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In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [459]--464.

Persistent URL: http://dml.cz/dmlcz/700126

ON THE SMOOTHNESS OF THE SOLUTIONS TO THE ELLIPTIC SYSTEMS

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Section B

Let us consider the system of quasilinear differential equations with respect to the unknown function $u = (u^{(1)},\dots,u^{(N)})$ in the form

$$L(u) \equiv \sum_{|\beta|\leq \ell} (-1)^{|\beta|} D^{\beta} a_{\beta}(x;u,\dots,D^{\ell}u) = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset R^m$ $(m \geq 2)$ with sufficiently smooth boundary Γ. Here $\beta = (\beta_1,\dots,\beta_m)$ is a multiindex, $D^{\beta} = D_1^{\beta_1}\dots D_m^{\beta_m}$ where D_j denotes the derivative over x_j, D_j^0 is the identity operator and $|\beta| = \beta_1+\dots+\beta_m$. We shall seek for the solution u of the system (1.1) which satisfies the boundary conditions

$$\left.\frac{\partial^k u}{\partial \nu^k}\right|_{\Gamma} = 0, \quad k = 0,1,\dots,\ell-1, \tag{1.2}$$

on Γ where ν is an exterior normal unit vector to Γ.

Suppose that each N-dimensional vector function $a_{\beta}(x;u,\dots,D^{\ell}u)$ (depending on x, u and all the derivatives of u up to the order ℓ) satisfies the following conditions:

1) For almost all $x \in \overline{\Omega}$ and for all $p_0,\dots,p_{\ell}$ bounded the functions $a_{\beta}(x;p_0,\dots,p_{\ell})$ are continuously differentiable with respect to the vector arguments p_s, $s = 0,1,\dots,\ell$.

2) The operator L defined by these functions is an elliptic operator with the bounded nonlinearities, i.e., for each vector $\xi \in R^M$ and for each $x \in \overline{\Omega}$ and $p_0,\dots,p_{\ell}$ the following inequalities hold:

$$\nu_0(1+T^2)^{\frac{p-2}{2}}|\xi|^2 \leq (A\xi,\xi) \leq \mu_0(1+T^2)^{\frac{p-2}{2}}|\xi|^2, \tag{1.3}$$

$$\|A\| \leq C(1+T^2)^{\frac{p-2}{2}}. \tag{1.3'}$$

By T^2 the sum $\sum_{k=1}^{N} \sum_{|\tau|\leq 1} |p_{\tau}^{(k)}|^2$ is denoted, where $p_{\tau}^{(k)}$ are the components of the vector $p_{|\tau|} \in R^{m^{|\tau|}}$ and A is an $M \times M$ matrix of the partial derivatives $\partial a_{\beta}^{(\tau)}/\partial p_{\tau}^{(k)}$. Here μ_0, ν_0 are positive constants and $p \in]1,2]$.

3) If the function $u \in W_q^{(\ell)}(\Omega)$ *) for $q > 1$ then $a_\beta(x;u,\dots,D^\ell u) \in$ $\in L_{q/(p-1)}(\Omega)$.

As usual, the function $u \in \overset{\circ}{W}{}_p^{(\ell)}(\Omega)$ is said to be a generalized solution of the boundary value problem (1.1), (1.2) if for each $v \in \overset{\circ}{W}{}_p^{(\ell)}(\Omega)$ the integral identity

$$\sum_{|\beta|\le\ell} \int_\Omega a_\beta(x;u,\dots,D^\ell u)D^\beta v\,dx = 0 \tag{1.4}$$

takes place.

The existence and the uniqueness of the generalized solution of the problem in question (under the natural restrictions) are nowdays a well-known facts which can be established -e.g.- by means of the theory of monotone operators.

In view of the problem of numerical solution, the following iterative process was suggested by the author (A.I.Koshelev [1]):

$$\sum_{i=0}^{\ell} \int_\Omega D^i u_{n+1} D^i v\,dx = \sum_{i=0}^{\ell} \int_\Omega D^i u_n D^i v\,dx -$$

$$- \varepsilon \sum_{|\beta|\le\ell} \int_\Omega a_\beta(x;u_n,\dots,D^\ell u_n)D^\beta v\,dx\,,$$

$$\left.\frac{\partial^k u_{n+1}}{\partial\nu^k}\right|_\Gamma = 0, \qquad k = 0,1,\dots,\ell-1. \tag{1.5}$$

Here ε is a sufficiently small positive constant and the notation is used. It is easy to see that this iterative process can be performed in each step. In fact, if $u_0 \in \overset{\circ}{W}{}_p^{(\ell)}(\Omega)$ then the conditions 1) - 3) quarantee that all the functions u_n belong to the same space. We shall suppose that the solution u of the problem (1.1) - (1.2) belongs to $\overset{\circ}{W}{}_2^{(\ell)}(\Omega)$. In this case we get the following assertion [2]:

Theorem 1.1 Let the conditions 1) - 3) be satisfied and let the solution $u \in \overset{\circ}{W}{}_2^{(\ell)}(\Omega)$ exist. Then the iterative process (1.5) converges to this solution u in $\overset{\circ}{W}{}_p^{(\ell)}(\Omega)$ for every initial value $u_0 \in \overset{\circ}{W}{}_p^{(\ell)}(\Omega)$.

