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In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [109]--113.

Persistent URL: http://dml.cz/dmlcz/700177

ON PROPERTIES OF OSCILLATORY SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS OF THE n–TH ORDER

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Consider the differential equation

(1) $$y^{(n)} = f(t,y,\dots,y^{(n-1)}), \quad n \ge 2$$

where $f : D \to R$ is continuous, $D = R_+ \times R^n$, $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, there exists a number $\alpha \in \{0,1\}$ such that

(2) $$(-1)^\alpha f(t,x_1,\dots,x_n)x_1 \ge 0 \quad \text{in } D.$$

Definition. The solution of (1) defined on R_+ is called proper if y is not trivial in any neighbourhood of ∞. The solution of (1) defined on $[0,b)$ is called non-continuable if either $b = \infty$ or $b < \infty$ and $\sum_{i=0}^{n-1} |y^{(i)}(t)| = \infty$.

The solution y of (1) defined on $[0,b)$, $b \le \infty$ is called oscillatory if there exists a sequence of its zeros tending to b and y is not trivial in any left neighbourhood of b.

Denote the set of all oscillatory solutions of (1), defined on $[0,b)$ by $O_{[0,b)}$. Let $O_{[0,\infty)} = O$ and $N = \{1,2,\dots\}$.

Many papers (see e.g. [6]) are devoted to the study of conditions under which oscillatory solutions exist. But the problem of behaviour of such solutions for $n > 2$ is not solved in a suitable way. We touch some problems concerning the behaviour of oscillatory solutions.

I. Definition. The point $c \in [0,b)$ is called H-point of y if there exist sequences $\{t_k\}_1^\infty$, $\{\bar{t}_k\}_1^\infty$ of numbers of $[0,b)$ such that $(t_k - c)(\bar{t}_k - c) > 0$, $y(t_k) = 0$, $y(\bar{t}_k) \ne 0$, $k \in N$.

In [4] some properties of zeros of $y \in O_{[0,\infty)}$ were studied for the linear case of (1). Especially, it was shown, that every zero of $y^{(i)}$, $i = 0,1,\dots,n-1$ is simple in some neighbourhood of $+\infty$. This result is generalized in [1] for the equation (1) if the interval $(0,b)$ does not contain H-points. Moreover, the following statement was proved:

Theorem 1. *Let either $n = 2n_0$, $n_0 + \alpha$ be odd, or n be odd and let $y \in O_{[0,b)}$. Then there exist at most two H-points in the interval $[0,b)$.*

If there exist two ones $c_1 < c_2$, *then* $y(t) \equiv 0$ *on* $[c_1, c_2]$.

If $n = 2n_0$, $n_0 + \alpha$ is even the statement of the theorem 1 is not valid as it is shown by the following

Theorem 2. *Let* $n = 2$, $\alpha = 1$. *There exist continuous functions* $f : D \to R$ *with the property* (2), $y \in O_{[0,\infty)}$ *and a sequence* $\{\tau_k\}_1^\infty$ *of numbers such that* $\tau_k \in R_+$, $\lim_{k\to\infty} \tau_k = \infty$ *and* τ_k *is the H-point of* y.

Proof. In [5] it is shown that there exist continuous function $a : R_+ \to (-\infty, 0)$ and numbers $b \in R_+$, $\lambda \in (0,1)$ such that the differential equation $y'' = a(t)|y(t)|^\lambda \operatorname{sgn} y(t)$ has an oscillatory solution on $[0,b)$ and $y(t) \equiv 0$ on $[b,\infty)$.

Let $\tau \in [0,b)$ be an arbitrary zero of y' and denote $h = b - \tau$. Define $\bar{a} : R_+ \to (-\infty, 0)$ and $\bar{y} : R_+ \to R_+$ in the following way: $\bar{a}, \bar{y}$ are periodic on $[\tau, \infty)$ with the period $2h$,

$$\bar{a}(t) = a(t),\ \bar{y}(t) = y(t) \quad \text{for } t \in [0,b)$$
$$\bar{a}(t) = a(2b - t),\ \bar{y}(t) = y(2b - t) \quad \text{for } t \in [b, b + h].$$

From this, according to $y'(\tau) = 0$ we get that $\bar{a} \in C^0(R_+)$, $\bar{y} \in C^1(R_+)$. By use of substitutions $t \to x$, $x = 2(b + ih) - t$, $t \in [b + (i-1)h, b + ih]$, $i = 0,1,2,\dots$ can be proved that $\bar{y}$ is a solution of $y'' = \bar{a}(t)|y(t)|^\lambda \operatorname{sgn} y(t)$. As b is H-point of y and $\bar{y}$, too, we can put $\tau_k = b + 2kh$, $k \in N$. The theorem is proved.

