EQUADIFF 6

Jean-Claude Nédélec

Mixed finite element in 3D in $H(\mathrm{div})$ and $H(\mathrm{curl})$

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [321]--325.

Persistent URL: http://dml.cz/dmlcz/700133

MIXED FINITE ELEMENT IN 3D IN H(div) AND H(curl)

J. C. NEDELEC
Ecole Polytechnique, Centre de Mathématiques Appliquées
91128 Palaiseau, France

I. INTRODUCTION.

Frayes De Venbeke first introduces the mixed finite element. Then P.A. Raviart and J.M. Thomas does some mathematics on these element in 2D and others do also : F. Brezzi V. Babuska ...
In 1980 we introduce a family of some mixed finite element in 3D and we use them for solving Navier Stokes equations.
In 1984 F. Brezzi, J. Douglass and L.D. Marini introduce in 2D a new family of mixed finite element conforming in H(div). That paper was the starting point for building new families of finite element in 3D.

II. FINITE ELEMENT IN H(div).

Notations.

K is a tetrahedron
∂K its boundary
n the normal
f a face which area is $\int_f d\gamma$
a is an edge which lenght is $\int_a ds$
$\text{curl } u = \nabla \wedge u \quad u = (u_1, u_2, u_3)$
$H(\text{curl}) = \{u \in L^2(\Omega))^3 \;;\; \text{curl } u \in (L^2(\Omega))^3\}$
$\text{div} = \nabla . u$
$H(\text{div}) = \{u \in (L^2(\Omega))^3 \;;\; \text{div } u \in L^2(\Omega)\}$

Spaces of polynomials.

P_k = polynomials of degree less or equal to k
$\tilde{P}_k$ = " homogeneous of degree k

$$\mathcal{D}_k = (P_{k-1})^3 + \tilde{P}_{k-1}\, r \qquad r = \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases}$$

$S_k = \{p \in (P_k) \;;\; (r.p) \equiv 0\}$
$\mathcal{R}_k = (P_{k-1})^3 \oplus S_k$

$\dim S_k = k(k + 2)$

$\dim \mathcal{D}_k = \frac{(k + 3)(k + 1)\, k}{2}$

$\dim \mathcal{R}_k = \frac{(k + 3)(k + 2)\, k}{2}$

We are now able to introduce the finite element conforming in H(div).

Definition. We define the finite element by

1) K is a tetrahedron
2) $P = (P_k)^3$ is a space of polynomials
3) The set of degrees of freedom which are

(3.1) $$\int_f (p \cdot n) q \, d\gamma \ ; \ \forall\, q \in P_k(f) \ ;$$

(3.2) $$\int_K (p \cdot q) \, dx \ ; \ \forall\, q \in \mathcal{R}_{k-1} \ .$$

we have the

Theorem.

The above finite element is unisolventand conforming in H(div). The associate interpolation operator Π is such that

$$\operatorname{div} \Pi p = \Pi^{\star} \operatorname{div} p \quad ; \quad \forall\, p \in H(\operatorname{div}) \quad ,$$

where $\Pi^{\star}$ is the L^2 projection on P_{k-1} .

When k = 1, the corresponding element has no interior moments and 12 degrees of freedom. Its divergence is constant.

Proposition.For a tetrahedron "regular enough" which diameter is k, we have

$$\| p - \Pi p \|_{(L^2(K))^3} \leqslant c\, h^{k+1} \|p\|_{(H^{k+1}(K))^3} \quad ;$$

$$\| D(p - \Pi p) \|_{(L^2(K))^3} \leqslant c\, h^{k} \|p\|_{(H^{k+1}(K))^3} \quad .$$

We are not going to prove this theorem. But we can recall that a finite element is said to be conforming in a functional space if the interpolate of an element of this space belong to this space.

In our case, the conformity in H(div) is equivalent to the continuity of the normal composent at each interface. This property is clearly true for our finite element since the unknowns on the face are

$$\int_f (p \cdot n)\, q \, d\gamma \quad ; \quad \forall\, q \in P_k(f)$$

and p.n is also $P_k(f)$.

III. FINITE ELEMENT IN H(curl).

A finite element is conforming in H(curl) if the tangential components are continue at the interface of the mesh.
We introduce the corresponding finite element.

Définition.

1) K is a tetrahedron
2) $P = (P_k)^3$ is the space of polynomials
3) The degrees of freedom are the following moments

3.1) $\int_a (p \cdot \tau)\, q\, ds \quad ; \quad \forall\, q \in P_k(a)$

3.2) $\int_f (p \cdot q)\, d\gamma \quad ; \quad \forall\, q \in \mathcal{D}_{k-1}(f)$ and tangent to the face f

3.3) $\int_K (p \cdot q)\, dx \quad ; \quad \forall\, q \in \mathcal{D}_{k-2}$.

We have the

Theorem.

