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ANALYTICAL AND COMPUTATIONAL PROBLEMS IN HIGHER SPECIAL FUNCTIONS

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Section D

1. What are "higher special functions"?

I use the term "higher special functins" to describe a certain collection of special functions which are distinguished from the more familiar functions in two ways:

(a) The differential equations which they satisfy ar more complicated than the more familiar hypergeometric equation, in that they contain more than three regular singularities: in a word, these functions are not of "hypergeometric type" and the methods by which we can study hypergeometric-type functions - solution in series, expression of solutions as definite or contour integrals, etc., are no longer available, or at least have to be modified.

(b) They arise from separation of variables in the Laplace or Helmholtz equation in more complicated coordinate systems than the familiar certesian, cylindrical or sphercial coordinates, which lead to Legendre and Bessel functions. An important feature of these more complicated coordinate systems is that when we carry out the process of separation of variables, the two separation parameters generally occur in all three of the separated equations, so that the determination of eigenvalues of these parameters becomes a "multi-parameter problem" instead of the standard Sturm-Liouville type.[Ref.2]

As a result of these differences, analytical problems, in constructing expressions for solutions and investigating their properties, are greater by at least an order of magnitude than the corresponding problems for, say, Bessel functions. The computational problems are also far greater: for instance, the ellipsoidal wave equation has been known in mathematical literature since at least 1925, but it was only in 1983 that the first general method for numerical computation of ellipsoidal wave functions was published.

I shall describe briefly two typical problems, each of which arises from the interest of physicists and engeneers in practical problem which can be expressed in terms of these functions.

2. The construction of ellipsoidal wave functions

The ellipsoidal wave equation arises when the Helmholtz equation is separated in ellipsoidal coordinates, so that its solution is needed for problems involving, say, diffraction by an ellipsoidal surface, or diffraction through an elliptic hole, or propagation of sound waves from an elliptic loudspeaker. In an algebraic form, the equation is

$$A(t)y''(t) + \tfrac{1}{2}A'(t)y'(t) + (\lambda + \mu t + \gamma t^2)y(t) = 0, \tag{1}$$

where $A(t) = t(t-1)(t-c), c \in (0,\infty)$, γ is fixed, while λ and μ are separation constants which we are free to choose. In fact, we must choose them so that the solution $y(t)$ may be an entire function of t: in other words, it must be finite at all of the three finite singularities $0,1,c$.

It can be proved that such a solution must be either an even function or an odd function of t: to be definite, let us consider only the even solution. We express the formal solution as a series

$$y = \sum_{r=0}^{\infty} \alpha_r t^{2r} \tag{2}$$

and when we substitute this in the differential equation we obtain, as usual, a recurrence relation between the coefficients, namely:

$$\lambda\alpha_0 + \tfrac{1}{2}c\alpha_1 = 0 , \tag{3}$$

$$\mu\alpha_0 + (\lambda - (1+c))\alpha_1 + 3c\alpha_2 = 0 , \tag{4}$$

$$\gamma\alpha_r + \{\mu + (r+1)(r+\tfrac{3}{2})\}\alpha_{r+1} + \{\lambda - (1+c)(r+2)^2\}\alpha_{r+2} + c(r+\tfrac{5}{2})(r+3)\alpha_{r+3} = 0, \quad r \geq 0. \tag{5}$$

We observe that the general relation involves four successive coefficients, which is in strong contrast to the usual situation for hypergeometric-type functions, where the recurrence relation is only two-term, or even for Mathieu functions, which have a three-term relation. Thus, we are seeking to solve a second-order differential equation by means of a third-order difference equation.

Standard difference equation theory shows that, asymprotically as $r \to \infty$,

$$\frac{\alpha_{r+1}}{\alpha_r} \sim 1 \quad \text{or} \quad \frac{1}{c}\ (<1) \quad \text{or} \quad \frac{-\gamma}{r^2}$$

but in order to ensure that y(t) is an entire function we must have

$$\frac{\alpha_{r+1}}{\alpha_r} \sim \frac{-\gamma}{r^2} \qquad (6)$$

The equation (5) may then be regarded as a linear third-order difference equation with (3), (4), as initial conditions and (6) as an end condition.

