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CONNECTIONS IN SCALAR REACTION DIFFUSION EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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We consider the flow of a one-dimensional reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1}$$

on the interval $x \in (0,1)$ with Neumann boundary conditions

$$u_x(t,0) = u_x(t,1) = 0. \tag{2}$$

Given two stationary solutions v, w of (1),(2) (i. e. solutions of

$$v'' + f(v) = 0, \ v'(0) = v'(1) = 0 \ ) \tag{3}$$

we say that v connects to w if there exists a solution u(t,x) of (1), (2) for $t \in (-\infty,\infty)$ such that

$$\lim_{t\to-\infty} u(t,.) = v, \quad \lim_{t\to\infty} u(t,.) = w. \tag{4}$$

For ordinary differential equations trajectories connecting stationary points have been studied in the context of shock waves [3,10] and travelling waves [10]. For(1),(2) the principal motivation for studying connections is somewhat different. As argued by Hale [4] the flow on the maximal compact invariant set A displays the essential qualitative features of the flow of (1),(2). Since (1),(2) is a gradient system, under mild growth conditions on f at infinity A consists of stationary solutions and connecting trajectories. Therefore, determining all stationary solutions and their connecting trajectories, we know the essential part of the flow.

For special classes of nonlinearities the problem of identification of pairs of stationary solutions admitting connections has been studied by Conley and Smoller [2, 10] and Henry [5, 6] who solved the problem completely for f satisfying f(0) = 0 and being qualitatively cubic-like. In [1] we have given an almost complete answer to the following question concerning equation (1) with Dirichlet boundary conditions for general f:

(Q) Given a stationary solution v, which stationary solutions does it connect to?

Similarly as in [1], to distinguish the w's to which v connects we introduce a scalar characteristics of the complexity of stationary solutions. However, while in [1] this is the maximal number of sign changes (called zero number, z), in our case its role will be played by the lap number l introduce by Matano [7]. For a given function v on $[0, 1]$ $l(v)$ is, by definition, the minimal number of intervals I_j into which $[0, 1]$ can be partitioned so that v is strictly monotone on each I_j and $l(v) = 0$ for v constant.

For v stationary we define the instability (Morse) index $i(v)$ as the number of negative eigenvalues of the problem

$$y'' + f'(v(x))+\lambda)\; y = 0 \tag{5}$$

$$y'(0) = y'(1) = 0. \tag{6}$$

By a Sturm-Liouville separation of zeros argument one obtains for $v \not\equiv$ cnst

$$l(v) \le i(v) \le l(v) + 1. \tag{7}$$

The stationary solution v is called hyperbolic if $\lambda = 0$ is not an eigenvalue of the problem (5),(6).

Given v hyperbolic, for $0 \le k \le l(v)$ we denote by $\tilde{v}_k(\underset{\sim}{v}_k)$ the stationary solution $\tilde{v}$ $(\underset{\sim}{v})$ satisfying $l(\tilde{v}) = k$ with smallest $\tilde{v}(0) >$ max Range v ($l(\underset{\sim}{v}) = k$ with largest $\underset{\sim}{v}(0) \subset$ Range v, respectively). By $\Omega(v)$ we denote the set of stationary solutions which v connects to. The following theorem is an almost complete answer to (Q):

Theorem. Let f be C^2 and let

$$\overline{\lim_{|s| \to \infty}} f(s)/s < 0 \tag{8}$$

Let v be a hyperbolic solution of (3).

(i) If v is constant or $i(v) = l(v)$ then

$$\Omega(v) = \{\tilde{v}_k, \underset{\sim}{v}_k : 0 \le k < i(v)\}$$

(ii) If $v(0) = \max v \ne \min v$ and $i(v) = l(v) + 1$ then

$$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_1 = \{\tilde{v}_k : 0 \le k < i(v)\},$$

$$\Omega_2 = \{\underset{\sim}{v}_k : 0 \le k < i(v) - 1\}$$

and either

$$\Omega_3 = \{\underset{\sim}{v}_k : k = i(v) - 1\}$$

or Ω_3 consists of one or several stationary solutions w with Range $w \subset$ Range v and $i(w) < i(v)$.

Note that from [7] it follows that there are no other cases possible except of (i) - (ii).

The proof of this theorem proceeds along the lines of the proof of the analogous theorem of [1] for the Dirichlet case. Therefore, the details of its outline given below can easily be completed from [1].

To establish connections we focus on the case $f(0) = 0$, $v \quad 0$ here for simplicity. Let the zero number $z(u(t,.))$ denote the number of sign changes of $x \mapsto u(t,x)$, $0 < x < 1$ - cf. [1]. Then $z(u(t,.))$ is decreasing with t [7, 8] and we may define the dropping times

$$t_k = \inf \{t \geq 0 : z(u(t_i)) \leq k\} \leq \infty$$

and

$$\tau_k = \tanh t_k \in [0, 1].$$

Note that $t_k \leq t_{k-1} \leq \ldots \leq t_0$. If $t_k < t_{k-1}$, the sign

$$\sigma_k = \text{sign } u(t,0), \quad t_k < t < t_{k-1}$$

is independent of t. We collect all this information in the map $y = (y_0, \ldots, y_k, \ldots)$ where

