METHOD OF ROTHE IN EVOLUTION EQUATIONS

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The aim of this paper is to present Rothe's method (also called method of lines , or the method of semidiscretization) as an efficient theoretical tool for solving a broad scale of evolution problems . Using time discretization , evolution problems are approximated by corresponding elliptic problems by means of which an approximate solution for the original evolution problem is constructed . By a relatively simple technique convergence of the approximate solution to the solution of the original evolution problem is proved . Thus , unlike some abstract methods for the analysis of existence and uniqueness problems for evolution equations , Rothe's method has a strong numerical aspect . At the same time it gives a first good insight into the structure of the solution of the investigated evolution problems .

Rothe's method introduced by E.Rothe in 1930 has been used and developed by many authors , e.g. O.A. Ladyženskaja ; T.D. Ventcel' ; A.M. Il-jin , A.S. Kalašnikov , O.A. Olejnik ; Š.J. Ibragimov ; P.P. Mosolov ; K. Rektorys [10] in linear and quasilinear parabolic problems . Nonlinear and abstract parabolic problems has been studied by J. Kačur [2]-[6]; J. Nečas [9] ; A.G. Kartsatos , E.M. Parrott [7],[8] etc. Linear and quasilinear hyperbolic equations has been considered by J. Jerome ; E. Martensen ; M. Pultar ; J. Kačur , etc. A modification of Rothe's method (discretization in x- variable) has been used by V.N. Faddeeva ; W. Walter ; C. Corduneanu , etc. Time and space discretization for solving evolution problems has been employed by many authors ,e.g. R. Glowinski , J.L. Lions , R. Trémoliéres [1] ; M. Zlámal [11] ; A. Ženíšek [12]; H.W. Alt , S. Luckhaus etc. , using similar technique to that of Rothe's method . For the more complete references we refer the reader to [6] .

Efficiency of Rothe's method we demonstrate in solving :

I. A nonlinear parabolic problems ;
II. Variational inequalities ;
III. Higher order equations .

I. A nonlinear parabolic problems.

Let V be a reflexive B-space with its dual V^* and let H be a Hilbert space. Let $||.||$, $|.|$ be the norms in V, H, respectively. We assume that $V \cap H$ is a dense set in V and H. By $\langle f,v \rangle$ we denote the duality for $f \in V^*$ and $v \in V$. Scalar product in H we denote by $(.,.)$. Let S_t be the interval $[-q,t]$ for $t \in [0,T] \equiv I$, $q \geq 0$.

An operator $F : L_\infty(S_T,H) \to L_\infty(S_T,H)$ is a Volterra operator iff $u(s) = v(s)$ a.e. in S_t implies $F(u)(t) = F(v)(t)$ for any $t \in S_T$. We assume $A : V \to V^*$ to be a coercive maximal monotone operator. Consider the equation

$$(1.1) \quad \frac{du(t)}{dt} + Au(t) = f(t,F(u)(t)) \quad \text{a.e. } t \in I, \ u = \phi \text{ in } S_0$$

where $\phi : S_0 \to H$ is a given Lipschitz continuous function ($\phi \in Lip(S_0 \to H)$) and $f \in Lip(I \times H \to H)$. Coerciveness of A we assume in the form

$$(1.2) \quad \langle Au,u \rangle \geq ||u||p(||u||) - C_1|u|^2 - C_2 \quad \forall u \in V$$

where $p : R_+ \to R_+$ and $p(s) \to \infty$ for $s \to \infty$. Lipschitz continuity of f is expressed in the form

$$(1.3) \quad |f(t,v) - f(t',v')| \leq C(|t-t'| + |t-t'||v| + |v-v'|) \ \forall t,t' \in I,$$

$\forall v,v' \in H$. We assume that F maps $Lip(S_T \to H)$ into $Lip(S_T \to H)$ and

$$(1.4) \quad ||F(u) - F(v)||_{C(S_T,H)} \leq C||u-v||_{C(S_T,H)}$$

$$(1.5) \quad |F(u)(t) - F(u)(t')| \leq |t-t'|L(||u||_{C(S_T,H)})(1 + ||\frac{du}{dt}||_{L_\infty(S_t,H)})$$

$\forall \ t,t' \in S_T$, $t' < t$ and $u \in Lip(S_T \to H)$ where $L:R_+ \to R_+$ is cont.f.

