Jochen W. Schmidt

Enclosing methods for perturbed boundary value problems in nonlinear difference equations

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [333]--338.

Persistent URL: http://dml.cz/dmlcz/700178

ENCLOSING METHODS FOR PERTURBED BOUNDARY VALUE PROBLEMS IN NONLINEAR DIFFERENCE EQUATIONS

J. W. SCHMIDT
Technical University of Dresden
Mommsenstr. 13, Dresden, DDR

1. In the lecture nonlinear equations $F_a(z) = 0$ are considered depending on an input parameter vector a which may be subjected to errors, shortly $a \in A$. In order to study the influence of the input $a \in A$ on the solutions z_a, by means of monotone enclosing methods intervals are constructed containing for each $a \in A$ at least one solution z_a. Such a type of methods can be developed if the operators F_a possess some monotony properties, see SCHMIDT/SCHNEIDER [1].

2. The FDM-discretization of the boundary value problem

$$u'' = 2\alpha_0 \sinh u - \varphi(t), \quad u(0) = p, \quad u(\gamma) = q \tag{2.1}$$

appearing in inner electronics is chosen as a model problem. Let the net density φ be given by

$$\varphi(t) = \varphi(t,\alpha_1,\dots,\alpha_7) = \tag{2.2}$$
$$= \alpha_1 10^{10} e^{-\alpha_2 t^2} + \alpha_3 10^8 e^{-\alpha_4 t^2} + \alpha_5 10^5 + \alpha_6 10^8 e^{-\alpha_7 (t-\gamma)^2} .$$

In general the parameter vector $a = (\alpha_0,\dots,\alpha_7)^T$ is affected with errors,

$$a = \tilde{a} \pm e, \; e = (\varepsilon_0,\dots,\varepsilon_7)^T . \tag{2. 3}$$

This vector interval represents the set A.

Applying the common finite difference method to (2.2) ($h = \gamma/(N + 1)$ step size, $t_i = i\,h$ nodes, ζ_i approximation to $u(t_i)$) one gets the following system of equations

$$F_a(z) = F_a^+(z) + F_a^-(z) \tag{2.4}$$

with

$$(F_a^+(z))_i = -\zeta_{i-1} + 2\zeta_i - \zeta_{i+1} + \alpha_0 h^2 e^{\zeta_i} \qquad (2.5)$$

$$(F_a^-(z))_i = -\alpha_0 h^2 e^{-\zeta_i} - h^2\varphi(t_i, \alpha_1, \ldots, \alpha_7) \quad (i = 1, \ldots, N)$$

and $\zeta_0 = p$, $\zeta_{N+1} = q$. Here the i-th component of a vector z is written as $(z)_i = \zeta_i$, and so on. Obviously, the operators F_a are offdiagonally antitone, the derivatives DF_a^+ are isotone and the derivatives DF_a^- are antitone if $\alpha_0 > 0$. These properties are essential in what follows.

3. Let R,S be finite dimensional linear spaces partially ordered by closed cones. Thus these cones are normal and regular, too. For a continuous operator

$$F : D = [y_1, x_1] \subset R \to S \qquad (3.1)$$

a mapping $\Delta F : D \times D \to L(R,S)$ is called an isotone-antitone divided difference operator if

$$F(x)-F(y) \le \Delta F(x,y)(x-y) \text{ for } y_1 \le y \le x \le x_1, \qquad (3.2)$$

$$\Delta F(x,y) \le \Delta F(u,v) \text{ for } y_1 \le v \le x \le u \le x_1 \qquad (3.3)$$

(i) For $F = F^+ + F^-$ the mapping

$$\Delta F(x,y) = DF^+(x) + DF^-(y)$$

is a divided difference operator if DF^+ is isotone and DF^- is antitone, see [10].

(ii) If, in addition, F is offdiagonally antitone

$$\Delta F(x,y) = \text{diag } DF^+(x) + \text{diag } DF^-(y)$$

is a divided difference operator, see [10].

(iii) In interval mathematics the maximal derivative

$$\Delta F(x,y) = (\max_{y \le z \le x} \partial_k F_i(z))$$

is widely used being also a divided difference operator, see [7].