Further we consider only non-degenerate systems, i.e., $p = 2$. Then the inequalities (1.3) and (1.3′) have the form

$$\nu_0|\xi|^2 \le (A\xi,\xi) \le \mu|\xi|^2 \tag{1.6}$$

$$\|A\| \le C\,. \tag{1.6′}$$

*) Throughout the paper we use the notation $W_q^{(\ell)}(\Omega)$ for the Sobolev space.

In this case we can get the estimate of the rate of the convergence of the process (1.5). Moreover, we obtain the existence of the solution of the problem (1.1), (1.2) in $\overset{\circ}{W}{}_2^{(\ell)}(\Omega)$ as a consequence of the convergence of the iteration process. Namely, the following theorem holds

Theorem 1.2. Let $p = 2$ and let 1) - 3) hold. Then there exists a generalized solution $u \in \overset{\circ}{W}{}_2^{(\ell)}(\Omega)$ of the problem (1.1),(1.2) to which the iterative process converges for the suitable choice of ε at a geometric rate for each initial value $u_0 \in \overset{\circ}{W}{}_2^{(\ell)}(\Omega)$.

Theorem 1.2 is proved in [2].

To establish the smoothness of a generalized solution it is convenient to study the convergence of the process (1.5) in spaces $H_{\ell,\alpha}(\Omega)$ introduced by Nirenberg [3] and Cordes [4]. The space $H_{\ell,\alpha}(\Omega)$ consists of the functions $u \in W_2^{(\ell)}(\Omega)$ having the finite form

$$\|u\|^2_{\ell,\alpha} = \sup_{x_0\in\Omega} \sum_{j=0}^{\ell} \int_\Omega |D^j u|^2 r^\alpha dx ,$$

$$r = |x - x_0|; \; x,x_0 \in \Omega \; . \tag{1.7}$$

As it is known, the space $H_{\ell,\alpha}(\Omega)$ is a Banach space and for $\alpha = 2 - m - 2\gamma$, $\gamma \in]0,1[$, it is embedded in the Hölder space $C^{(\ell-1+\gamma)}(\Omega)$. This property makes it convenient for the proof of the regularity of the solutions.

Let us denote $w_n = u_n - u_{n-1}$. From two consequent equations (1.5) we obtain - using the mean-value theorem - the following relation

$$\sum_{i=0}^{\ell} \int_{\Omega'} D^i w_{n+1} D^i v dx = \int_{\Omega'} (I - \varepsilon\bar{A}) W_n V dx , \tag{1.8}$$

Here the bar over A means that this matrix is calculated in some point located between u_n and u_{n-1}, function v belongs to $\overset{\circ}{W}{}_q^{(\ell)}(\Omega')$ where Ω' is an arbitrary strictly interior subdomain of Ω and W_n,V are the vectors, whose components are the derivatives of the functions w_n,v, respectively.

Let $A = A^+ + A^-$ where A^+ and A^- are symmetric and antisymmetric parts of A, λ_i are the eigenvalues of A^+, $\lambda = \inf\limits_{i,x,p} \lambda_i$, $\Lambda = \sup\limits_{i,x,p} \lambda_i$, $s_\lambda = \lambda_1 + \ldots + \lambda_M$, $S_{\lambda^2} = \lambda_1^2 + \ldots + \lambda_M^2$; Let σ denote the upper bound of eigen-values of symmetric matrix $A^+A^- - A^-A^+ - (A^-)^2$. Suppose that $\sigma \geq 0$. Let us consider two positive constants

$$K_\varepsilon^2 = \sup_{i,x,p} |1 - \varepsilon\lambda_i|^2 + \sigma\varepsilon^2, \; L_\varepsilon^2 = \sup_{x,p} |M - 2\varepsilon S_\lambda + \varepsilon^2 (S_{\lambda^2} + \sigma)| .$$

From (1.8) we get the inequality

$$\left|\sum_{i=0}^{\ell}\int_{\Omega} D^i w_{n+1} D^i v dx\right| \le K_\varepsilon \|w_n\|_{\ell,\alpha} \cdot \|v\|_{\ell,-\alpha} \qquad (1.9)$$

and the analogous relation with L_ε instead of K_ε. The inequality (1.9) allows to prove the convergence of the iterative process (1.5) in $H_{\ell,\alpha}(\Omega')$ choosing a suitable test function v.

We recall the results of [5].

Put

$$A_\ell = \begin{cases} [(1+\frac{m-2}{m-1})(1+(m-2)(m-1))]^{\ell_1/2}, & \text{if } \ell = 2\ell_1 \\ [(1+\frac{m-2}{m-1})(1+(m-2)(m-1))]^{\ell_1/2}[1+\frac{(m-2)^2}{m-1}]^{1/2}, & \text{if } \ell = 2\ell_1+1. \end{cases}$$

Theorem 1.3. If the conditions 1) - 3) hold and, moreover, one of the inequalities

$$K_\varepsilon A_\ell < 1, \qquad (1.10)$$

$$L_\varepsilon A_\ell < 1, \qquad (1.11)$$

is true, then the iterative process converges in $H_{\ell,\alpha}(\Omega')$ (here $\alpha = 2 - m - 2\gamma$, $\gamma \in (0,1)$) at a geometric rate for sufficiently small γ and sufficiently smooth initial value u_0.

