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In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [203]--208.

Persistent URL: http://dml.cz/dmlcz/700170

ON THE MOUNTAIN PASS LEMMA

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In this paper, I propose to describe a generalized Mountain Pass Lemma (MPL, in short), which extends the original MPL due to Ambrosetti and Rabinowitz [1] in two aspects:

(a) from a Banach space to a closed convex subset,

(b) from the strong separation condition of values of functions to a weaker one.

Three applications on multiple solutions of variational inequality, semilinear elliptic BVP, and minimal surface are presented.

1. Let $\mathfrak{X}$ be a Banach space. Let C be a closed convex subset of $\mathfrak{X}$. Let Q and S be two closed subsets of C.

We say that the boundary ∂Q and S link w.r.t. C, if

(1) $\quad \partial Q \cap S = \phi,$

(2) $\quad$ for each $\phi : Q \to C$ continuous, satisfying

$$\phi|_{\partial Q} = id|_{\partial Q},$$

we have

$$\phi(Q) \cap S \neq \phi .$$

Suppose that $f : C \to \mathbf{R}^1$ is a restriction of a C^1 function defined on a neighborhood of C. According to the variational inequality theory, we say $x_0 \in C$ a critical point of f w.r.t. C, if

$$\langle f'(x_0), x - x_0 \rangle \geq 0 \qquad \forall x \in C ,$$

where $\langle , \rangle$ is the duality between $\mathfrak{X}^*$ and $\mathfrak{X}$.

For $x^* \in \mathfrak{X}^*$, and $x_1 \in \mathfrak{X}$, let us define

$$\| x^* \|_{x_1} = \operatorname{Sup}\{\langle x^*, x - x_1 \rangle \mid x \in C \text{ with } \|x - x_1\| \leq 1\} .$$

We extend the Palais Smale (P.S. in short) Condition w.r.t. C as following:

For any sequence $\{x_n\} \subset C$, such that $f(x_n)$ is bounded, and $\|-f'(x_n)\|_{x_n} \to 0$ has a convergent subsequence.

THEOREM 1. Suppose that f satisfies the P.S. Condition w.r.t. C, and that $\exists\, \alpha \in \mathbf{R}^1$ such that

$$\operatorname{Sup}\{f(x) \mid x \in \partial Q\} \leq \alpha ,$$

$$\operatorname{Sup}\{f(x) \mid x \in Q\} < +\infty ,$$

and

$f(x) > \alpha, \quad \forall x \in S.$

Then one of the three possibilities occurs:

(1) α is an accumulate point of critical values.

(2) α is a critical value with uncountable K_α.

(3) $c = \inf_{A \in F} \operatorname{Sup}_{x \in A} f(x) > \alpha$ is a critical value,

where $F = \{A = \phi(Q) \mid \phi \in C(Q,C), \text{ with } \phi|_{\partial Q} = id|_{\partial Q}\}$.

The proof depends on [6] and the following deformation lemma.

Let K be the critical set of f. $\forall\, a \in \mathbf{R}^1$, denote $K_a = f^{-1}(a) \cap K$ and $f_a = \{x \in C \mid f(x) \le a\}$.

DEFORMATION LEMMA. Suppose that c is the unique critical value of f in the interval $[c,b)$ and that K_c is countable, then f_c is a strong deformation retract of $f_b \setminus K_b$.

Proof. It is a combination of the proofs given in K.C. Chang [5], Chang, Eells [7] and Z.C. Wang [19]. A pseudo gradient vector field and an associate flow were constructed in [7] for $f \in C^{2-0}$ and finite K_c, it was proved in [5]. An improvement which enables to cover our conditions, was given in [19].

Proof of Theorem 1. If non of these cases occurred, then there would exist $\varepsilon > 0$ and $\phi_0 \in C(Q,C)$ such that:

$$\alpha = c, \quad f^{-1}(c, c+\varepsilon] \cap K = \phi, \quad K_c \text{ is countable and}$$

$$\phi_0(Q) \subset f_{c+\varepsilon} .$$

According to the deformation lemma, there is a continuous ϕ: $f_{c+\varepsilon} \to f_c$. Since $\phi \circ \phi_0 \in C(Q,C)$ with $\phi \circ \phi_0|_{\partial Q} = id|_{\partial Q}$, we have $(\phi \circ \phi_0)(Q) \cap S \neq \phi$. It implies

$$\operatorname{Sup}\{f(x) \mid x \in \phi \circ \phi_0(Q)\} > \alpha = c .$$

This is a contradiction.

As corollaries, we have

COROLLARY 1. Suppose that $x_0 \in C$ is a local minimum, and that $\exists\, x_1 \in C$ such that $f(x_0) \ge f(x_1)$, then f has a critical point other than x_0.