The above finite element is unisolvent and conforming in H(curl). Moreover if Π is the corresponding interpolation operator and Π^* the interpolation operator associate to the H(div) finite element introduce previously for degree k-1 we have

$$\text{curl}\ \Pi p = \Pi^* \text{curl}\ p$$

IV. APPLICATION TO THE EQUATION OF STOKES.

The Stokes'equation is usually written in the (u,p) variable in a bounded domain Ω of R^3 as

$$\begin{cases} -\nu\, \Delta u + \text{grad}\ p = f \quad , & \text{in } \Omega \\ \text{div}\ u = 0 & \text{in } \Omega \\ u|_\Gamma = 0 \end{cases}$$

We introduce the vector potential ϕ as

$$\begin{cases} -\Delta\phi = \text{curl}\ u \quad , & \text{in } \Omega \\ \text{div}\ \phi = 0 \quad , & \text{in } \Omega \\ \phi \wedge n|_\Gamma = 0 \end{cases}$$

Then the Stokes equation can be written in the (ϕ,ω) variables where

$$\omega = \text{curl}\ u$$

We introduce

$$H(\text{div}^0) = \{\ v \in (L^2(\Omega))^3\ ;\ \text{div}\ v \in 0\ ,\ v.n|_\Gamma = 0\ \}$$

$$\mathcal{H} = \{\ \psi \in H(\text{curl})\ ;\ \text{div}\ \psi = 0\ ;\ \psi \wedge n|_\Gamma = 0\ \}$$

Then a variational formulation of the Stokes equation is

$$\begin{cases} \nu \int_{\Gamma} (\text{curl } \omega . \text{curl } \psi) dx = \int_{\Omega} (f . \text{curl } \psi) dx \ ; \ \forall \ \psi \in H \\ \int_{\Omega} (\omega . \Pi) dx - \int_{\Omega} (\text{curl } \phi . \text{curl } \Pi) dx = 0 \ ; \ \forall \ \Pi \in H(\text{curl}) \end{cases}$$

Let C_h be a mesh covering Ω .

We can introduce some finite element spaces

$$W_h = \{ \omega_h \in H(\text{curl}) \ ; \ \omega_h|_K \in (P_k)^3 \ ; \forall \ K \in C_k \ \}$$

$$W_h^0 = \{ \omega_h \in W_h \ ; \ \omega_h \wedge n|_{\Gamma} = 0 \ \}$$

$$V_h = \{ v_h \in H(\text{div}) \ ; \ v_h|_K \in (P_{k-1})^3 \ ; \ \forall \ K \in C_h \ \}$$

$$U_h = V_h \cap H(\text{div}^0)$$

The approximate problem become then

$$\begin{cases} \nu \int_{\Omega} (\text{curl } w_h \cdot v_h) dx = \int_{\Omega} (f . v_h) dx \ ; \ \forall \ v_h \in U_h \ ; \\ \int_{\Omega} (w_h . \Pi_h) dx - \int_{\Omega} (u_h . \text{curl } \Pi_h) \ dx = 0 \ ; \ \forall \ \Pi_h \in W_h \ . \end{cases}$$

We can also use a vector potential ϕ_h.
This goes like that

$$\Theta_h = \{ \ \theta_h \in H^1(\Omega) \ ; \ \theta_h|_K \in P_{k+1} \ ; \ \forall \ K \in C_h \ \}$$

$$\Theta_h^0 = \Theta_h \cap H_0^1(\Omega)$$

We have the

Theorem.

If the transgulation is regular, for every $v_h \in U_h$, there exist a unique $\psi_h \in W_h^0$ such that

$$\begin{cases} \text{curl } \psi_h = v_h \\ \int_{\Omega} (\psi_h \ . \ \text{grad } \theta_h) dx = 0 \ ; \ \forall \ \theta_h \in \Theta_h^0 \end{cases}$$

and we have also

$$\|\psi_h\|_{H(\text{curl})} \leqslant c \ \|v_h\|_{(L^2(\Omega))^3} \ .$$

This theorem can be use to transfer the above approximate problem in one in (ψ,ω) and also to find a local basis in the space U_h.

BIBLIOGRAPHY

F. BREZZI, On the existence, uniqueness and approximation of saddle point problems ausing from Lagrangian multipliers. RAIRO 8 : 129 - 151 (1974).

F. BREZZI, J. DOUGLASS & L.D. MARINI, Two families of mixed finite elements for second order elliptic problems. To appear in Numerische Mathematik.

P.G. CIARLET, The finite element method for elliptic problems. North Holland Amsterdam (1978).

P.G. CIARLET & P.A. RAVIART, A mixed finite element method for the biharmonic equation. Mathematical aspects in finite element method (C de Boor ed.) pp. 125 - 145 Academic Press New York (1974).

M. FORTIN, An analysis of the convergence of mixed finite element method. RAIRO 11 : 341 - 354 (1977).

J.C. NEDELEC, Mixed finite element in $\mathbb{R}^3$. Numerische Mathematik 35 : 315 - 341 (1980).

J.C. NEDELEC, Elements finis mixtes incompressibles pour l'equation de Stokes dans $\mathbb{R}^3$. Numerische Mathematik 39, 97 - 112 (1982).

P.A. RAVIART & J.M. THOMAS, A mixed finite element method for 2nd order elliptic problems. In Dold A Eckmann B (eds), Mathematical aspects of finite element methods. Lecture Notes 606 Springer Berlin (1977).

J.M. THOMAS, Thesis Paris (1977).

J.C. NEDELEC, A new family of mixed finite element in $\mathbb{R}^3$ (à paraître).