4. It is assumed that for any operator

$$F_a : D = [x_1, y_1] \subset R \rightarrow S, \; a \in A \tag{4.1}$$

an isotone-antitone divided difference operator ΔF_a exists. Because, in general, F_a and ΔF_a are not explicitly available, bounds of theirs are used. Suppose there exist mappings $U, V : D \rightarrow S$ such that

$$U(z) \leq F_a(z) \leq V(z) \text{ for } z \in D, \; a \in A \; . \tag{4.2}$$

The bounds U and V are assumed to be sharp in the following sense,

$$F_a(z) \leq 0 \text{ for all } a \in A \text{ implies } V(z) \leq 0 \; , \tag{4.3}$$

$$F_a(z) \geq 0 \text{ for all } a \in A \text{ implies } U(z) \geq 0 \; , \tag{4.4}$$

valid for every $z \in D$. Further, for ΔF_a let exist an upper bound $B : D \times D \rightarrow L(R,S)$ characterized by

$$\Delta F_a(x,y) \leq B(x,y) \text{ for } y_1 \leq y \leq x \leq x_1, \; a \in A, \tag{4.5}$$

$$B(x,y) \leq B(u,v) \text{ for } y_1 \leq v \leq y \leq x \leq u \leq x_1 . \tag{4.6}$$

Now, the iterative process can be formulated.

Method [1]: Determine x_{n+1}, y_{n+1} such that

$$U(x_n) + B(x_n,y_n)(x_{n+1} - x_n) = 0 \; , \tag{4.7}$$

$$V(y_n) + B(x_n,y_n)(y_{n+1} - y_n) = 0, \; n = 1,2,\dots \; .$$

If ΔF_a is taken according to (i) or (ii) one gets a Newton-type method or a Jacobi-Newton-type method, respectively.

5. Monotone enclosing theorem: Let $x_1, y_1 \in R$, $y_1 \leq x_1$ be such that

$$V(y_1) \leq 0 \leq U(x_1) \; . \tag{5.1}$$

Suppose that the linear operators $B(x,y)$ are invertible and that

$$B(x,y)^{-1} \geq 0 \text{ for } y_1 \leq y \leq x \leq x_1 . \tag{5.2}$$

Then the sequence (x_n) and (y_n) are well-defined by (4.7), any of the operators F_a, $a \in A$, possesses a zero $z_a \in [y_1, x_1]$, and for such zeros the monotone enclosing

$$y_1 \leq .. \leq y_{n-1} \leq y_n \leq z_a \leq x_n \leq x_{n-1} \leq .. \leq x_1, \; n = 1,2\dots \tag{5.3}$$

is valid.

A proof shall be skatched. The operator T,

$$T(z) = z - B(x_1,y_1)^{-1}F_a(z),\ z \in R$$

is isotone since for $y_1 \le y \le x \le x_1$ one gets

$$T(x) - T(y) = x - y - B(x_1,y_1)^{-1}\{F_a(x) - F_a(y)\},$$

$$F_a(x) - F_a(y) \le B(x,y)(x - y) \le B(x_1,y_1)(x - y)$$

implying $T(y) \le T(x)$. Further, $T(x_1) \le x_1$ and $T(y_1) \ge y_1$ hold. Thus, a fixed-point theorem of Kantorovich assures $z_a = T(z_a)$ for some vector $z_a \in [y_1,x_1]$, and hence $F_a(z_a) = 0$ follows.

Next, beinning with $y_n \le z_a \le x_n$, $F_a(z_a) = 0$, $V(y_n) \le 0 \le U(x_n)$ one gets immediately $x_{n+1} \le x_n$, further $x_{n+1} \le z_a$ because of

$$x_{n+1} - z_a = x_n - z_a - B(x_n,y_n)^{-1}\{U(x_n) - F_a(z_a)\},$$

$$U(x_n) - F_a(z_a) \le F_a(x_n) - F_a(z_a) \le B(x_n,y_n)(x_n - z_a),$$

and $U(x_{n+1}) \ge 0$ in consequence of

$$F_a(x_n) \ge U(x_n) = B(x_n,y_n)(x_n - x_{n+1}) \ge F_a(x_n) - F_a(x_{n+1}),\ a \in A.$$

Analogously $y_n \le y_{n+1} \le z_a$ and $V(y_{n+1}) \le 0$ is derived.

6. The assumption (5.2) can be weakened as follows: There esists a mapping $G \in L(S,R)$ with $\ker G = \{0\}$ and

$$G \ge 0,\ G\,B(x_1,y_1) \le I\ ,$$

see [1].

7. In the model problem (2.4),(2.5) let the input parameters be

$$\alpha_0 = 1 \pm \varepsilon_0,\ \alpha_1 = 2.8 \pm \varepsilon_1,\ \alpha_2 = 2321.385 \pm \varepsilon_2,\ \alpha_3 = 11/3 \pm \varepsilon_3,$$

$$\alpha_4 = 1121.918 \pm \varepsilon_4,\ \alpha_5 = 2 \pm \varepsilon_5,\ \alpha_6 = 20/3 \pm \varepsilon_6,\ \alpha_7 = 869.9157 \pm \varepsilon_7$$

and $\gamma = 0.25717$, $p = \operatorname{arsinh}(\varphi(0)/2)$ and $q = \operatorname{arsinh}(\varphi(\gamma)/2)$. In order to demonstrate the different influence of these parameters on the respective components of the zeros some typical examples are given computed by the Newton-type method (4.7),(i). The dimension in all cases is N = 30. Further, the notation $x^* = (\zeta_1,\dots,\zeta_{30})^T = \lim x_n$, $y^* = (\eta_1,\dots,\eta_{30})^T = \lim y_n$ is used.

