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HARDY SPACE METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

STEFAN MÜLLER

ABSTRACT. Hardy space methods have lead to remarkable progress on nonlinear partial differential equations with critical growth. The results obtained by a variety of authors include the regularity theory for weakly harmonic maps, existence results for the two-dimensional instationary Euler equations with vortex sheet initial data and the Lipschitz parametrization of $W^{2,2}$ surfaces. This paper gives a quick review of the basic tools needed and discusses their application.

1. Introduction

In recent years research by a variety of authors lead to remarkable progress in the study of nonlinear partial differential equations with critical growth through the use of Hardy space methods. The idea is simple. A typical difficulty when dealing with such equations is that the nonlinear term is a priori only known to be in L^1 while there is no good elliptic theory in L^1. There is, however, a well-established regularity theory in the slightly smaller Hardy space $\mathcal{H}^1$ and Coifman, Meyer, Lions and Semmes [CLMS] recently discovered that certain nonlinear quantities which at first glance only seem to be in L^1 are in fact in $\mathcal{H}^1$. In the following I will briefly discuss applications of this idea to the (longstanding) regularity problem for harmonic maps, good parametrizations of surfaces and the two-dimensional Euler equations. Further applications can be found in [CLMS]. First, it is time to recall the definition of $\mathcal{H}^1$ and some of its properties.

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2. The Hardy space $\mathcal{H}^1$

The *Hardy spaces* $\mathcal{H}^p$ (with $0 < p < \infty$) have been an object of interest in complex analysis for a long time. A breakthrough came in the early 70's when it was discovered that they possess a simple real variable characterization and that various seemingly unrelated definitions are equivalent. A classic reference is the article by Fefferman and Stein [FS] (see also [Se] for a short introduction to Hardy spaces). In the following we only consider $\mathcal{H}^1(\mathbb{R}^n)$ which can be defined as follows.

Let ϕ we a smooth function with compact support in the unit ball and suppose that $\int \phi = 1$. For a function $f \in L^1(\mathbb{R}^n)$ define the regularized maximal function by

$$f^*(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n} \varepsilon^{-n} \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)\, dy \right| . \tag{1}$$

DEFINITION 1. The function f is in $\mathcal{H}^1(\mathbb{R}^n)$ if and only if $f^* \in L^1(\mathbb{R}^n)$.

The expression $\|f^*\|_{L^1}$ defines a norm on $\mathcal{H}^1$ and different choices of ϕ lead to equivalent norms. Note that f^* is closely related to the Hardy–Littlewood maximal function but that it takes into account possible cancellations since the absolute value is taken outside the integral. By the Lebesgue point theorem $\mathcal{H}^1$ is a subspace of L^1. Note also that f can only be in $\mathcal{H}^1$ if $\int f = 0$ since otherwise f^* only decays like $|x|^{-n}$ at ∞.

A crucial property of $\mathcal{H}^1$ is that elliptic operators (or more generally singular integral operators) behave well on that space while the same is not true in L^1. We will only need the following result.

THEOREM 2. ([FS]). *Let $f \in \mathcal{H}^1$ and let u be a solution of*

$$-\Delta u = f \quad \text{in} \quad \mathbb{R}^n .$$

Then u can be written in the form $u = u_0 + H$ where H is harmonic and u_0 satisfies

$$\|\nabla^2 u_0\|_{\mathcal{H}^1} \le C \|f\|_{\mathcal{H}^1} . \tag{2}$$

If $n = 2$ then one can use the embeddings (see [Ad], Lemma 5.8) $W^{2,1} \hookrightarrow C^0$ and $W^{2,1} \hookrightarrow W^{1,2}$ to show that $u_0 \in W^{1,2} \cap C^0$ and

$$\|\nabla u_0\|_{L^2} + \|u_0\|_{L^\infty} \le \|f\|_{\mathcal{H}^1} . \tag{3}$$

The space $\mathcal{H}^1$ has various other interesting properties. In particular its dual is the John–Nirenberg space BMO of functions of bounded mean oscillation, see [FS], [Se], [Tor] ... for many further details.

In the last section we will also need local versions of the Hardy space. To this end let

$$f^{**}(x) = \sup_{1>\varepsilon>0} \left| \int_{\mathbb{R}^n} \varepsilon^{-n} \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)\, dy \right| .$$

The local Hardy space h^1 introduced by Goldberg [Go] consists of all functions f for which $\|f^{**}\|_{L^1}$ is finite and this quantity gives a norm on h^1. Finally $\mathcal{H}^1_{\text{loc}}$ consists of all functions f for which $f^{**} \in L^1_{\text{loc}}$ (see also [Zhe] and [EM], Section 5).

