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ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR SEMILINEAR PARABOLIC PROBLEMS WITH NONSMOOTH DATA

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We shall survey some recent work on the numerical solution of the semilinear initial boundary value problem

$$(1) \qquad u_t - \Delta u = f(u) \quad \text{in } \Omega\times I, \quad I=(0,t^*],$$
$$u=0 \quad \text{on } \partial\Omega\times I,$$
$$u(0)=v \quad \text{in } \Omega,$$

where Ω a bounded domain in R^d with a sufficiently smooth boundary $\partial\Omega$, and f is a smooth function on R for which we assume for simplicity that f and f' are bounded. Such an assumption is normally reasonable only if the solution of (1) is known a priori to be bounded, $|u|\le B$, say, but if this is the case f may be modified if necessary for $|u|>B$ to satisfy our assumption, without changing the solution of (1).

For spatial discretization of (1), let $S_h\subset H_0^1 = H_0^1(\Omega)$ be a family of finite-dimensional spaces parametrized by a small positive parameter h and let the semidiscrete solution $u_h:\bar{I}\to S_h$ be defined by

$$(2) \qquad (u_{h,t},\chi)+(\nabla u_h,\nabla\chi) = (f(u_h),\chi), \quad \text{for } \chi\in S_h, \ t\in I,$$
$$u_h(0)=v_h \in S_h,$$

where $(.,.)$ is the standard inner product in $L_2(\Omega)$.

In order to discuss the error in (2) we assume that S_h is such that the corresponding linear elliptic problem admits an $O(h^r)$ error estimate in $L_2=L_2(\Omega)$. More precisely, we assume that the elliptic projection P_1, i.e. the orthogonal projection onto S_h with respect to the Dirichlet inner product $(\nabla v, \nabla w)$, satisfies, for some $r\geq 2$ and some constant M,

(3) $$\|P_1 v-v\| \leq Mh^r\|v\|_{H^r}, \qquad \text{for } v\in H_0^1\cap H^r,$$

where $\|.\|$ denotes the norm in L_2. It is then well known that if u is sufficiently smooth on the closed interval $\bar{I}$, and if the discrete initial data v_h are suitably chosen, then

$$\|u_h(t)-u(t)\| \leq C(u,M)h^r, \qquad \text{for } t\in\bar{I}.$$

To guarantee that u is smooth enough for this result, both smoothness of v and compatiblity conditions between v and the differential equation at $\partial\Omega$ for $t=0$ are necessary. For instance, in the linear homogeneous case ($f\equiv 0$ in (1)) it was shown in Bramble, Schatz, Thomée and Wahlbin [3] that

$$\|u_h(t) - u(t)\| \leq Ch^r\|v\|_{H^r} \qquad \text{for } v\in D((-\Delta)^{r/2}),\ t\in I,$$

which thus requires $\Delta^j v|_{\partial\Omega}=0$ for $j<r/2$. Such requirements are not always satisfied in practice and it is therefore of interest to analyze the error for nonsmooth or incompatible data. Note that the solution of (1) will always be smooth for positive time. For the linear homogeneous equation this may be expressed by saying that the Laplacian generates an analytic semigroup $E(t)=\exp(\Delta t)$ and that $u(t)=E(t)v$ satisfies

(4) $$\|E(t)v\|_{\dot{H}^\beta} \leq Ct^{-(\beta-\alpha)/2}\|v\|_{\dot{H}^\alpha} \qquad \text{where } \|v\|_{\dot{H}^\alpha}=\|(-\Delta)^{\alpha/2}v\|.$$

For the linear homogeneous equation the nonsmooth data situation has been investigated in Blair [2], Helfrich [5], Bramble, Schatz, Thomée and Wahlbin [3] and later papers (cf. Thomée [7]). In this case, it may be shown using the smoothness property (4) that if v_h is chosen as $P_0 v$, the L_2 projection of v onto S_h. then

(5) $$\|u_h(t)-u(t)\| \leq Ch^{\alpha+\sigma}t^{-\sigma/2}\|v\|_{\dot{H}^\alpha}, \qquad \text{for } 0\leq\alpha\leq\alpha+\sigma\leq r.$$

In particular, optimal order convergence is attained for t positive even if v is only in L_2. A similar result showing $O(h^r)$ convergence for positive time without initial regularity is known also for the linear inhomogeneous problem, cf. Thomée [7].

In the semilinear situation the following result has been proved in Johnson, Larsson, Thomée and Wahlbin [6].

Theorem 1. Let u be a solution of (1) with $\|v\|\le\rho$. Assume further that (3) is satisfied (with $r\ge2$) and let u_h be the solution of (2) with $v_h=P_0v$. Then there exists a constant $C=C(\rho,M)$ such that

$$\|u_h(t)-u(t)\| \le Ch^2(t^{-1}+|\log(h^2/t)|), \quad \text{for } t\in\bar{I}.$$

The above result thus shows that for r=2 the error in the semilinear case is essentially of the same order as for the linear homogeneous equation. For $r>2$, however, the result of Theorem 1 is weaker than the case $\alpha=0$ of (5). The reason why the above argument fails to yield higher order convergence than second is related to the lack of integrability of the right hand side of (5) for $\sigma>2$, $\alpha=0$. In spite of this, it may be shown that an analogue of (5) holds, in the sense that the convergence rate in L_2 at positive time is almost two powers of h higher than the order of regularity of the initial data (up to the optimal order $O(h^r)$).