Solving (1.1) we apply Rothe's method in the following way: Let n be a positive integer, $h = Tn^{-1}$, $t_i = ih$. Successively for $i=1,\dots,n$ we look for the solution $u_i \equiv u_{i,n} \in V \cap H$ of the elliptic equation

$$(1.6) \quad (\frac{u-u_{i-1}}{h},v) + \langle Au,v \rangle = (f(t_i,F(\tilde{u}_{i-1})(t_i)),v) \quad \forall v \in V \cap H$$

where $u_0 = \phi(0)$ and $\tilde{u}_{i-1} \in Lip(S_T \to H)$ is defined by

$$\tilde{u}_{i-1} \equiv \tilde{u}_{i-1,n} = \begin{cases} \phi & \text{on } S_0 \\ \phi(0) & \text{on } [0,h] \\ u_{j-1} + (t-t_{j-1})h^{-1}(u_j - u_{j-1}), & t_j \leq t \leq t_{j+1} \end{cases}$$

$$\begin{cases} & \text{for } j=1,\dots,i-1 \\ u_{i-1} & \text{on } [t_i, T] \end{cases} .$$

The existence of $u_i \in V \cap H$ is assured bv the following argument. The element $u_{i-1}h^{-1} + f(t_i, F(\tilde{u}_{i-1})(t_i))$ is in H and the operator $A_h : V \cap H \to V^* + H$ defined bv $[A_h u, v] = \frac{1}{h}(u,v) + \langle Au, v\rangle$ is a coercive maximal monotone . Hence theory of monotone operators guarantee the existence of u_i . Uniqueness of u_i is a consequence of strict monotonicitv of A_h . By means of $u_i \equiv u_{i,n}$ we construct Rothe's function $u_n(t)$ and the corresponding step function $\bar{u}_n(t)$

$$(1.7) \quad u_n(t) = u_{i-1} + (t-t_{i-1})h^{-1}(u_i - u_{i-1}), \quad t_{i-1} \le t \le t_i \ , \ i=1,\dots,n$$

$$(1.8) \quad \bar{u}_n(t) = u_i \quad \text{for } t_{i-1} < t \le t_i \ , \ i=1,\dots,n \ , \quad \bar{u}_n(0) = u_o \ .$$

Then (1.6) can be rewritten in the form

$$(1.9) \quad \left(\frac{du_n(t)}{dt}, v\right) + \langle A\bar{u}_n(t), v\rangle = (f_n(t, F(\tilde{u}_{n-1})(t)), v) \quad \forall\, v \in V \cap H$$

where $f_n(t,v) = f(t_i, v)$ for $t_{i-1} < t \le t_i$, $i=1,\dots,n$. First , we prove a priori estimates for $\{u_n\}$ (see Lemmas 1,2,3) and then we take the limit as $n \to \infty$ in (1.9) . We obtain

Theorem 1. Let $A : V \to V^*$ be maximal monotone and let $A\phi(0) \in H$. If (1.2) - (1.5) are satisfied then there exists the unique solution u of (1.1) in the following sense : $u \in L_\infty(I,V)$, $u \in \mathrm{Lip}(I \to H)$, $\frac{du}{dt} \in L_\infty(I,H)$ and $Au \in L_\infty(I,H)$. Moreover , the estimate

$$\| u_n - u \|^2_{C(I,H)} \le \frac{C}{n}$$

takes place where $\{u_n\}$ is from (1.7).

A priori estimates we obtain in the following way.

Lemma 1. The estimate $|u_i| \le C$ takes place for all n , $i=1,\dots,n$.

Proof. We put $u=u_i$, $v = hu_i$ into (1.6) . We sum it up for $i=1,\dots,j$. Using (1.2) -(1.4) we estimate

$$|u_j|^2 \le C_1 + C_2 \sum_{i=1}^{j} \max_{1 \le k \le i} |u_k|^2 h$$

and hence

$$\max_{1\le k\le j} |u_k|^2 \le C_1 + C_2 \sum_{i=1}^{j} \max_{1\le k\le i} |u_k|^2 h \quad .$$

Thus Gronwall's Lemma implies the required result.

Lemma 2. The estimates

(1.10) $$\left|\frac{u_i-u_{i-1}}{h}\right| \le C \;, \quad ||u_i|| \le C$$

hold for all n, $i=1,\dots,n$.

