Example: One considers the first example described in section **1** on a regular net, i.e, $h_k = h = 1/(N + 1)$. $P = 2$, $\lambda = \pi^2$, $\alpha = h/6$, $w = h/2$, $\vartheta = 2/h$; theorem 1 gives the bounds:

$$\| F \|_2 \leqq h, \quad \| F^{-1} \|_2 \leqq 6/h, \quad C_2(F) \leqq 6;$$

$$\| H \|_2 \leqq 4/h, \quad \| H^{-1} \|_2 \leqq 6/(\pi^2 h), \quad C_2(H) \leqq 24/(\pi^2 h^2);$$

direct computations show that $C_2(F) \sim 3$, $C_2(H) = 4/(\pi^2 h^2)$ as $h \to 0$. The same asymptotic behaviour is also valid for almost uniform meshes; more specifically one considers a set of decompositions of [0, 1] in elements; for each decomposition let h be the length of the largest element; one supposes the existence of a constant γ independant of the decompositions such that for each decomposition the ratio

h_i/h_j of the length of two elements is $\leqq \gamma$; then by theorem 1 one gets easily the results:

$$C_2(F) = O(1), \quad C_2(H) = O(h^{-2}) \quad \text{as} \quad h \to 0.$$

Because the notion of finite element is not well defined, it is difficult to formulate a general theorem; however by using on the usual finite element functions the same technique as in the preceeding example, one gets for uniform and almost uniform meshes the asymptotical results:

$$C_2(F) = O(1), \quad C_2(H) = O(h^{-2m}) \quad \text{as} \quad h \to 0; \tag{9}$$

here h denotes, for a particular decomposition of D, the maximum of the diameters of the elements; we recall that m is the order of the bilinear form Λ; one must remark that the notion of "almost uniform mesh" is more complicated when the dimension p of G is > 1 than for the one-dimensional case; for example, for decompositions in triangles, all the angles have to remain bounded above a fixed positive constant. It is interesting to note that (9) means that the asymptotic behaviour of the condition numbers of the discretizations matrices are the same for the finite element method and for the finite differences method.

3. CONDITION NUMBER FOR THE UNIFORM NORM

The asymptotic results are essentially the same as for the euclidean norm, but less general and more complicated to obtain. The following theorems are proved in [11]; other results are contained in [10].

Besides the notations of section **1**, we introduce the following ones. For a subset $Z \subset G$, $m(Z)$ is its measure, $d(Z)$ its diameter; c_p is the measure of the unit sphere in $\boldsymbol{R}^p$; let M = maximum number of supports C_i covering a same element;

$$\frac{d^p(C_j)}{m(e_k)} \leqq \gamma, \quad i = 1, \ldots, N, k = 1, \ldots, E;$$

$$\frac{d^p(e_i)}{d^p(e_j)} \leqq \delta, \quad i, j = 1, 2, \ldots, E.$$

Theorem 2. $$\| F^{-1} \|_\infty \leqq s^{-1}\alpha^{-1}(Mc_p n^p \gamma)^{\frac{1}{2}},$$

where s is any number between 0 *and* 1 *and n is the smallest integer for which*

$$\mu^{-2}Mc_p\gamma p(1-\mu)^{n-1} n^{p-1} \leqq (1-s)^2.$$

Theorem 3. *One supposes that Λ satisfies the following coerciveness relation*

$$\Lambda(u, u) \geqq \varkappa(\int_G u^2)^{\frac{1}{2}} \max_{x \in G} | u(x) |, \quad u \in V, \quad \varkappa > 0;$$

then $\| H^{-1} \|_\infty \leqq \varkappa^{-1}(c_p 2^{-p}\mu^{-2}\delta Mm(D))^{\frac{1}{2}} \| F^{-1} \|$.

The simplest way to evaluate $\| F \|_\infty$ and $\| H \|_\infty$ is by direct inspection of the matrices; however one can also use the bounds of theorem 1:

$$\| F \|_\infty \leqq \sqrt{Q} \, \| F \|_2, \qquad \| H \|_\infty \leqq \sqrt{Q} \, \| H \|_2, \tag{10}$$

where Q is the maximum number of supports C_i having a intersection of positive measure.

