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THE ROTHE METHOD FOR NONLINEAR HYPERBOLIC PROBLEMS

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The ROTHE method or the horizontal method of lines, if it is applied to parabolic as well as to hyperbolic evolution problems, reduces these problems to a sequence of elliptic problems. That from a former point of view, such an approach has appeared more natural in the case of parabolic than of hyperbolic problems, may serve as an explanation for the considerable delay of time in studying the method for both classes of problems. So after ROTHE [11] has introduced his method in the early thirties of our century, numerous parabolic differential equation problems, linear as well as nonlinear ones, have been treated by it successfully; the names of LADYSHENSKAJA, REKTORYS, NEČAS, and KAČUR may stand here for many others (references, for instance, may be seen from the book of REKTORYS [10]). On the other hand, efforts for applying the ROTHE method to hyperbolic problems firstly have been started during the last decade. Results have been given mainly for certain linear problems of mathematical physics, so as for the wave equation [1,2,5], the continuity equation [3], and the MAXWELL equations [4]; recently the vibrating string problem with discontinuous data has been completely solved by the ROTHE method [6]. Further linear hyperbolic problems have been investigated by REKTORYS [10]. With regard to nonlinear hyperbolic problems, however, one is standing at the very beginning. First results of MUNZ [8,9] concerning the quasilinear scalar conservation equation, especially have shown the ROTHE method as a suitable tool for approximation of shocks and rarefaction waves.

In the following we shall consider the CAUCHY problem for the BURGERS equation

$$u_t + \frac{1}{2}(u^2)_x = 0 \quad , \quad (x,t) \in \mathbb{R} \times (0,\infty) \quad , \tag{1}$$

where the initial values

$$u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R} \quad , \tag{2}$$

are assumed to be piecewise continuous with at most a finite number of discontinuities and existing limits for $x \to \mp\infty$. The ROTHE method for a fixed chosen time step length $h > 0$ leads to the ordinary differential equation

$$u + \frac{h}{2}(u^2)' = u_o \qquad x \in \mathbb{R} \tag{3}$$

which for given $u_o(x)$, $x \in \mathbb{R}$, has to be solved successively according to the next time step. On the solutions $u(x)$, $x \in \mathbb{R}$, of (3), there are imposed piecewise continuity with at most a finite number of discontinuities and existing limits for $x \to \mp\infty$; furthermore, the square $[u(x)]^2$, $x \in \mathbb{R}$, is asked as a piecewise continuously differentiable function (as a consequence of the foregoing, for the derivative there may occur at most a finite number of discontinuities). Without mentioning in detail, the following assertions will concern to solutions of (3) having at least these properties.

Theorem 1 (Behaviour at the infinity). For a solution $u(x)$, $x \in \mathbb{R}$, of (3) it holds

$$\lim_{x\to\mp\infty} u(x) = \lim_{x\to\mp\infty} u_o(x) \quad . \tag{4}$$

Proof follows immediately from (3) in connection with the second L'HOSPITAL rule:

$$0 = \lim_{x\to\mp\infty} \frac{h[u(x)]^2}{2x} = \lim_{x\to\ \infty}\left\{\frac{h}{2}\frac{d}{dx}[u(x)]^2\right\} = \lim_{x\to\mp\infty} u_o(x) - \lim_{x\to\mp\infty} u(x) \quad .$$

Remark. The proof of Theorem 1 makes only use of the conservation property of the underlying partial differential equation (1). Thus the accordance of the limits (4) will be obtained analogously for other hyperbolic problems when they are given in conservation form. For instance, this holds for the EULER equations.

Theorem 2 (Global uniqueness). There exists at most one continuous solution $u(x)$, $x \in \mathbb{R}$, of (3).