Proof. We subtract (1.6) for $u=u_j$, $v=\delta_h u_j \equiv \frac{u_j-u_{j-1}}{h}$ from (1.6) for $u=u_{j-1}$, $v=\delta_h u_j$. Owing to the monotonicity of A and (1.3) -(1.5) we obtain

(1.11) $$|\delta_h u_j| \le |\delta_h u_{j-1}| + C(h + \max_{1\le k\le j} |\delta_h u_k| h)$$

because of Lemma 1. From (1.6) for $i=1$, $u=u_1$, $v=\delta_h u_1$ we conclude

$$|\delta_h u_1| \le C$$

since $u_0=\phi(0)$ and $Au_0 \in H$. Thus successively from (1.11) we obtain

$$|\delta_h u_j| \le C_1 + C_2 \sum_{i=1}^{j} \max_{1\le k\le i} |\delta_h u_k| h$$

which similarly as above implies $|\delta_h u_i| \le C$. Using this estimate and (1.10) in (1.6) for $u=u_i$, $v=u_i$ we obtain $||u_i|| \le C$ because of (1.2).

As a consequence of (1.10) and (1.6) for $u=u_i$ we have

$$|\langle Au_i,v\rangle| \le C|v| \quad \forall\, n \;,\; i=1,\dots,n \;.$$

The previous a priori estimates can be rewritten in the form

(1.12) $$\left|\frac{du_n(t)}{dt}\right| \le C \;, \quad ||u_n(t)||_{V\cap H} \le C \;, \quad |\langle A\bar{u}_n(t),v\rangle| \le C|v|$$

(1.13) $$|u_n(t) - u_n(t')| \le C|t - t'| , \quad |u_n(t) - \bar{u}_n(t)| \le \frac{C}{n} \quad .$$

Lemma 3. There exists an $u \in L_\infty(I,V)$, $u \in \mathrm{Lip}(I \to H)$ with $\frac{du}{dt} \in L_\infty(I,H)$ such that

$$||u_n-u||^2_{C(I,H)} \le \frac{C}{n} \,, \quad \frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L_2(I,H) \text{ and } A\bar{u}_n(t) \rightharpoonup Au(t) \text{ in } V^*$$

(also in H) $\forall\, t \in I$.

Proof. Subtract (1.9) for $n=r$ from (1.9) for $n=s$ where $v = \bar{u}_r(t) - \bar{u}_s(t)$. Using (1.12) and (1.13) we estimate

$$(1.14)\quad \frac{d}{dt}|u_r-u_s|^2 \le C\left(\frac{1}{r} + \frac{1}{s} + ||\tilde{u}_{r-1}-u_r||^2_{C(S_t,H)} + ||\tilde{u}_{s-1}-u_s||^2_{C(S_t,H)} + ||u_r-u_s||^2_{C(S_t,H)}\right).$$

Integrating (1.14) over $(0,t)$ and taking into account the estimate

$$||\tilde{u}_{r-1} - u_r||^2_{C(S_t,H)} \le \frac{C}{r^2} \sup_{\tau\in[0,t]} \left|\frac{du_r(\tau)}{d\tau}\right|^2 \le \frac{C}{r^2}$$

we conclude $u_n \to u$ in $C(I,H)$ and the estimate $||u_n-u||^2_{C(I,H)} \le \frac{C}{n}$.
Hence and from (1.12) , (1.13) we obtain

$\bar{u}_n(t) \rightharpoonup u(t)$, $||A\bar{u}_n(t)||_* \le C$, $(|A\bar{u}_n(t)| \le C)$ and

$$\langle A\bar{u}_n(t),\ \bar{u}_n(t) - u(t)\rangle \to 0 \qquad \forall\ t\in I .$$

Hence maximal monotonicity of A implies $A\bar{u}_n(t) \rightharpoonup Au(t)$ in V^* (moreover in H).

Proof of Theorem 1. We integrate (1.9) over (τ_1,τ_2) and take the limit as $n \to \infty$. Owing to Lemma 3 we conclude that u is a solution of (1.1) since $\tau_1,\tau_2 \in I$ are arbitrary . Uniqueness follows from (1.1) by standard arguments .