In case $C = \mathfrak{X}$, this was obtained in K.C.Chang [2,4] in 1982. Obviously, it implies some results in D.G. de Figueiredo [8], D.G. de Fiqueiredo S.Solmini [9], and Pucci-Serrin [12].

COROLLARY 2. Suppose that f has two local minima, then there exists a third critical point.

2. We present three applications of Theorem 1 (or its corollaries).

(1) Variational Inequality

Let Ω be an open subset in $\mathbf{R}^3$, and let g be a nonnegative measurable function defined on Ω.

THEOREM 2. The functional

$$f(u) = \int_\Omega [\frac{1}{2}(\nabla u)^2 - \frac{1}{3}u^3 + gu] \qquad (1)$$

has at least two critical points w.r.t. the positive cone P in $H_0^1(\Omega)$.

THEOREM 3. Let $\psi \in H^1(\Omega)$, and let $C = \{u \in H_0^1(\Omega) \mid 0 \le u(x) \le \psi(x)$ a.e.$\}$. Assume that

$$\inf\{f(u) \mid u \in C\} < 0 . \qquad (2)$$

Then f(u) has at least three critical points w.r.t. C.

Outline of the proof. It is easy to see that $u_1 = 0$ is a local minimum, and that the global minimum u_2 of f is attainable. The condition (2) implies $u_1 = u_2$. Corollary 2 implies the conclusion of Theorem 2.Similarly, Corollary 1 implies the conclusion of Theorem 2.

REMARK 1. The condition (2) is satisfied, if $\psi(X)$ is large enough.

REMARK 2. For similar considerations, see C.Q. Zhung [20] and A. Szulkin [18].

(2) A combination of the variational method and the sub - and super-solutions.

Let Ω be an open bounded domain with smooth boundary $\partial\Omega$ in $\mathbf{R}^n$, and let $g \in C^\gamma(\Omega \times \mathbf{R}^1, \mathbf{R}^1)$, for some $o < \gamma < 1$, be a function satisfying

$$|g(x,t)| \le c(1 + |t|^\alpha)$$

for some constants $C > 0$ and $\alpha < \frac{n+2}{n-2}$ if $n \ge 3$.

THEOREM 4. Let $G(x,t) = \int_0^t g(x,\xi)d\xi$. Assume that the functional

$$f(u) = \int_\Omega [\frac{1}{2}(\nabla u)^2 - G(x,u(x))]dx$$

satisfies the P.S. condition in the space $H_0^1(\Omega)$, and that f is unbounded below. Moreover if there exists a pair of strict sub- and super-solutions of the equation

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega , \\ u|_{\partial\Omega} = 0 . \end{cases}$$

Then the equation has at least two distinct solutions.

For a proof, cf. K.C.Chang [2]. A considerable simplification can be found in K.C.Chang [5].

Many applications derived from this theorem, which includes the superlinear Ambrosetti Prodi type problem, a nonlinear eigenvalue problem, Amann three solution theorem, and a resonance problem. See K.C.Chang [3] . The superlinear Ambrosetti Prodi type problem was rediscussed in de Figueiredo [8] and de Figueiredo Solimini [9] .

(3) Minimal surfaces

Let M be a compact oriented surface of type (p,k), and let (N,h) be a compact Riemannian manifold with nonpositive sectional curvature. If μ is a conformal structure on M compactible with its orientation, then we write (M,μ) for the associated Riemann surface.

For a map $\phi : (M,\mu) \to (N,h)$, the energy is

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dxdy \ .$$

Let $\Gamma = \{\Gamma_i\}_1^k$ be a set of disjoint oriented Jordan curves in N satisfying an irreducibility condition, which prevents the degeneracy of topological type.

THEOREM 5. If $\phi_i : (M,\mu_i) \to (N,h)$, i = 0,1 are homotopic admissible conformal isolated E-minima, then there is a conformal structure μ on M and an admissible conformal harmonic map $\phi : (M,\mu) \to (N,h)$ homotopic to both, which is not an E-minimum.

A special case, in which M is a borded planar domain and N is Euclidean space $\mathbf{R}^n$, is due to Morse-Tompkins and Shiffman [13,14,15] . If M is a disc or an annulus and $N = \mathbf{R}^n$, that special case has been reproved by struwe [16,17] .

In proving this theorem, corollary 2 is applied. The closed convex set is the following

$$C = \mathfrak{M}^k \times \mathfrak{T}(p.k),$$

where $\mathfrak{M} = \{u \in C^\circ \cap H^{1/2}([0,2\pi],\mathbf{R}^1) | u$ is weakly monotone, and $u(\frac{2k\pi}{3}) = \frac{2k\pi}{3}$, for $k = 0,1,2,3\}$,

and $\mathfrak{T}(p,k)$ denotes the Teichmüller space of compact oriented surface M of type (p,k). The Munford compactness theorem is applied to verify the P.S. Condition.

For details see Chang Eells [6,7] .

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