3. Some nonlinearities are special

It has been known for some time now that certain nonlinear expressions such as the Jacobien $\det\nabla u$ of a map $u : \mathbb{R}^n \to \mathbb{R}^n$ behave particular well under weak convergence and this observation has been crucial in such diverse fields as nonlinear elasticity and quasiregular maps (see [Mo], [Re], [Ba]).

More recently it emerged that the Jacobien also enjoys special integrability properties. The following result was shown in [Mu1], [Mu2] (inspired by results of Zhang [Zh]): if $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and if $\det\nabla u \geq 0$ a.e. then $\det\nabla u \log(2 + \det\nabla u)$ is locally integrable (see [IS], [BFS], [IL], [Mi] for further developments). The space $L \log L$ is well known in harmonic analysis and Coifman, Lions, Meyer and Semmes, partially motivated by the above results, established the crucial connection to $\mathcal{H}^1$.

THEOREM 3. ([CLMS]) *If* $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ *then* $\det \nabla u \in \mathcal{H}^1$ *and*

$$\|\det \nabla u\|_{\mathcal{H}^1} \leq C \|\nabla u\|^n_{L^n} . \tag{4}$$

The connection between weakly continous quantities and quantities that enjoy higher integrability does not seem to be coincidental. See [CLMS] for results similar to the one above for the quantities that appear in the theory of compensated compactness which was developed by Murat and Tartar (see [Ta1]). Combining Theorems 2 and 3 one can recover the remarkable earlier results of Wente [We] (see also Brezis and Coron [BC] and Tartar [Ta2], [Ta3], [Ta4]) on the equation $-\Delta u = \det\nabla v$ (Wente considered a slightly more special case but the same ideas apply).

A very short proof of the theorem which starts from the definition of $\mathcal{H}^1$ given in Section 2 and uses only the fact that the Jacobien can be written as a divergence (see [Mo], [Re], [Ba]), the Sobolev–Poincare inequality and L^p ($p > 1$) estimates for the Hardy–Littlewood maximal function (see [St]) appears in [Li] (see also [Ev]).

4. Weakly harmonic maps

Let B be the unit ball in $\mathbb{R}^2$ and let $u \in W^{1,2}(B, \mathbb{R}^{n+1})$. Suppose that

$$u(x) \in S^n \quad \text{a.e.}$$

We say that u is weakly harmonic if

$$-\Delta u = u|\nabla u|^2 \tag{5}$$

in the weak sense, i.e, if u is a stationary point of the Dirichlet integral subject to the constraint $u \in S^n$. It had been open for a long time whether all such maps are smooth although that was known under additional assumptions (see, e.g., the references in [He1], [He2], [Ri]).

THEOREM 4. ([He1]) *Weakly harmonic maps are smooth.*

Hélein also showed that the regularity theorem still holds if the target S^n is replaced by an arbitrary Riemannian manifold ([He2]). His results where generalized by Evans [Ev] and Bethuel [Be] who proved partial regularity of weakly harmonic maps from higher dimensional domains provided that a monotonicity formula holds. Rivière [Ri] gave counterexamples to regularity if that condition is not imposed.

A sketch of the proof. The first observation is that equation (5) can be written in divergence form. This is essentially a consequence of the symmetries of S^n and Noether's theorem and to the author's knowledge was first observed in the derivation of the Eriksen–Leslie theory of liquid crystals (see [Le]), see also Shatah ([Sh]). Specifically a map $u \colon B \to S^n$ (in $W^{1,2}$) is a weak solution of (5) if and only if

$$\operatorname{div} b^{ij} = 0, \quad \text{for all} \quad i, j = 1, \ldots, n+1,$$

where

$$b^{ij} = u^i \nabla u^j - u^j \nabla u^i.$$

In particular there exist functions w^{ij} such that $b^{ij} = \nabla^{\perp} w^{ij}$, where $\nabla^{\perp} = (-\partial/\partial x^2, \partial/\partial x^1)$. A short calculation using $|u| = 1$ shows that

$$u^i |\nabla u|^2 = \sum_j \det(\nabla w^{ij}, \nabla u^j)$$

and (after a suitable localization argument) it follows from Theorems 2 and 3 that u is continuous. Standard results for harmonic maps then imply that u is smooth. □

It should be noted Helein's original proof does not use $\mathcal{H}^1$, in fact the result in the appendix of [BC] gives slightly more precise results. The proof for general targets, as well as the partial regularity results do, however, make use of $\mathcal{H}^1$ (see also the references in [Ev] and [Be]).