Remark 1. Theorem 1 holds true also when $A : V \to V^*$ is nonstationary under the following assumptions :

$A(t) : V \to V^*$ is maximal monotone $\forall\ t\in I$;

$A(t)u = \nabla\Phi(t,u)$, i.e. $A(t)$ are potential $(\Phi\colon I\times V \to R)$

$\langle A(t)u,u\rangle \ge ||u||p(||u||)$, $p(t) \to \infty$ for $t \to \infty$

$$||\frac{d}{dt} A(t)u||_* + ||\frac{d^2}{dt^2} A(t)u||_* \le C_1 + C_2 p(||u||) .$$

For the proof it suffices to combine the techniques used in the proof of Theorem 1 with those used in [3] .

Remark 2. A modification of Theorem 1 with m-accretive operators $A(t) : D \subset V \to V^*$ $(t\in I)$ satisfying

$$||A(t)v - A(t')v|| \le |t-t'|L(||u||)(1 + ||A(t)v||)$$

and with the right hand side $f = G(t,u_t)$ (at fixed t the operator G transforms the values of $u_t(s) = u(t + s)$, s $[-q,0]$ into H) satisfying Lipschitz like condition has been obtained by A.G. Kartsatos and M.E. Parrott in [7] .

II. Variational inequalities .

Let Φ be a proper ($\Phi: V \to (-\infty,\infty]$, $\Phi \not\equiv \infty$), convex and lower semicontinuous (l.s.c.) function on V . We assume $A : V \to V^*$ to be a bounded maximal monotone operator . Consider the variational inequality

$$(2.1) \qquad \left(\frac{du(t)}{dt}, v-u(t)\right) + \langle Au(t), v-u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq$$

$$(f(t,F(u)(t)), v-u(t)) \quad \forall\, v \in V\cap H \text{ , a.e. } t \in I \text{ , } u = \phi \text{ on } S_o$$

where ϕ and F are the same as in Section I. We use the approximation scheme

$$(2.2) \qquad (\delta_h u_i, v-u_i) + \langle Au_i, v-u_i\rangle + \Phi(v) - \Phi(u_i) \geq (f(t_i,F(\tilde{u}_{i-1})(t_i)), v-u_i) \text{ ,} \forall\, v \in V\cap H$$

where $\tilde{u}_{i-1}$ is the same as in Section I . It is elliptic variational inequality with respect to u_i provided $u_1,\dots,u_{i-1}$ are known . Coerciveness of A is assumed in the form : There exists $v_o \in V$ with $\Phi(v_o) < \infty$ such that

$$(2.3) \qquad (\langle Au, u-v_o\rangle + \Phi(u)).||u||^{-1} \to \infty \quad \text{for} \quad ||u|| \to \infty \text{ .}$$

Then , existence of $u_i \in V\cap H$ satisfying (2.2) is guaranteed by the well-known results from elliptic variational inequalities . Here , instead of $Au_o \in H$ we assume : There exists $z_o \in H$ such that

$$(2.4) \quad (z_o, v-u_o) + \langle Au_o, v-u_o\rangle + \Phi(v) - \Phi(u_o) \geq (f(0,F(\tilde{u}_o)(0)), v-u_o)$$

$$\forall\, v \in V\cap H \quad \text{where} \quad u_o = \phi(0)$$

Since we have less possibilities in the test function v than in Section I , we assume

$$(2.5) \quad \begin{cases} \text{either} \quad \Phi(0) < \infty \quad \text{, or} \\ |F(u)(t) - F(u)(t')| \leq C|t-t'| \; ||\frac{du}{dt}||_{L_\infty(S_t,H)} \end{cases}$$

for $u \in Lip(S_T \to H)$. Let us put $i=j$, $v=u_{j-1}$ into (2.2) and then $i=j-1$, $v=u_j$. Adding these inequalities the values with Φ are eliminated and we are in the same situation as in the case of equations . Thus , we obtain the same a priori estimates (except of $|\langle Au_i,v\rangle| \leq C|v|$) as in Section I . Since $\bar{u}_n(t) \rightharpoonup u(t)$ we have $\Phi(u(t)) \leq \liminf \Phi(\bar{u}_n(t))$ (Φ is also weakly l.s.c. on V). From this information and from

$$(2.6) \quad \langle A\bar{u}_n(t), \bar{u}_n(t)-v\rangle \leq \Phi(v) - \Phi(\bar{u}_n(t)) + \left(\frac{du_n(t)}{dt}, v-\bar{u}_n(t)\right) - (f(t,F(\tilde{u}_{n-1})(t)), v-\bar{u}_n(t))$$

for $v=u(t)$ we conclude $\limsup \langle A\bar{u}_n(t),\bar{u}_n(t)-u(t)\rangle \le 0$. Since A is a pseudomonotone operator we obtain

$$\langle Au(t),u(t)-v\rangle \le \liminf \langle A\bar{u}_n(t),\bar{u}_n(t)-v\rangle \quad \forall\, v \in V .$$