It may be shown that Theorem 1 is, in fact, essentially sharp in the sense that an estimate of the form

$$(6) \qquad \|u_h(t_0)-u(t_0)\| \le C(\rho,M,t_0)h^\sigma, \qquad |u(x,t)| \le B,$$

cannot hold for any $\sigma>2$ and $t_0>0$, regardless of the value of r. (Note that the requirement that u is bounded is more stringent than boundedness of $\|v\|$.) We shall sketch an example to indicate this.

Consider thus the problem

$$(7) \qquad \begin{aligned} u_t - u_{xx} &= f(u) \qquad \text{for } x\in J=[0,1],\ t\in I,\\ u(0,t) &= u(1,t) = 0,\\ u(x,0) &= v(x), \end{aligned}$$

where $f(u) = u^2$ for $|u| \le B$. Let $h=1/N$, $x_j=jh$, $J_n=(x_n,x_{n+1})$ and consider the semidiscrete analogue using the finite dimensional space

$$S_h = \{\chi\in C(J);\ \chi|_{J_n} \in \Pi_{r-1} \text{ for } n=0,\dots,N-1;\ \chi(0)=\chi(1)=0\}.$$

For the initial values we choose

$$v(x) = v_N(x) = \Psi(Nx),$$

where φ is a not identically vanishing function of the form

$$\varphi(x) = \Sigma_{j=1}^{r+1} \varphi_j \sin \pi j x,$$

which is orthogonal to Π_{r-1} on J. Note that v_N is then orthogonal to S_h, and also that, independently of N,

$$\|v_N\|_{L_\infty} \leq \Sigma_1^{r+1} |\varphi_j| = \rho,$$

where ρ may be chosen smaller than B. The exact solution of (7) is then also smaller than B in modulus on $\bar{I}=[0,t^*]$ with t^* suitably small, independently of N. Since $v_h=v_{N,h}=P_0v_N=0$, by the construction of v_N, we have $u_h(t)\equiv 0$ on I and hence $e(t)=u_h(t)-u(t)=-u(t)$. Using comparison theorems and some Fourier series arguments, one may show for $u=u_N$ that

$$\|e(t)\| = \|u_N(t)\| \geq C/N^2 = Ch^2.$$

Hence an inequality such as (6) is not possible for $\sigma>2$. We may think of this as an example of nonlinear interaction of Fourier modes.

We shall now briefly consider the discretization of equations such as (1) and (2) with respect to the time variable. Consider thus a semilinear problem of the form

(8) $$du/dt + Au = f(u) \qquad \text{for } t\in I,$$
$$u(0) = v,$$

where A is a positive definite selfadjoint linear but not necessarily bounded operator in a Hilbert space H, and where f is bounded together with its Fréchet derivative.

For the approximate solution of (8) we introduce a time step k and let $U_n \in H$ be the approximation of $u(t_n)$, $t_n=nk$, defined by a scheme of the form

(9) $$U_{n+1} = E_k U_n + kF(k,U_n), \qquad n=0,1,2,...$$
$$U_0 = v.$$

Here $E_k=r(kA)$ where $r(\lambda)$ is a rational function which is such that for some $p\geq 1$,

(10) $$r(\lambda) = e^{-\lambda} + O(\lambda^{p+1}) \quad \text{as } \lambda\to 0,$$

and such that

(11) $$|r(\lambda)| < 1 \qquad \text{for } \lambda \geq 0.$$

Further $F(k,\phi)$ is such that (9) is consistent with (8). More precisely, assume for small k, with $\|.\|$ the norm in H,

(12) $$\|F(k,\phi) - F(k,\varphi)\| \le C\|\phi - \varphi\|$$

and

(13) $$\|A^{-1}(F(k,\phi) - f(\phi))\| \le Ck(\|A\phi\| + 1) \qquad \text{for } \phi\in D(A).$$

A simple example is provided by the linearized backward Euler method,

$$(U_{n+1}-U_n)/k + AU_{n+1} = f(U_n),$$

which is of this form with $r(\lambda) = 1/(1+\lambda)$ and $F(k,\varphi)=E_k f(\varphi)$ and which satisfies (10) with $p=1$, as well as (11), (12) and (13).

We first recall a nonsmooth data error estimate by Baker, Bramble and Thomée [1] (see also [7]) for the linear homogeneous equation, $f\equiv 0$ in (8) and the corresponding discrete scheme (9) with $F(k,\varphi)=0$:

$$\|U_n - u(t_n)\| \le Ck^p t_n^{-p}\|v\| \qquad \text{for } v\in H,\ t_n\in I.$$

This result may be combined with the corresponding result for discretization in space of (1) to yield error bounds for totally discrete schemes of order $O(h^r+k^p)$ for t positive without smoothness assumptions on the initial data.

In the semilinear situation we have the following nonsmooth data error estimate by Crouzeix and Thomée [4].

Theorem 2. Under our present assumptions we have

$$\|U_n - u(t_n)\| \le C(\rho)k\{t_n^{-1}\log(t_{n+1}/k) + (\log(t_{n+1}/k))^2\} \text{ for } \|v\|\le\rho.$$

This result may again be combined with Theorem 1 concerning discretization in space to show an essentially $O(h^2+k)$ convergence result for the complete discretization of (1), without any other requirements for the initial data than $v\in L_2(\Omega)$.

In the same way as for the semidiscrete equation, the nonlinearity limits the order of convergence possible in the case of non-smooth data. Thus, in particular, one may show by an example that for a Runge-Kutta type method of order of accuracy $p>1$ and if $s>1$ then it is not possible to show

$$\|U_n - u(t_n)\| \le C(\rho)k^s \qquad \text{for } \|v\|\le\rho,\quad t_n=t>0.$$

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