5. Lipschitz parametrization of $W^{2,2}$ graphs

Let $w \in W^{2,2}(\mathbb{R}^2)$ and let $\Gamma \subset \mathbb{R}^3$ be the graph of w. Although $W^{2,2}$ functions need not to be Lipschitz, Toro established the striking fact that Γ has a bilipschitz parametrization.

THEOREM 5. ([To]) *There exists a parametrization $f : \mathbb{R}^2 \to \Gamma$ and a constant C (depending only on $\|\nabla^2 w\|_{L^2}$) such that for all $x, y \in \mathbb{R}^2$*

$$C^{-1}|x-y| \leq \big|f(x) - f(y)\big| \leq C|x-y|\,.$$

Toro's proof proceeds by an explicit iterative construction. Here I would like to sketch the alternative proof of [MS].

Sketch of proof. For convenience we assume that w is smooth with compact support and show that the Lipschitz constant C only depends on $\|\nabla^2 w\|_{L^2}$. Let f be a conformal parametrization of Γ, i.e., $f_{x_1} \cdot f_{x_2} = 0$ and $|f_{x_1}| = |f_{x_2}| = e^u$. Since w has compact support u is harmonic outside a large ball and hence has a limit at ∞. Replacing $f(x)$ by $f(\lambda x)$ if necessary we may assume that this limit is zero.

The Gauss curvature K of Γ satisfies

$$-\Delta u = e^{2u}(K \circ f)\,.$$

Let $N\colon \Gamma \to S^2$ be the Gauss map that associates to each point $p \in \Gamma$ its normal $N(p)$ (given by $f_{x_1} \wedge f_{x_2}/|f_{x_1} \wedge f_{x_2}|$) and let $\phi = N \circ f$. Since K is the Jacobien of the Gauss map we have

$$e^{2u}(K \circ f) = \det \nabla\phi\,.$$

By the conformal invariance of the Dirichlet integral and a short calculation

$$\int_{\mathbb{R}^2} |\nabla\phi|^2\,dx = \int_{\Gamma} |\nabla N|^2 \leq \int_{\mathbb{R}^2} |\nabla^2 w|^2\,dx\,.$$

It now follows from Theorems 2 and 3 (and (4)) that

$$\sup|u| \leq C \int_{\mathbb{R}^2} |\nabla^2 w|^2\,dx\,. \tag{6}$$

Here we used the fact that bounded harmonic functions are constant and that u approaches zero at infinity. In applying Theorem 3 we glossed over the fact that ϕ takes its values in S^2 rather than $\mathbb{R}^2$. This causes no problem since Γ is a graph and therefore ϕ only takes its values in the upper half-sphere which can be mapped to a ball in $\mathbb{R}^2$ by a volume-preserving diffeomorphism.

From (6) we obtain the upper bound in the theorem. For the lower bound a short argument is needed to compare the extrinsic and the intrinsic distance on Γ (see [MS], Section 5). □

Similar techniques can be applied to (smooth) immersed surfaces $M \hookrightarrow \mathbb{R}^n$ which need not be graphs. The fundamental assumption in this case is that the second fundamental form A satisfies

$$\int_M |A|^2 < \infty . \tag{7}$$

By a result of Huber [Hu] such surfaces admit a conformal parametrization $f : S \setminus \{a_1, \dots, a_q\} \to M \hookrightarrow \mathbb{R}^n$ where S is a compact Riemann surface.

THEOREM 6. ([MS]) *Let M be as above and suppose that M is complete, connected and non-compact. Then the immersion f is proper, i.e, $f(x) \to \infty$ if $x \to a_i$. If, in addition, $\int_M |A|^2 < 4\pi$, then the conformal type of M is $\mathbb{C}$ and $f : \mathbb{C} \to M \hookrightarrow \mathbb{R}^n$ is an embedding.*

6. Instationary Euler equations

We are interested in solutions of the two-dimensional instationary Euler equations

$$u_t + \operatorname{div} u \otimes u = -\nabla p , \tag{8}$$

$$\operatorname{div} u = 0 , \tag{9}$$

with "rough" initial data. More precisely we suppose that $u_0(x) := u(0, x)$ satisfies

$$\omega_0 := \operatorname{curl} u_0 \in \mathcal{M} ,$$

where $\mathcal{M}$ denotes the space of Radon measures on $\mathbb{R}^2$. The condition $\operatorname{div} u_0 = 0$ and ω_0 determine u_0 up to a gradient of a harmonic function. Choose a normalization by requiring that u_0 can be written as $u_0(x) = \lambda(|x|)x^\perp + u_1(x)$ where λ is smooth and where $u_1 \in L^2(\mathbb{R}^2; \mathbb{R}^2)$.