This is solved by means of an iterative technique: having fixed c and γ, we choose a moderate value N, take estimated initial values of λ_0 and μ_0, set $\alpha_{N+1} = 1$, $\alpha_{N+2} = \alpha_{N+3} = 0$, and compute back by equation (6) to obtain $\alpha_0, \alpha_1, \alpha_2$. We then use (3) and (4) to obtain new α_0', μ_0'. From these, and the initially chosen λ_0, μ_0 we obtain a second estimate for λ, μ, and repeat the process. Either a two-dimensional Newton method, or a secant method, is suitable for this step. Details are in [1].

This process is satisfactory and has been used to construct the $\lambda(\gamma)$, $\mu(\gamma)$ eigenvalue curves, but it suffers from a major defect, namely, the process only converges if we start with a very good approximation to the true eigenvalue-pair λ, μ. The only methods at present known, are to work along a pair of curves starting either with $\gamma = 0$ or with a large value of γ, using the asymptotic expressions for $\lambda(\gamma)$, $\mu(\gamma)$ which fortunately are known.

A major problem in this field of work is to devise a process which will enable us to estimate eigenvalue-pairs for a given γ: the variational methods which are convenient in the case of one-parameter problems cannot, unfortunately, be used here.

3. The recessive solution of Mathieu's equation

In the course of investigating a problem in linear elasticity, namely that of an infinite strip punch indenting a half space, it was found necessary to make computations with the recessive solution of the modified Mathieu equation. This problem is as follows.
We have the equation

$$y''(t) = Q(t)y(t), \quad Q(t) = \lambda + 2h^2 \cosh 2t, \; \lambda + 2h^2 > 0$$

It can easily be shown that this equation has a unique solution, normalized by the condition $y(0) = 1$, and such that $y(t) \to 0$ as $t \to \infty$: every other solution in such that $|y(t)| \to \infty$ as $t \to \infty$. The problem is to compute the value of $y'(0)$.

Because of the rapid increase of the function Q(t), a forward shooting method is likely to be unstable, so a backwards method was employed. We make first the transformation

$$v(t) = y(t)/y'(t)$$

leading to the first-order equation

$$v'(t) = 1 - Q(t)v^2(t).$$

We take a moderate value t_0, set $v(t_0) = 0$ and integrate numerically back to $t = 0$, giving $v(0)$, and hence $y'(0)$.

The literature of numerical analysis seems to indicate that problems of this kind have not received as much attention as one would expect, in view of the large category of equations of the form

$$y''(t) = Q(t)y(t)$$

which have recessive solutions of this kind. As a check on the method used, it was tested on the well-known Airy equation, for which $Q(t) = t$, and the solution $y(t)$ is a multiple of the Airy function $Ai(t)$. The value of $Ai(0)/A_i'(0)$ is known to be $-3^{5/6}[\Gamma(\frac{2}{3})]^2/2\pi$, numerically -1.371721165. Using the method indicated above, with $t_0 = 20$, and employing a fourth-order Runge-Kutta process with step size 0.1, a small HP 97 calculator was sufficient to give the numerical value -1.371721143.

This example shows that, even in the relatively well-studied area of Mathieu functions, the demands of practicality require us to tackle analytical or numerical problems of surprising difficulty and of a surprisingly fundamental analytical nature. The value of $y'(0)$ to give the unique recessive solution to be no general method of finding it without laborious construction of at least one solution.

In the same area, the behaviour of the recessive solution $y(t)$ with respect to the parameters λ, h^2 is very important: one would like to have a "comparison" theory for such recessive non-oscillatory solutions, comparable to the deep theory we already have for oscillatory equations.

References

[1] F.M. Arscott, P.J. Taylor, R.V.M. Zahar, *Mathematics of Computation*, Vol. 40, Jan. 1983, 367-380.

[2] *"The land beyond Bessel: a survey of higher special functions"*, F.M. Arscott, Springer Lecture Notes Series, No. 819.