Proof. Assuming that there exist two different continuous solutions $u(x), v(x)$, $x \in \mathbb{R}$, so the continuous function $w(x) := u(x) - v(x)$, $x \in \mathbb{R}$, does not vanish everywhere. Note that because of Theorem 1 it holds

$$\lim_{x\to\mp\infty} w(x) = \lim_{x\to\mp\infty} u(x) - \lim_{x\to\mp\infty} v(x) = 0 \tag{5}$$

Let now $x_o \in \mathbb{R}$ be a point with $w(x_o) \neq 0$. If $w(x)$ has at least one zero in the open interval $(-\infty, x_o)$, then for continuity there exists a maximum zero in this interval and we denote it by $a < x_o$; if, however, there are no zeroes in $(-\infty, x_o)$, we put $a = -\infty$. Analogously let $b > x_o$ denote the minimum zero for $w(x)$ in (x_o, ∞) or stand for ∞, respectively. Together with (5) we get

$$\lim_{x\to a} w(x) = \lim_{x\to b} w(x) = 0 \quad . \tag{6}$$

Observing the continuity and piecewise continuous differentiability of $[u(x)]^2, [v(x)]^2$, $x \in \mathbb{R}$, it follows from (3) and (6) by improper integration that

$$\int_a^b w(x)\,dx = \int_a^b \{u(x) - v(x)\}\,dx = -\frac{h}{2}\int_a^b \left\{\frac{d}{dx}[u(x)]^2 - \frac{d}{dx}[v(x)]^2\right\} dx$$

$$= -\frac{h}{2}\Big[[u(x)]^2 - [v(x)]^2\Big]_a^b = -\frac{h}{2}\Big[w(x)(u(x) + v(x))\Big]_a^b = 0$$

This is a contradiction to $w(x) \not\equiv 0$, $x \in (a,b)$.

Theorem 3 (Local uniqueness). Let $(a,b) \subseteq \mathbb{R}$ be an arbitrary finite or infinite open interval and let the above ordinary differential equation problem be formulated analogously for (a,b) instead of $\mathbb{R}$. Let further $u(x), x \in (a,b)$, be a positive continuous (negative continuous) solution of (3) which has a positive limit for $x \to a$ (negative limit for $x \to b$). Then there does not exist another continuous solution of (3) with the same limit for $x \to a$ $(x \to b)$.

Proof only for the first case. Assume that there exists a continuous solution $v(x)$, $x \in (a,b)$, different from $u(x)$, $x \in (a,b)$, but with the same limit for $x \to a$. Then the difference $w(x) := u(x) - v(x)$, $x \in (a,b)$, forms a continuous function satisfying

$$\lim_{x\to a} w(x) = 0 \quad . \tag{7}$$

Next we are able to find a point $x_o \in (a,b)$ with properties

$$w(x_o) \not\equiv 0 \quad , \quad u(x_o) + v(x_o) \geqq 0 \quad . \tag{8}$$

Indeed, if $u(x) + v(x)$, $x \in (a,b)$, has no zeroes, from continuity and

$$\lim_{x\to a} \{u(x) + v(x)\} = 2 \lim_{x\to a} u(x) > 0$$

it follows that $u(x) + v(x) > 0$, $x \in (a,b)$, and so it is trivial to find $x_o \in (a,b)$ satisfying (8); if, however, $u(x) + v(x)$, $x \in (a,b)$, has a zero $x_o \in (a,b)$, so this zero immediately fulfills the second condition in (8) and the first condition follows from $v(x_o) = -u(x_o)$ as

$$w(x_o) = u(x_o) - v(x_o) = 2\,u(x_o) > 0$$

Now we denote by $a^* < x_o$ the maximum zero for $w(x)$ in the open interval (a,x_o) if there exists a zero at all, otherwise we put $a^* = a$. So in any case when observing (7), we get

$$\lim_{x\to a^*} w(x) = 0 \quad . \tag{9}$$

Then by improper integration, it follows from (3) and (9) that

$$\int_{a^*}^{x_o} w(x)\,dx = \int_{a^*}^{x_o} \{u(x) - v(x)\}\,dx = -\frac{h}{2}\int_{a^*}^{x_o} \left\{\frac{d}{dx}[u(x)]^2 - \frac{d}{dx}[v(x)]^2\right\} dx$$

$$= -\frac{h}{2}\left[w(x)(u(x) + v(x))\right]_{a^*}^{x_o} = -\frac{h}{2}\, w(x_o)(u(x_o) + v(x_o))$$

here because of (8), we have the contadiction, that the left hand side has the sign of $w(x_o) \neq 0$ whilst the right hand side either has the opposite sign or vanishes.