Then integrating (2.6) and taking $\liminf_{n\to\infty}$ in (2.6) we obtain the solution of (2.1) by the same arguments as in Section I .

Theorem 2. Let $A : V \to V^*$ be a bounded maximal monotone operator . If (1.3) , (1.4), (2.3)-(2.5) are satisfied then there exists the unique solution of (2.1) with the same properties as in Theorem 1 .

Remark 3. Theorem 2 holds true also in the case of the operator A being nonstationary under the assumptions of Remark 1 .

Similarly , the following types of evolution variational inequalities can be solved

a) $$\left(\frac{d^2u(t)}{dt^2},v-\frac{du(t)}{dt}\right) + b\left(t;\frac{du(t)}{dt},v-\frac{du(t)}{dt}\right) + a\left(t;u(t),v-\frac{du(t)}{dt}\right) +$$

$$\Phi(v) - \Phi\left(\frac{du(t)}{dt}\right) \ge \left(f(t),v-\frac{du(t)}{dt}\right) \qquad u(0) = U_o\ ,\ \frac{du(0)}{dt} = U_1\ ;$$

b) $$\left(\frac{du(t)}{dt},v-\frac{du(t)}{dt}\right) + a\left(t;u(t),v-\frac{du(t)}{dt}\right) + \Phi(v) - \Phi\left(\frac{du(t)}{dt}\right) \ge$$

$$\left(f(t),v-\frac{du(t)}{dt}\right) \qquad u(0) = U_o\ ;$$

c) $$a\left(t;u(t),v-\frac{du(t)}{dt}\right) + \Phi(v) - \Phi\left(\frac{du(t)}{dt}\right) \ge \left(f(t),v-\frac{du(t)}{dt}\right)\ , u(0)=U_o$$

$$\forall\ v \in V\cap H\ ,\ \text{a.e. } t \in I\ .$$

Here V , H are Hilbert spaces and $b(t;u,v)$, $a(t;u,v)$ are continuous bilinear forms in $u,v \in V$ ($t \in I$) . We use the approximation scheme

a_1) $$\frac{1}{h}(\delta_h u_i-\delta_h u_{i-1},v-\delta_h u_i) + b(t_i;\delta_h u_i,v-\delta_h u_i) + a(t_i;u_i,v-\delta_h u_i) +$$

$$\Phi(v) - \Phi(\delta_h u_i) \ge (f(t_i),v-\delta_h u_i)$$

b_1) $$(\delta_h u_i,v-\delta_h u_i) + a(t_i;u_i,v-\delta_h u_i) + \Phi(v) - \Phi(\delta_h u_i) \ge (f(t_i),v-\delta_h u_i)$$

c_1) $$a(t_i;u_i,v-\delta_h u_i) + \Phi(v) - \Phi(\delta_h u_i) \ge (f(t_i),v-\delta_h u_i)$$

$\forall\ v \in V\cap H$, $i=1,\dots,n$. If we express $u_i = u_o + \delta_h u_i h + \sum_{j=1}^{i-1} \delta_h u_i h$ and $a(t_i;u_i,v) = ha(t_i;\delta_h u_i,v) + \sum_{j=1}^{i-1} ha(t_i;\delta_h u_j,v) + a(t_i;u_o,v)$ in a_1), b_1), c_1) then we obtain the elliptic variational inequalities with respect to $\delta_h u_i$. We shall assume

(2.7) $a(t;u,v) = a(t;v,u)$;

(2.8) $a(t;u,u) + \alpha|u|^2 \geq C||u||^2$ $(\alpha \geq 0)$;

(2.9) $b(t;u,u) \geq -C|u|^2$;

(2.10) $|\frac{d^p}{dt^p} a(t;u,v)| \leq C||u|| \, ||v||$ (p=1,2 in the cases a),b) p=1 in the case c)) ;