We give a brief proof of (10). Let A be F or H; each row of A has at most Q elements different from zero; for the row i let I be the set of indices j with $a_{ij} \neq 0$; let A^* be the square submatrix of order $\leqq Q$ corresponding to I; for $x \in \boldsymbol{R}^N$ let x^* be the subvector corresponding to I; one has

$$| (Ax)_i | \leqq \| A^* x^* \|_\infty \leqq \| A^* x^* \|_2 \leqq \| A^* \|_2 \, \| x^* \|_2 \leqq \| A^* \|_2 \sqrt{Q} \, \| x^* \|_\infty \leqq$$
$$\leqq \| A \|_2 \sqrt{Q} \, \| x \|_\infty .$$

Example: Again we consider the first example described in section 1 with a uniform mesh, i.e., $h_k = h = 1/(N + 1)$, $k = 1, 2, \ldots, N + 1$. We have $p = 1$, $c_p = 2$, $M = 2$, $\gamma = 2$, $\delta = 1$, $\alpha = h/6$, $\mu = 1/3$, $\varkappa = 2\pi$, $Q = 3$; theorem 2 with $s = 0.75$, theorem 3 and relation (10) give

$$\| F \|_\infty \leqq 1.73/h, \quad \| F^{-1} \|_\infty \leqq 98.5/h, \quad C_\infty(F) \leqq 170;$$
$$\| H \|_\infty \leqq 5.92/h, \quad \| H^{-1} \|_\infty \leqq 66.5/h, \quad C_\infty(H) \leqq 394/h^2;$$

direct computations show that $2.8 \leqq C_\infty(F) \leqq 3$ and, as $h \to 0$, $C_\infty(H) \sim 1/(2h^2)$.

As for the euclidean norm, the method of finite elements leads for uniform and almost uniform meshes to the results

$$C_\infty(F) = O(1), \quad C_\infty(H) = O(h^{-2m}) \quad \text{as} \quad h \to 0;$$

we have to emphasize the fact that this last result relative to H supposes that the coerciveness condition of theorem 3 is satisfied; it is a conjecture that it should be possible to relax considerably this restriction.

4. FINITE ELEMENTS ON NON UNIFORM MESHES

In [9] Fried considers decompositions of G with the presence of two adjacent elements having very different sizes. Since the mass and stiffness matrices are not equilibrated in this case, the discussion of their numerical stability supposes a proper scaling. With the help of various examples, Fried shows that in some cases the numerical stability of the stiffness matrix is as good as in the case of a uniform mesh but in other cases it can be much worse.

Here we adopt a different point of view which will lead to similar results for the stiffness matrix. We consider a set of decompositions of the domain G in elements; for the sake of simplicity we suppose that all the elements contained in any support

C_i have a point in common; we also suppose the existence of the numbers α^*, β^*, γ^* independant of the decompositions such that for each decomposition one has

$$\alpha^* m(e_k)\, x^{kt}x^k \leqq x^{kt}F_k x^k \leqq \omega^* m(e_k)\, x^{kt}x^k, \tag{11}$$

$$m(e_i) \leqq \gamma^* m(e_j) \quad \text{if} \quad e_i \cap e_j \neq \emptyset \tag{12}$$

$m(e_i)$ denotes the measure of e_i; x^* and F_k have been defined in section **1.** (12) means that, in a decomposition, two elements having a point in common cannot be too different in size. (11) is satisfied by the usual finite elements. Consider a particular decomposition and a support C_i; let $\mathscr{E}$ be the set of the elements contained in C_i and n be the number of elements of $\mathscr{E}$; $q_i = \left(\sum_{e \in \mathscr{E}} m(e)\right)/n$ is the average measure of the elements in $\mathscr{E}$; finally let D be the diagonal matrix of order N with diagonal elements $q_i^{-\frac{1}{2}}$ and D_k be the diagonal submatrix of D relative to the indices of I_k (see definition in section 1); we introduce the vector y of order N with components y_i and the sub-vector y^k defined by the relations

$$x = Dy, \quad x_i = y_i/\sqrt{q_i}, \quad x^k = D_k y^k;$$

if $i \in I_k$; then $\gamma^{*-1} \leqq m(e_k)/q_i \leqq \gamma^*$; replacing in (11) we get

$$\alpha^*\gamma^{*-1} y^{kt}y^k \leqq \alpha^* m(e_k)\, y^{kt}D_k^2 y^k \leqq y^{kt}D_k F_k D_k y^k \leqq \omega^* m(e_k)\, y^{kt}D_k^2 y^k \leqq \omega^*\gamma^* y^{kt}y^k;$$

the arguments used in theorem 2 and the relation

$$\sum_{k=1}^{E} y^{kt}D_k F_k D_k y^k = y^t DFDy$$

prove the following result:

$$\| DFD \|_2 \leqq P\omega^*\gamma^*; \quad \| (DFD)^{-1} \|_2 \leqq \gamma^*/\alpha^*; \quad C_2(DFD) \leqq P\omega^*\gamma^{*2}/\alpha^*; \tag{13}$$

so we have proved that $C_{op2}(F)$ is bounded by a constant independant of the decompositions.