Remark 1. Theorem 3 gives a hint how to proceed for solving the differential equation (3) uniquely. So if starting at some point with a positive or negative initial value, one has to integrate to the right or to the left, respectively. On the other hand, the sign of the exact solution analogously indicates the direction of the characteristics. So it turns out that local uniqueness for the ROTHE solution is assured by integrating into the direction of characteristics.

Remark 2. As it can be seen from the example $u_o(x) = 1$, $x \in \mathbb{R}$, the sign condition in Theorem 3 plays a significant role. So the solution $u(x) = 1$, $x \in \mathbb{R}$, is the only one of (3) with limit 1 for $x \to -\infty$, but there exist an infinite number of further solutions with limit 1 for $x \to \infty$; indeed, with an arbitrary real constant C, such a solution $u(x)$, $x \in \mathbb{R}$, may be obtained as the inverse of the monotonously decreasing function

$$x(u) = -h\{u + \ln(u-1)\} + C \quad , \quad u \in (1,\infty) \quad .$$

We shall make use of the foregoing theorems when discussing the following four examples.

Example 1 (MUNZ [8]). If $u_o(x)$, $x \in \mathbb{R}$, is the step function with value 2 for negative or 1 for positive x, respectively, the exact solution $u(x,t), (x,t) \in \mathbb{R} \times [0,\infty)$, of the evolution problem (1) and (2) is given as a shock wave at $x = \frac{3}{2}t$ with value 2 left or 1 right of the shock, respectively. Assume that for an arbitrary time step a ROTHE solution exists which, for convenience, will be denoted by $u_o(x), x \in \mathbb{R}$; besides the general properties mentioned above let this solution be monotonously nonincreasing with lower bound 1, let it have the value 2 for $x \in (-\infty,0)$, and let it be continuous for $x \in (0,\infty)$. Note that for such solution the limits for $x \to \mp\infty$ exist and that everything holds for the given initial function. The next ROTHE step $u(x)$, $x \in \mathbb{R}$, then may be computed from (3) as a continuous solution with value 2 for $x \in (-\infty,0)$; for $x \in [0,\infty)$ the solution follows by means of the initial condition $u(0) = 2$ in connection with the lower function $u_o(x)$ and the upper function 2. This especially yields

$$1 \leq u_0(x) \leq u(x) \quad , \quad x \in (0,\infty) \tag{10}$$

From the differential equation (3) together with (10) it follows that $u'(x) \leq 0$, $x \in (0,\infty)$; so $u(x)$, $x \in \mathbb{R}$, is monotonously nonincreasing and because of (10), it has the lower bound 1. Theorem 2 as well as the first case of Theorem 3 say that there is no further continuous solution, so the next ROTHE step is well-defined. Finally by induction, all ROTHE solutions are uniquely determined. Because of Theorem 1, for every ROTHE solution the limit 2 for $x \to -\infty$ or 1 for $x \to \infty$ is obtained, respectively.

Example 2 (MUNZ [8]). Here $u_0(x)$, $x \in \mathbb{R}$, is considered as a step function with value 1 for negative or 2 for positive x, respectively. The exact solution is a rarefaction wave with values $\frac{x}{t}$ for $t \leq x \leq 2t$, $0 < t < \infty$ and value 1 left or 2 right of the wave, respectively. As it turns out quite similarely to Example 1, the ROTHE method again can be carried out uniquely.