(2.11) $|\frac{d}{dt} b(t;u,v)| \leq C||u|| \, ||v||$

We assume that there exist $s_o \in H$, $z_o \in H$ such that

$(2.12)_a$ $(s_o, v-U_1) + b(0;U_1,v-U_1) + a(0;U_o,v-U_1) + \Phi(v) - \Phi(U_1) \geq (f(0),v-U_1)$;

$(2.12)_b$ $(z_o,v-z_o) + a(0;v-z_o) + \Phi(v) - \Phi(z_o) \geq (f(0),v-z_o)$;

$(2.12)_c$ $a(0;U_o,v-z_o) + \Phi(v) - \Phi(z_o) \geq (f(0),v-U_o)$

$\forall v \in V \cap H$ .By the same way as above (see also [5]) we obtain

Theorem 3. If (2.7) - (2.12) are satisfied and if $f, \frac{df}{dt}, \frac{d^2f}{dt^2} \in L_2(I,V^*)$ (or $f, \frac{df}{dt} \in L_2(I,H)$ in a), b)) then there exists the unique solution of a), b) , c) , respectively , with the following properties :

$$u \in C(I,V) \ , \ \frac{du}{dt} \in L_\infty(I,V) \ , \ ||u_n - u||^2_{C(I,V)} \leq \frac{C}{n}$$

where $\{u_n\}$ is the corresponding sequence of Rothe's function . Moreover , in the cases a) , b) we have

$$\frac{du}{dt} \in C(I,H) \ , \ \frac{d^2u}{dt^2} \in L_\infty(I,H) \ , \ \left|\frac{du_n(t)}{dt} - \frac{du(t)}{dt}\right|^2 \leq \frac{C}{n} \qquad \forall t \in I \ .$$

Remark 4. In the fact in the cases a) , b) a perturbed symmetry of $a(t;u,v)$ can be assumed . Let $a_o(t;u,v)$ be continuous bilinear form in $u,v \in V$ $(t \in I)$ satisfying $|a_o(t;u,v)|$ $C||u|| \, |v|$. It suffices to assume $a(t;u,v) + a_o(t;u,v)$ is symmetric .

Remark 5. In the cases of variational inequalities a) , b) a more general problem (corresponding to problem (2.1)) with a right hand side $f(t,F(u)(t))$ can be considered . If (1.3),(1.4),(2.5) are satisfied then Theorem 3 holds true .

Remark 5. Using time and space discretization the variational inequalities a) , b) have been solved in [1]. A special case of (2.1) (A is asymptotically linear , $\Phi \equiv \Phi_K$ - indicatrix of the closed convex set K in V) have been solved in [12] .

III. Higher oder evolution equations.

In this section we apply Rothe's method to the equations of the form

$$(3.1)\quad G(t)\frac{d^m w(t)}{dt^m} + \sum_{k=0}^{m-1} A_k(t)\frac{d^k w(t)}{dt^k} = g(t,w,\dots,\frac{d^{m-1}w}{dt^{m-1}})$$

$$\frac{d^k w(0)}{dt^k} = W_k\ ,\ k=0,\dots,m-1\ ,\quad \text{where}\quad A_k(t)\in\mathcal{L}(V,V^*),\ G(t)\in\mathcal{L}(H,H)$$

$(\in\mathcal{L}(V,V^*))$, $g\in Lip(I\times[V]^m\to H)$ and V , H being Hilbert spaces with $V\cap H$ dense in V and H . The equations of type (3.1) include the governing equations of quasistatic and dynamic problems of viscoelastic plates and shallow shells (see [13]). We assume that either i) $A_{m-1}(t)$ is V-elliptic , or ii) $A_{m-2}(t)$ is V-elliptic . Operator $G(t)$ is supposed to be symmetric and H-elliptic . Using transformation

$$u = \frac{d^{m-1}w}{dt^{m-1}} \quad \text{in the case i) , or}\quad u = \frac{d^{m-2}w}{dt^{m-2}}\quad (m\ge 2)\ \text{in the case ii)}$$

the equation (3.1) can be reduced to the form

$$E)_i\quad G(t)\frac{du(t)}{dt} + A(t)\,u(t) = f(t,u,F(u)(t))\qquad \text{or}$$

$$E)_{ii}\quad G(t)\frac{d^2u(t)}{dt^2} + B(t)\frac{du(t)}{dt} + A(t)u(t) = f(t,u,\frac{du}{dt},F(u)(t))$$

where

$$(3.2)\quad F(u)(t) = (\int_0^t u\,ds,\dots,\int_0^t (t-s)^{p}\,u(s)\,ds)\quad (p=m-2\ \text{in i)}\ ,\ p=m-3\ \text{in ii)}\ )$$

The problem $E)_i$ has been considered in Section I . Now , we formulate Problem 3.1 which includes the problem $E)_{ii}$.