$$y_0 = \sigma_0 (1 - \tau_0)^{1/2}$$

$$y_k = \sigma_k (\tau_{k-1} - \tau_k)^{1/2}.$$

Taking $n = i(v) - 1$ and a small sphere Σ^n around v in the unstable manifold $W^u(v)$,

$$y: \quad \Sigma^n \to S^n$$

turns out to be a continuous and essential mapping into the standard n-sphere S^n. In particular, y is surjective. Therefore, for any $0 \leq k < i(v)$, $\sigma \in \{-1, 1\}$ there exists a $u_0 \in W^u(v)$ such that $y(u_0) = \sigma e_k$ where e_k denotes the k-th unit vector. Hence the trajectory $u(t)$ through u_0 connects v to a stationary w with

$$z(w - v) = k \text{ and sign } (w(0) - v(0)) = \sigma .$$

The fact that $w = v_k$ for $\sigma = 1$ and $w = v_k$ for $\sigma = -1$ (the latter in case (i) follows from the following two lemmas:

Lemma 1. The stationary solution v does not connect to w if there is a $\bar{w}$ stationary with $\bar{w}(0)$ between $v(0)$ and $w(0)$ such that

$$z(v - \bar{w}) \leq z(w - \bar{w}).$$

Lemma 2. For stationary solutions v, w

$$z(v - w) = \begin{cases} l(v) \geq 1 & \text{if Range } w \subset \text{Range } v \\ 0 & \text{if Range } v \cap \text{Range } w = \emptyset \end{cases} \tag{9}$$

Note that up to interchange of v and w all possible cases are taken care of in Lemma 2.

The proofs of these lemmas can easily be obtained by adapting those of the corresponding lemmas from [1]. The first one is based on the maximum principle, the second employs the phase plane portraits of (v, v') and (w, w'): one notes that between two successive local extrema of v there is precisely one intersection point of v and w in case Range w $\subset$ Range v.

Concluding we note that $\tilde{v}_k$ and $\underset{\sim}{v}_k$ can easily be identified from the global bifurcation diagram of the parametric equation

$$u_t = u_{xx} + a^2 f(u)$$

as given e. g. in [9]. Also, we note that by further analysis we can identify the members of Ω_3 more precisely.

Figure 1 illustrates the Theorem for a particular f. Points on one curve represent stationary solutions with the same lap number which increases from curve to curve by one from left to right starting with $l = 1$. Case (i) applies to v in the left part of Figure 1 with $l(v) = 0$, $i(v) = 3$. Case (ii) applies to v in the right part of Figure 1. In this case $l(v) = 8$, $i(v) = 9$, $v_8 \notin \Omega_3$ and all candidates for Ω_3 not excluded by (ii) are marked by "?". By further analysis we are able to show that connections do exist to those solutions marked by "!" and do not exist to those solutions marked by "x".

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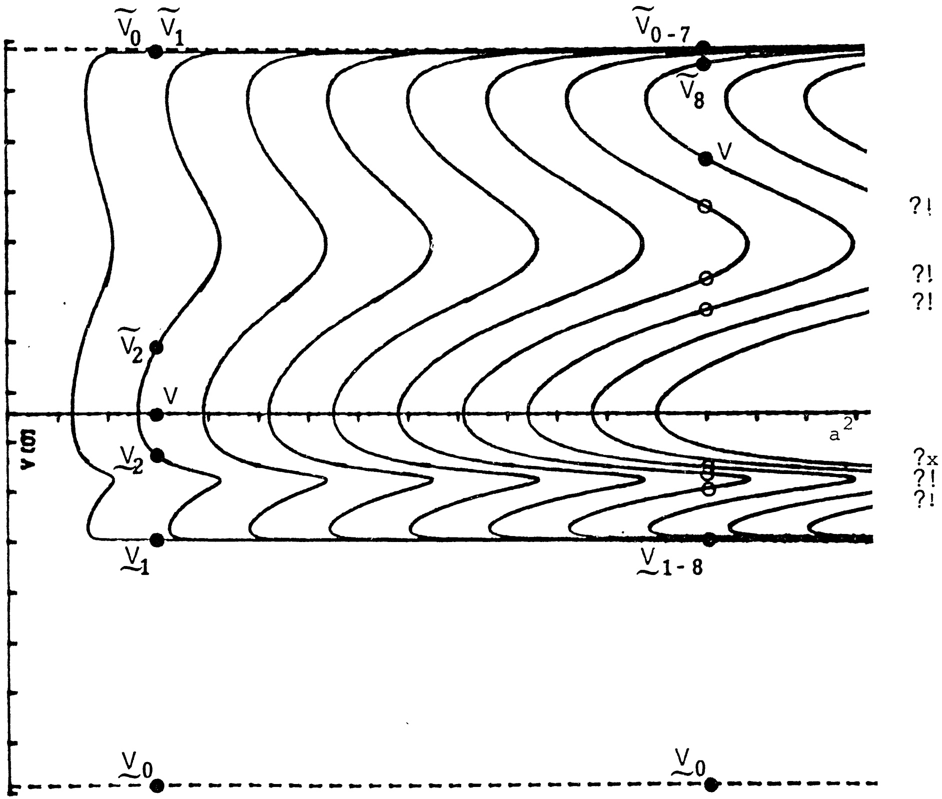

Fig. 1 Bifurcation diagram for $f(u) = (u+10.2)u((u-4)^2 + 1.75^2)(u-10)$