Example 3 (MARTENSEN [7]). The initial values $u_0(x)$, $x \in \mathbb{R}$, are given as -1 for negative or 1 for positive x, respectively. The exact solution is a rarefaction wave with values $\frac{x}{t}$ for $-t \leq x \leq t$, $0 < t < \infty$ and value -1 left or 1 right of the wave, respectively. Evidently Theorem 3 is not applicable with respect to both the infinities. If beginning with the first time step, the ROTHE solutions $u(x)$, $x \in \mathbb{R}$, are further asked to be continuous, monotonously increasing, and skew-symmetric with respect to the origin, then such solutions can be constructed successively by means of a fixed point method. Uniqueness is now assured by Theorem 2. As a secondary result it turns out that all the ROTHE solutions (contrarily to their squares) are not from each side differentiable at the origin.

Example 4 (MUNZ [9]). If $u_0(x)$, $x \in \mathbb{R}$, has the value 2 for negative or -1 for positive x, respectively, the exact solution is obtained as a shock wave at $x = \frac{1}{2}t$ with value 2 left or -1 right of the shock, respectively. For the piecewiese continuous ROTHE solution $u(x)$, $x \in \mathbb{R}$, beginning with the first time step, the further supposition is made that the square $[u(x)]^2$, $x \in \mathbb{R}$, remains continuous when passing through a discontinuity; in such a way there is made use of the conservation property governing the ROTHE differential equation (3). In particular, with a well-defined discontinuity $x^* \in (0,\infty)$, the ROTHE solution $u(x)$, $x \in \mathbb{R}$ is obtained with constant value 2 for $x \in (-\infty,0)$, as a monotonously decreasing solution of the differential equation (3) for $x \in [0,x^*]$ satisfying the initial condition $u(0) = 2$ and the free boundary condition

$u(x^*) = 1$, and with constant value -1 for $x \in (x^*, \infty)$. Here Theorem 3 leads to local uniqueness for the left interval $(-\infty, x^*)$ as well as for the right one (x^*, ∞); furthermore by means of Theorem 3, this ROTHE solution turns out to be the only one with exactly one discontinuity whilst a continuous solution does not exist. With regard to the complete ROTHE method, the discontinuities form a monotonously increasing sequence.

For the examples mentioned before numerical computations have been done by standard methods, where the results have shown a high accuracy in comparison with the exact solutions [7,8,9]. Recently for such nonlinear hyperbolic problems the L_1-convergence of the ROTHE method with respect to any compactum in the upper (x,t)-plane has been proved [9]. The pointwise convergence, however, remains still as an open question.

References

[1] Gerdes, W.; Martensen, E.: Das Rotheverfahren für die räumlich eindimensionale Wellengleichung. ZAMM 58 (1978) T367-T368

[2] Halter, E.: Das Rotheverfahren für das Anfangs-Randwertproblem der Wellengleichung im Außenraum. Dissertation, Karlsruhe 1979

[3] Halter, E.: The convergence of the horizontal line method for the continuity equation with discontinuous data. ZAMP 35 (1984) 715-722

[4] Martensen, E.: The convergence of the horizontal line method for Maxwell's equations. Math. Methods Appl. Sci. 1 (1979) 101-113

[5] Martensen, E.: The Rothe method for the wave equation in several space dimensions. Proc. Roy. Soc. Edinburgh 84A (1979) 1-18

[6] Martensen, E.: The Rothe method for the vibrating string containing contact discontinuities. Meth. Verf. math. Phys. 26 (1983) 47-67

[7] Martensen, E.: Approximation of a rarefaction wave by discretization in time. Applications of Mathematics in Technology, V. Boffi and H. Neunzert eds. Stuttgart: Teubner 1984, 195-211

[8] Munz, C.-D.: Über die Gewinnung physikalisch relevanter Stoßwellenlösungen mit dem Rotheverfahren. Dissertation, Karlsruhe 1983

[9] Munz, C.-D.: Approximate solution of the Riemann problem for the Burgers equation by the transversal method of lines. To appear in ZAMP

[10] Rektorys, K.: The Method of Discretization in Time and Partial Differential Equations. Dordrecht/Boston/London: Reidel Publishing Company 1982

[11] Rothe, E.: Zweidimensionale parabolische Randwertaufgeben als Grenzfall eindimensionaler Randwertaufgaben. Math.Ann. 102 (1930) 650-670