Let V , V_1 ; H , H_1 be Hilbert spaces and let $\langle u,v\rangle_V$, $\langle x,y\rangle_H$ be the continuous pairings between $u\in V_1$, $v\in V$ and $x\in H_1$, $y\in H$, respectively . Let $a(t;u,v)$, $b(t;u,v)$ be the same as in Section I and let $G(t;u,v)$ be a continuous bilinear form for $u,v\in H$. Consider the operators $f_V\in Lip(I\times V\to V_1)$, $f_H\in Lip(I\times V\times H\to H_1)$ and Volterra type operators F_V: $Lip(S_T\to V)\to Lip(S_T\to V)$, F_H: $Lip(S_T\to H)\to Lip(S_T\to H)$.

<u>Problem 3.1.</u> To find $u\in C(I,V\cap H)$ with $\frac{du}{dt}\in L_\infty(I,V\cap H)$, $\frac{du}{dt}\in C(I,H)$ $\frac{d^2u}{dt^2}\in L_\infty(I,H)$ such that

$$(3.3)\quad G(t;\frac{d^2u(t)}{dt^2},v) + b(t;\frac{du(t)}{dt},v) + a(t;u(t),v) =$$

$$\langle f_V(t,F(u)(t)),v\rangle_V + \langle f_H(t,F_V(u)(t),F_H(\tfrac{du}{dt})(t)),v\rangle_H$$

holds for all $v\in V\cap H$ and $u=\phi$, $\frac{du}{dt}=\psi$ on S_o where $\phi\in Lip(S_o\to V)$, $\psi\in Lip(S_o\to H)$ are given functions .

To solve Problem 3.1 we use the approximation scheme

(3.4) $\frac{1}{h} G(t_i;\delta_h u_i-\delta_h u_{i-1},v) + b(t_i;\delta_h u_i,v) + a(t_i;u_i,v) =$

$\langle f_V(t_i,F(\tilde{u}_{i-1})(t_i)),v\rangle_V + \langle f_H(t_i,F_V(\tilde{u}_{i-1})(t_i),F_H(\delta_h\tilde{u}_{i-1})(t_i)),v\rangle_H$

$\forall\ v\in V\cap H$ where $\tilde{u}_{i-1}$ is the same as in Section I and $\delta_h\tilde{u}_{i-1}$ is constructed by means of ψ, $\delta_h u_1,\ldots,\delta_h u_{i-1}$ by the same way as $\tilde{u}_{i-1}$. Similarly as above (3.4) can be transformed to the elliptic equation with respect to $\delta_h u_i$ provided $\delta_h u_1,\ldots,\delta_h u_{i-1}$ are known .

The solution of Problem 3.1 and the convergence of our approximation scheme we obtain under the following assumptions

(3.5) $G(t;u,v) = G(t;v,u)$;

(3.6) $C_1|u|^2 \le G(t;u,u) \le C_2|u|^2$;

(3.7) $|\frac{d}{dt} G(t;u,v)| \le C\ |u||v|$;

(3.8) $\|F_R(u) - F_R(v)\|_{C(S_T,R)} \le C\|u - v\|_{C(S_T,R)}$ for $R = V$, H ;

(3.9) $\|F_R(u)(t)-F_R(u)(t')\|_R \le |t-t'|L(\|u\|_{C(S_T,R)})(1+\|\frac{du}{dt}\|_{L_\infty(S_t,R)})$

where $L : R_+ \to R_+$ is continuous , $t,t'\in I$, $t'<t$ and $F(u)$ is from (3.2). Analogously to (2.12) we assume : There exists $s_0\in H$ such that

(3.10) $G(0;s_0,v) + b(0;\psi(0),v) + a(0;\phi(0),v) = \langle f_V(0,0),v\rangle_V +$

$\langle f_H(0,F_V(\tilde{\phi})(0),F_H(\tilde{\psi})(0)),v\rangle_H \quad \forall\ v\in V\cap H$.