II. Let n_0 be the entire part of $\frac{n}{2}$. Put for $y \in C^{n_0}(R_+)$, $m \in N$ $J_m(t;y) = \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} y(\tau_1)d\tau_1 \dots d\tau_m$, $J_0(t;y) = y(t)$, $t \in R_+$

$$(3) \qquad Z(t;y) = \sum_{i=0}^{n-n_0-1} (-1)^{\alpha+i} \binom{n-i}{n} \frac{n}{2(n-i)} J_{2i}(t;[y^{(i)}]^2).$$

The following Lemma was proved in [1]:

Lemma. *Let* y *be a solution of* (1) *defined on* R_+ *and let either* $n = 2n_0$, $n_0 + \alpha$ *be odd or* n *be odd. Then*

$$Z^{(n-1)}(t;y) = \sum_{i=0}^{n_0-1} (-1)^{\alpha+i} y^{(n-i-1)} y^{(i)}(t) + \frac{1}{2}(-1)^{n_0+\alpha}(n - 2n_0)[y^{(n_0)}(t)]^2,$$

$$Z^{(n)}(t;y) = (-1)^\alpha y^{(n)}(t)\, y(t) + (-1)^{n_0+\alpha}[y^{(n_0)}(t)]^2 (n - 2n_0 - 1) \ge 0, \quad t \in R_+.$$

In the present part we shall study the asymptotic behaviour of proper oscillatory solutions of (1) under the assumptions

(4) $\quad n = 2n_0 + 1,\ n_0 \in N$.

Definition. Let $y \in O$ and $\lim_{t\to\infty} Z^{(n-1)}(t;y) = c$. Then $y \in O^1$ ($y \in O^2$) if $c = \infty$ ($c < \infty$).

It is shown in [1] that for $y \in O^1$ $\limsup_{t\to\infty} |y(t)| = \infty$ holds. The behaviour of $y \in O^2$ is different.

Theorem 3. *Let (4) be valid and let continuous functions* g, $g_1: R_+ \to R_+$ *exist such that* $g(x) > 0$ *in some neighbourhood of* $x = 0$, $\liminf_{x\to\infty} g(x) > 0$ *and*

(5) $\quad g(|x_1|) \le |f(t,x_1,\dots,x_n)| \le g_1(|x_1|)$ *in* D

holds. Let $y \in O^2$. *Then* $c = 0$, $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0,1,\dots,n-2$ *and* $y^{(n-1)}$ *is bounded.*

Proof. Let $M \in (0,\infty)$ be a number such that $M_1 = \min_{M\le x<\infty} g(x) > 0$.

Let $D_1 = \{t : t \in R_+,\ |y(t)| \le M\}$, $D_2 = R_+ - D_1$, $y_i(t) = y(t)$ for $t \in D_i$, $y_i(t) = 0$ for $t \in R_+ - D_i$, $i = 1,2$. It is clear (by use of (5)) that $y_1 \in L^\infty(R_+)$, $y_1^{(n)} \in L^\infty(R_+)$. According to Lemma and (5)

(6) $$\infty > Z^{(n-1)}(\infty;y) - Z^{(n-1)}(0;y) = \int_0^\infty (-1)^\alpha y^{(n)}(t)y(t)dt \ge$$

$$\ge \int_0^\infty g(|y(t)|)|y(t)|dt \ge \int_0^\infty g(|y_2(t)|)|y_2(t)|dt \ge$$

$$\ge M_1 \int_0^\infty |y_2(t)|dt\ ;$$

$$\int_0^\infty |y_2^{(n)}(t)|dt \le \frac{1}{M}\int_0^\infty |y_2^{(n)}(t)y_2(t)|dt \le \frac{1}{M}\int_0^\infty |y^{(n)}(t)y(t)|dt <$$

$$< \infty .$$

Thus $y_2 \in L^1(R_+)$, $y_2^{(n)} \in L^1(R_+)$ and according to [3, V, §4] and (5)