As an illustration we take the second example of section **1** (boundary condition $u(0) = 0$) with $h_k = a^{k-1}(1-a)/(1-a^N)$, $a < 0$; we have $\gamma^* = 1/a$, $m(e_k) = h_k$, $\alpha^* = 1/6$, $\omega^* = 1/2$, $P = 2$; from (13) we get $C_{\mathrm{op2}}(F) \leqq 6/a^2$; in fact direct computations show that $\lim_{a \to 0} C_{\mathrm{op2}}(F) = 1$ uniformely in N.

(13) is a very satisfactory result for the mass matrix. The following three examples show that it is not possible to get simple results for the stiffness matrix.

a) We consider example 1 of section **1** (boundary condition $u(0) = u(1) = 0$) with $h_k = a^{k-1}(1-a)/(1-a^{N+1})$, $k = 1, 2, \ldots, N+1$, $a < 1$. Direct computations and the property stated in section 1 on optimal conditionning give the following result

$$C_{\mathrm{op2}}(H) \leqq C_{\mathrm{op}\infty}(H) \leqq \left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^2 \qquad \text{(independantly of } N\text{);}$$

we recall that for $a = 1$ we got in section **2**: $C_{op2}(H) \sim 4N^2/\pi^2$ as $N \to \infty$; in particular we have the surprising result: $\lim_{a \to 0} C_{op\infty}(H) = 1$ uniformely in N.

b) We consider example 2 of section **1** (boundary condition $u(0) = 0$) with $h_k = a^{k-1}(1 - a)/(1 - a^N)$, $k = 1, 2, \ldots, N$, $a < 1$. By direct computation we get the following result

$$a^{-N}\left\{\frac{2a(1+\sqrt{a})^2}{(1+a)(1-a)^2} - \varepsilon_N\right\} \leqq C_{op\,2}(H) \leqq a^{-N}\left\{\frac{2a\{(1+\sqrt{a})^2+1\}}{(1+a)(1-a)^2} + \varepsilon_N\right\}$$

with $\lim_{N \to \infty} \varepsilon_N = 0$; we have therefore $C_{op\,2}(H) = O(a^{-N})$ whereas for $a = 1$ theorem 1 gives $C_2(H) = O(N^2)$.

c)*) We consider example 1 of section **1** (boundary condition $u(0) = u(1) = 0$) for N odd, $N + 1 = 2q$, $h_k = h_{2q+1-k} = a^{k-1}(1 - a)/(1 - a^q)$, $k = 1, 2, \ldots, q$ (figure 3); the elements are concentrated around $x = 0.5$. Denoting by $C_b(N)$ the optimal condition number obtained for H in the preceeding example b, one gets easily the following relations

$$0.5C_b(q) \leqq C_{op\,2}(H) \leqq 2C_b(q);$$

therefore $C_{op\,2}(H) = O((\sqrt{a})^{-N})$ whereas for $a = 1$ we have $C_2(H) \sim 4N^2/\pi^2$ as $N \to \infty$.

Remark: Instead of computing the asymptotic growth of $C_{op\,2}(H)$ with respect to N, we can consider it with respect to the length h_{min} of the smallest element; for examples b) and c) we then have the comforting results

$$C_{op\,2}(H) = O(h_{min}^{-1}) \qquad \text{and} \qquad C_{op\,2}(H) = O(h_{min}^{-1}).$$

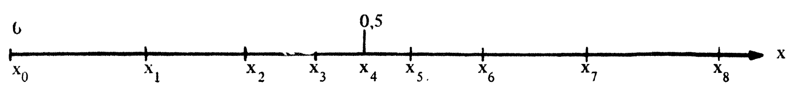

Figure 3

5. CONCLUDING REMARKS

1. In [13] Fix and Strang have obtained the results of sections **2** and **3** for uniform meshes by using Fourier transforms.

2. Since results on the condition numbers of stiffness matrices are essentially equivalent to the usual stability properties for the finite differences method, it is possible to deduce from them results on convergence. However for the finite element method properties of consistency are not easy to establish.

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* Suggested by Prof. CH. BLANC.

3. One can use the results on the condition number of stiffness matrices for studying perturbation problems, for example the effect of numerical integration in the computation of the elements of the stiffness matrix; however one does not get optimal results in this way (see [14]).

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