The case where ω_0 is a one-dimensional measure concentrated on a curve corresponds to so-called vortex-sheet initial data which are believed to be of great practical importance and have been studied extensively numerically (see, e.g., [Ma1] and the references therein).

A standard procedure to obtain solutions for such rough initial data u_0 is to approximate them by smooth data $u_{0\varepsilon}$ and to pass to the limit $\varepsilon \to 0$ in the corresponding solutions u_ε. By the usual estimates (see [Yu], [Ka], [DM], [De]) one obtains bounds (independent of ε) for $u_\varepsilon(t,\cdot) \in L^2_{\text{loc}}$, $\omega_\varepsilon(t,\cdot) \in L^1_{\text{loc}}$, uniformly for $t \in [0,T]$ and there exists a sequence $\varepsilon_j \to 0$ such that

$$\begin{aligned} u_{\varepsilon_j} &\rightharpoonup \bar{u} \quad \text{weakly in} \quad L^2_{\text{loc}}([0,T] \times \mathbb{R}^2) \\ u_{\varepsilon_j} &\to \bar{u} \quad \text{a.e. in} \quad [0,T] \times \mathbb{R}^2 . \end{aligned}$$

If instead of weak convergence in L^2_{loc} one had strong convergence then one could easily show that $\bar{u}$ is a solution of (8) and (9). Such strong convergence may, however not hold in general (cf. [Ma1]). Di Perna and Majda (see [DM], where further references can be found) begun a detailed investigation into the question whether a limit of (exact or approximate) solutions of Euler's equation is again a solution if no strong convergence holds. In particular they showed that this is the case for the stationary Euler equation (in two dimensions) by a careful analysis of the "concentration set" where strong convergence fails. It has so far not been possible to extend that approach to the time-dependent case, mainly due to poor control in time (but see [Lo], [Bi]).

Delort [De] took a different approach. First note that u is a weak solution of (8), (9) if and only if

$$\omega_t = \left((u^2)^2 - (u^1)^2\right)_{x_1x_2} + (u^1u^2)_{x_2x_2} - (u^1u^2)_{x_1x_1}, \tag{10}$$

where $\omega = \operatorname{curl} u$ or

$$u = K * \omega, \qquad K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2}. \tag{11}$$

Delort then showed the following, using the representation (11) of u.

THEOREM 7. *If $\omega_{0\varepsilon} \geq 0$ then one has with the above notation*

$$u^1_{\varepsilon_j} u^2_{\varepsilon_j} \rightharpoonup \bar{u}^1\bar{u}^2, \quad (u^1_{\varepsilon_j})^2 - (u^2_{\varepsilon_j})^2 \rightharpoonup (\bar{u}^1)^2 - (\bar{u}^2)^2$$

in the sense of distributions. In particular $\bar{u}$ is a solution of (8), (9).

For variants and generalizations of Delort's argument see [Ge] and [Ma2].

In [EM] (see also [Se]) an attempt was made to understand Delort's result in the context of Hardy spaces. The main observation is the following.

THEOREM 8. *Let $\psi \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ and suppose that $\Delta\psi \geq 0$. Then $\psi_{x_1}\psi_{x_2}$ and $(\psi_{x_1})^2 - (\psi_{x_2})^2$ are in $\mathcal{H}^1_{\text{loc}}$ and for each $\phi \in C_0^\infty(\mathbb{R}^2)$ one has*

$$\|\phi\psi_{x_1}\psi_{x_2}\|_{h^1} + \|\phi((\psi_{x_1})^2 - (\psi_{x_2})^2)\|_{h^1} \leq C \int_{B(0,R)} |D\psi|^2 \, dx ,$$

where C and R only depend on ϕ (but not on ψ).

The proof is similar in spirit to Delort's and relies on the definition of $\mathcal{H}^1_{\text{loc}}$ given above and the representation (11) of u.

For the application to the Euler equation note that due to the equation $\operatorname{div} u = 0$, u can be expressed as $u = \nabla^{\perp}\psi$ so that $\omega = \operatorname{curl} u = \Delta\psi$.

Delort's result can then be recovered from the following fact (see [EM] for details of the argument).

THEOREM 9. ([JJ], [EM]). *Suppose that $f_j \to f$ a.e. and that $\|f_j\|_{h^1}$ is bounded. Then $f_j \rightharpoonup f$ in the sense of distributions.*

Note that the result fails if h^1 is replaced by L^1. Using the Hardy space approach one also obtains a new estimate for the streamfunction ψ, namely $\psi_t \in L^\infty([0,T], L^1_{\text{loc}})$ or more precisely

$$\|\phi\psi_t(t,\cdot)\|_{h^1} \leq C(\phi, T) , \qquad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^2), \quad t \in [0,T] .$$

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