<u>Theorem 4</u>. Suppose $f_V\in Lip(I\times V\to V_1)$, $f_H(I\times V\times H\to H_1)$ (see (1.3)) and $\psi(0)\in V$. If (3.5) -(3.10) are satisfied then there exists the unique solution of Problem 3.1 . Moreover, the estimates

(3.11) $\|u_n - u\|^2_{C(I,V)} \le \frac{C}{n}$, $\|\frac{du_n}{dt} - \frac{du}{dt}\|^2_{L_\infty(I,H)} \le \frac{C}{n}$

hold where $\{u_n\}$ is the corresponding sequence of Rothe's functions .

By a similar technique used in Sections I and II, successively we obtain a priori estimates

$$|u_i| \le C\ ,\ |\delta_h u_i| \le C$$

and then

$$|\delta_h^2 u_i| \le C\ ,\|\delta_h u_i\| \le C\ ,\ \|u_i\| \le C\ .$$

Similarly as in Lemma 3 a priori estimates (3.11) can be proved .Then taking the limit as $n\to\infty$ in approximation scheme (3.4) we conclude Theorem 4 .

Example . Problem 3.1 can be interpreted in the following way . We put $V = \overset{\circ}{W}{}_2^2(\Omega)$, $V_1 = W_2^{-2}$, $H = L_2(\Omega) = H_1$ where $\Omega \subset R^N$. Consider

$$a(t;u,v) = \sum_{|i|\ |j|\leq 2} \int_\Omega a_{ij}(x,t)\, D^i u\, D^j v\, dx \quad \text{for} \quad u,v \in \overset{\circ}{W}{}_2^2(\Omega) \quad ;$$

$$b(t;u,v) = \sum_{|i|\ |j|\leq 2} \int_\Omega b_{ij}(x,t)\, D^i u\, D^j v\, dx \quad (\text{or} \quad b(t;u,v) = \int_\Omega uv\, dx);$$

$$<f_V(t,F(u)(t)),v>_V = \int_\Omega \Delta v \int_0^t (t-s)^p\, \Delta u(s)\, ds\, dx \quad (p \geq 1)\ ;$$

$$<f_H(t,F_V(u)(t),F_H(\tfrac{du}{dt})(t)),v>_H = \begin{cases} \int_\Omega v\, \Delta u(\omega(t))\, dx & \omega \in \mathrm{Lip}(S_T \to S_T) \\ & \omega(t) \leq t\ ; \\ \int_\Omega v \int_{-q}^{\omega(t)} K(s,t)\, \Delta u(s)\, ds\, dx\ ; & \\ \int_\Omega v \int_{-q}^{\omega(t)} K(s,t)\, \frac{du(s)}{ds}\, ds\, dx\ . & \end{cases}$$

Bilinear form $G(t;u,v)$ can be interpreted in the following way .

1) $H = L_2(\Omega) = H_1$, $G(t;u,v) = \int_\Omega uv\, dx$.

Then the first term in (3.1) is of the form $\frac{d^m w}{dt^m}$;

2) $H = L_{2,\alpha}(\Omega) = \{u\ ; \int_\Omega \alpha u^2\, dx < \infty\}$, $H_1 = L_{2,\frac{1}{\alpha}}(\Omega)$, $<u,v>_H = \int_\Omega uv\, dx$ where $\alpha(x) > 0$, $\alpha \in L_1(\Omega)$. We consider $C_1\alpha(x) \leq g(x,t) \leq C_2\alpha(x)$ Then $G(t;u,v) = \int_\Omega g(x,t)\, uv\, dx$ $(u,v \in L_{2,\alpha}(\Omega))$ generates a degenerate first term in (3.1) in the form $g(x,t) . \frac{d^m w}{dt^m}$;

3) $H = V = \overset{\circ}{W}{}_2^2$, $H_1 = V_1 = W_2^{-2}$. Then $G(t;u,v)$ generates the first term in (3.1) in the form $G(t)\, \frac{d^m w}{dt^m}$ where $G(t) \in \mathcal{L}(V,V^*)$ is a symmetric , V-elliptic operator .

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