Example 1: $\varepsilon_0 = 0.01$, $\varepsilon_i = 0 \ (i \neq 0)$

i	η_i	ζ_i
5	19.943..	19.963..
9	-12.509..	-12.489..
10	0.453..	0.473..
11	10.618..	10.638
15	12.197..	12.217..
20	13.420..	13.440..
25	18.155..	18.175..

Example 2: $\varepsilon_1 = 0.01$, $\varepsilon_i = 0 \ (i \neq 1)$

i	η_i	ζ_i
5	19.949..	19.957
9	-12.500..	-12.498..
10	0.4625..	0.4646..
11	10.6282..	10.6286..
15	12.20699609	12.20699609
20	13.43032446	13.43032446
25	18.16511803	18.16511803

Example 3: $\varepsilon_5 = 0.01$, $\varepsilon_i = 0 \ (i \neq 5)$

i	η_i	ζ_i
5	19.953080..	19.953085..
9	-12.505..	-12.493..
10	0.413..	0.513..
11	10.602..	10.654..
15	12.201..	12.211..
20	13.428..	13.431..
25	18.165105..	18.165130..

Example 4: $\varepsilon_6 = 0.01$, $\varepsilon_i = 0 \ (i \neq 6)$

i	η_i	ζ_i
5	19.95308310	19.95308310
9	-12.49914874	-12.49914874
10	0.46357969	0.46357969
11	10.62844138	10.62844139
15	12.206994..	12.206997..
20	13.429..	13.431..
25	18.163..	18.166..

References

[1] SCHMIDT,J.W., SCHNEIDER,H., *Enclosing methods in perturbed non-linear operator equations*, Computing 32, 1-11 (1984).

[2] COLLATZ,L., *Aufgaben monotoner Art*, Arch. Math. (Basel)3, 366-376 (1952).

[3] NICKEL,K., *The construction of a priori bounds for the solution of a two point boundary value problem with finite elements*, Computing 23, 247-265 (1979).

[4] SCHRÖDER,H., *Operator Inequalities*, New York-London: Academic Press 1980.

[5] SPREUER,H., *A method for the computation of bounds for ordinary linear boundary value problems*, J. Math. Anal. Appl. 81, 99-133 (1981).

[6] WILDENAUER,P., *A new method for automatical computation of error bounds for nonlinear boundary value problems*, Computing 34, 131-154 (1985).

[7] ALEFELD,G., *Quadratisch konvergente Einschließung von Lösungen nichtkonvexer Gleichungssysteme*, Z. Angew. Math. Mech. 54, 335-345 (1974).

[8] ORTEGA,J.M., RHEINBOLDT,W.C., *Monotone iterations for nonlinear equations with application to Gauss-Seidel methods*, SIAM J. Numer. Anal. 4, 171-190 (1967).

[9] SCHMIDT,J.W., LEONHARDT,H., *Eingrenzung von Lösungen mit Hilfe der Regula falsi*, Computing 6, 318-329 (1970).

[10] SCHMIDT,J.W., SCHNEIDER,H., *Monoton einschließende Verfahren bei additiv zerlegbaren Gleichungen*, Z. Angew. Math. Mech, 63, 3-11 (1983).

[11] SCHNEIDER,N., *Monotone Einschließung durch Verfahren vom Regula falsi-Typ*, Computing 26, 33-44 (1981).

[12] TÖRNIG, W., *Monoton konvergente Iterationsverfahren zur Lösung nichlinearer Differenzen-Randwertprobleme*, Beiträge Numer. Math. 4, 245-257 (1975).

[13] TÖRNIG,W.,*Monoton einschließend konvergente Itartionsprozesse vom Gauss-Seidel-Typ*, Math. Meth. Appl. Sci. 2, 489-503 (1980).

[14] GROẞMANN,CH.. KRÄTZSCHMAR.M., *Monotone Diskretisierung und adaptive Gittergenerierung für Zwei-Punkt-Randwertaufgaben*, Z. Angew Math. Mech. 65, T264-266 (1985).

[15] GROẞMANN,CH., KRÄTZSCHMAR,M., ROOS,H.-G., *Uniformly enclosing discretization methods of high order for boundary value problems* Math. Meth. Appl. Sci. (submitted 1985).