(7) $\quad |y^{(i)}(t)| \le K < \infty,\ t \in R_+,\ i = 0,1,2,\dots,n-1.$

Let $\{t_k\}_1^\infty$, $\{\tau_k\}_1^\infty$ be sequences, such that $0 \le t_k < \tau_k < t_{k+1}$, $\lim_{k\to\infty} t_k = \infty$, $y(t_k) = 0$, $y'(\tau_k) = 0$, $y(t) \neq 0$ on (t_k,τ_k), $k \in N$. Then, by use of (6) and (7)

$$\infty > \int_0^\infty g(|y(t)|)|y(t)|dt \ge \frac{1}{K}\sum_{k=1}^\infty \int_{t_k}^{\tau_k} g(|y(t)|)|y(t)||y'(t)|dt \le$$

$$\leq \frac{1}{K} \sum_{k=1}^{\infty} {}_0\int^{|y(\tau_k)} g(s)sds \ .$$

Thus $\lim_{t\to\infty} y(t) = 0$ and according to Kolmogorov-Horny Theorem ([4], p. 167) and (7) we can conclude that

(8) $\lim_{t\to\infty} y^{(i)}(t) = 0, \quad i = 0,1,2,\dots,n-2$.

Let $c \neq 0$. By integration and by use of Lemma we get the existence of $\bar{t} \in R_+$ such that

(9) $|Z(t;y)| \geq \frac{|c|}{4(n-1)!} t^{n-1}, \ t \in [\bar{t},\infty)$.

As according to (8) $\lim y^{(n_0)}(t) = 0$, it follows from (3) that

$$|Z(t;y)| \leq A(t)t^{n-1}, \quad \lim_{t\to\infty} A(t) = 0$$

which contradicts to (9). The theorem is proved.

III. This paragraph contains some remark concerning the existence of proper oscillatory solutions of (1). The case $\alpha = 0$ was investigated in [7].

Definition. The equation (1) has Property A_0 if every proper solution of (1) is oscillatory for n even and is either oscillatory or

(10) $\lim_{t\to\infty} y^{(i)}(t) = 0$,

$i = 0,1,\dots,n-1$ for n odd. The equation (1) has Property A_1 if every proper solution is either oscillatory or (10) holds for $i = 1,2,\dots$ $\dots,n-1$.

The following theorem gives us sufficient conditions for the existence of proper oscillatory solutions if $\alpha = 1$.

Theorem 4. *Let* $\alpha = 1$ *and both* n, n_0 *be even* (*n be* odd). *Let* (1) *have Property* A_0 (*Property* A_1). *Let continuous functions* $h : R_+ \to R_+$, $\omega : R_+ \to (0,\infty)$ *exist such that* ω *is non-decreasing,* ${}_0\int^{\infty} \frac{dt}{\omega(t)} = \infty$ *and*

(11) $|f(t,x_1,\dots,x_n)| \leq h(t)\omega\left(\sum_{i=1}^{n} |x_i|\right)$ *in* D

hold. Then every non-continuable solution y *of* (1), *satisfying* $Z^{(n-1)}(0;y) > 0$ *is oscillatory and proper.*

Proof. Let y be a non-continuable solution of (1) for which

(12) $Z^{(n-1)}(0;y) > 0$.

According to the assumptions of Theorem and [6, Th. 12.1] y is either proper or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0,1,2,\dots,n-1$. As by virtue of Lemma the function $z^{(n-1)}(0;y)$ is non-decreasing, we can conclude that y is proper.

Further, in both cases, it follows from Lemma of Kiguradze ([5], Lemma 14.1) that in case of y be non-oscillatory $\lim_{t\to\infty} z^{(n-1)}(t;y) = 0$ holds. Thus we get the contradiction to (12) and Lemma. The theorem is proved.

Remark 1. The conditions, under which (1) has Property A_0 or A_1 were studied by many authors, see e.g. [6].

2. For the linear case of (1) the existence of oscillatory solutions from the set 0^2 was proved in [5].

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