Theorem 1.3 and the following Theorem 1.4 were proved first for the second order systems ($\ell = 1$) in a number of the author's papers (see for example [2]).

The quotient of the geometric sequence mentioned above is given by the left hand side of (1.10) or (1.11) and it depends on ε. The rate of the convergence will be the best for such an ε for which K_ε attaines its infimum. It is proved in the paper [2] that

$$K \equiv \inf_{\varepsilon>0} K_\varepsilon = \begin{cases} [(\Lambda-\lambda)^2 + 4\sigma]^{1/2}/(\Lambda+\lambda), & \sigma \le \frac{(\Lambda-\lambda)\lambda}{2} \\ [\sigma/(\sigma+\lambda^2)]^{1/2}, & \sigma \ge \frac{(\Lambda-\lambda)\lambda}{2} \end{cases} \qquad (1.12)$$

For a symmetric matrix A the relation (1.12) takes the form

$$K = \frac{\Lambda-\lambda}{\Lambda+\lambda}, \qquad (1.13)$$

and this value of K_ε is obtained for $\varepsilon = \frac{2}{\Lambda+\lambda}$. Let us mention that K is always less than 1. For second order systems, the optimal conditions for the convergence of the process (1.5) is given by

$$K[1 + \frac{(m-2)^2}{m-1}]^{1/2} < 1 \ . \qquad (1.14)$$

The convergence of the iterative process established in theorem 1.3 allows us to write the conditions guaranteeing the regularity of solutions of problem (1.1), (1.2).

Theorem 1.4. If the coefficients of system (1.1) satisfy 1) - 3) and the inequality

$$K\ A_{\ell} < 1 \tag{1.15}$$

holds, then a generalized solution of problem (1.1),(1.2) has Hölder continuous derivatives up to the order $\ell - 1$ in every strictly interior subdomain Ω' of Ω.

Let us mention that condition (1.15) in case of the symmetric matrix A gives bounds for dispersion of eigenvalues. For nonsymmetric matrix A it expresses a subordination of the antisymmetric part A^- of the matrix A to the symmetric part A^+.

Thus we can conclude that the condition on A guaranteeing the optimal convergence of process (1.5) in $C^{(\ell-1+\gamma)}(\Omega)$ is the same as the condition guaranteeing the existence of a solution of the problem in $C^{(\ell-1-\gamma)}(\Omega)$. Moreover, as it was proved in the paper [6], the condition (1.15) (having in this case the form (1.14)) is sharp for $\ell = 1$ i.e. for second order systems. It means that violation of this condition can imply the existence of nonregular solutions.

It seems that the conditions (1.10) and (1.15) are not sharp for every $\ell > 1$. It was proved in [7] (because of the possibility of better choice of a test function) that the Hölder continuity of first derivatives of solutions of fourth order systems ($\ell = 2$) is guaranteed under weaker conditions on the dispersion of eigenvalues of A. We get

Theorem 1.5. If $\ell = 2$ and the conditions 1) - 3) and the inequality (1.16) hold

$$K\{1 + \frac{(m-2)^2}{m-1}\,[1 + \max\{0, \frac{2m(m-2)(m-4)}{9(m^2-1)}\}]\}^{1/2} < 1 \tag{1.16}$$

then every generalized solution of problem (1.1),(1.2) has first Hölder continuous derivatives in any strongly interior subdomain Ω' of Ω. Moreover, the iterative process (1.5) converges for a suitable ε to a solution of problem (1.1),(1.2) in Hölderian norm.

The conditions of Theorem 1.5 are sharp in the same sense as for $\ell = 1$.

Put $L = \inf_{\varepsilon>0} L_\varepsilon$. The condition of convergence of problem 1.5 as well as the condition for the existence of regular solution can be written also in the form

$$L\ A_{\ell} < 1. \tag{1.17}$$

It can be proved that this condition is too strong in contrast to (1.15). However, the constant L can be effectively calculated as it can be expressed in terms of invariants of matrix A. This fact can be decisive for numerical solution of the problem. The condition (1.17) is studied in details in [2],[8]. The case of a matrix A satisfying Cordes condition is studied there, too.

For second order systems ($\ell = 1$) the condition (1.16) guarantees the smoothness of solutions up to the boundary. We have

Theorem 1.6. Let $\ell = 1$ and the assumptions of Theorem 1.4 be fulfilled. If the coefficients a_β satisfy certain natural smoothness conditions and the inequality (1.16) holds, than the generalized solution of (1.1),(1.2) belongs to $W_q^{(2)}(\Omega)$ for a $q > m$.

Analogous results on the existence of solutions with Hölder continuous derivatives of second order and on the higher differentiability of solutions are true under stronger natural conditions on the data of the problem.

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