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On oscillation of solutions of linear ordinary differential equations


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Consider the linear differential equation

\[(1) \quad u^{(n)}(t) + p(t)u = 0\]

where \(n \geq 3\) and the function \(p : \mathbb{R}_+ \to \mathbb{R}\) is locally integrable. In addition, we assume that \(p\) satisfies one of the following inequalities

\[(2) \quad p(t) > 0 \quad \text{for} \quad t \in \mathbb{R}_+\]
\[(3) \quad p(t) < 0 \quad \text{for} \quad t \in \mathbb{R}_+\]

A nontrivial solution of the equation (1) is said to be oscillatory if it has infinitely many zeros and nonoscillatory otherwise.

The equation (1) is said to be oscillatory if it has an oscillatory solution and nonoscillatory if all its nontrivial solutions are nonoscillatory.

The equation (1) has property A if for even \(n\) each nontrivial solution of this equation is oscillatory and for odd \(n\) either oscillatory or satisfying the condition

\[(4) \quad |u^{(i)}(t)| \to 0 \quad \text{for} \quad t \to +\infty \quad (i = 0, \ldots, n - 1).\]

The equation (1) has property B if for even \(n\) each nontrivial solution of this equation is either oscillatory, or satisfying condition (4) or the condition

\[(5) \quad |u^{(i)}(t)| \not\to 0 \quad \text{for} \quad t \to +\infty \quad (i = 0, \ldots, n - 1)\]

and for odd \(n\) either oscillatory, or satisfying the condition (5).

Let \(1 \in \{1, \ldots, n - 1\}\). The equation (1) is said to be \((1, n - 1)\) conjugate (at a neighborhood of +\(\infty\)) if for any \(t_0 \geq 0\) there exist \(t_2 > t_1 \geq t_0\) and a nontrivial solution of this equation such that

\[u^{(i)}(t_1) = 0 \quad (i = 0, \ldots, l - 1),\]
\[u^{(i)}(t_2) = 0 \quad (i = 0, \ldots, n - l - 1).\]

Otherwise the equation (1) is said to be \((1, n - 1)\) disconjugate (at a neighborhood of +\(\infty\)).

With regard to the presence of properties A and B, oscillation of equations and \((n/2, n/2)\) conjugacy of equations of even order, see [1 - 5].
In what follows we study the connection of \((l, n - 1)\) conjugacy with oscillation as well as with the presence of property A or B.

**THEOREM 1.** Let the inequality \(2\) (the inequality \(3\)) hold. Then the following statements are equivalent:

a) the equation \(1\) has property A (property B);

b) for any \(l \in \{1, \ldots, n - 1\}\) such that \(l + n\) is odd (even), the equation \(1\) is \((l, n - l)\) conjugate;

c) the equation \(1\) is \((n - 1, 1)\) conjugate \((1/2(3 + (-1)^n), n - 1/2(3 + (-1)^n))\) conjugate).

Define the numbers \(\ell_n^a\) and \(\ell_n^s\) by the following equalities.

\[
\ell_n^a = \begin{cases} 
\frac{n}{2} - 1 & \text{if } n \equiv 0 \pmod{4}, \\
\frac{n}{2} & \text{if } n \equiv 2 \pmod{4}, \\
\frac{n}{2} - \frac{1}{2} & \text{if } n \equiv 1 \pmod{4}, \\
\frac{n}{2} + \frac{1}{2} & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

\[
\ell_n^s = \begin{cases} 
\frac{n}{2} - \frac{1}{2} & \text{if } n \equiv 0 \pmod{2}, \\
\frac{n}{2} & \text{if } n \equiv 1 \pmod{2}.
\end{cases}
\]

**THEOREM 2.** Let the inequality \(2\) (the inequality \(3\)) hold. Then the equation \(1\) is oscillatory if and only if it is \((\ell_n^a, n - \ell_n^a)\) conjugate \((\ell_n^a, n - \ell_n^a)\) conjugate).

To prove this assertion the following statements are used:

**LEMMA 1.** Let the inequality \(2\) (the inequality \(3\)) hold, \(p\) be not trivial in any neighborhood of \(\pm \infty, l \in \{1, \ldots, n - 1\}\) and \(l + n\) is odd (even). Then the equation \(1\) is \((l, n - l)\) disconjugate if and only if there exists a solution of this equation such that

\[
 u(l)(t)u(t) > 0 \quad \text{for } t \geq t_0 \quad (i = 0, \ldots, l - 1),
\]

\[
(-1)^{l+i} u(l)(t)u(t) > 0 \quad \text{for } t \geq t_0 \quad (i = l, \ldots, n - 1).
\]

**COROLLARY.** Let the inequality \(2\) (the inequality \(3\)) hold, \(l \in \{1, \ldots, n - 1\}\) and \(l + n\) is odd (even). Then the equation \(1\) has the solution satisfying the condition \(8_l^a\) if and only if the equation

\[
 u(n) + (-1)^n \cdot p(t)u = 0
\]

has the solution satisfying \(B_n - \ell\).

**LEMMA 2.** Let the inequality \(2\) (the inequality \(3\)) hold, \(l \in \{2, \ldots, [n/2]\}\), \(l + n\) is odd (even) and there exists a solution of the equation \(1\) satisfying the condition \(8_n^a\). Then the equation

\[
 u(n) - p(t)u = 0
\]

has a solution satisfying the condition \(B_n - \ell - 1\).
By means of theorems 1 and 2 and by the results of [4,5] we can derive the sufficient conditions under which the equation (1) is \((l,n - 1)\) conjugate.

THEOREM 3. Let \(l \in \{1, \ldots, n - 1\}\) and either the inequality (2) hold and \(n + l\) is odd, or the inequality (3) hold and \(n + l\) is even. Then the equation (1) is \((l,n - l)\) conjugate if one of the following conditions is fulfilled:

1) \(\limsup_{t \to +\infty} t \int_0^\infty \frac{n-2}{(n-1)!} |p(s)| ds > (n - 1)! ;

2) \(n + l\) is odd and

\(\liminf_{t \to +\infty} t \int_0^\infty \frac{n-2}{n!} |p(s)| ds > M_n^\alpha ;\)

3) \(n\) and \(l\) are even and

\(\liminf_{t \to +\infty} t^2 \int_0^\infty \frac{n-3}{n!} |p(s)| ds > 1/2M_n^\alpha ;\)

4) \(n\) and \(l\) are odd and

\(\liminf_{t \to +\infty} t \int_0^\infty \frac{n-2}{n!} |p(s)| ds > M_{n,n}^\alpha ,\)

where \(M_n^\alpha\) and \(M_{n,n}^\alpha\) are the largest local maximum of the polynomials

\(P_n^\alpha(x) = -x(x - 1) \ldots (x - n + 1)\)

and

\(P_n^{\alpha+1}(x) = x(x - 1) \ldots (x - n + 1)\), respectively.

THEOREM 4. Let the inequality (2) hold. The equation (1) is \((\ell_n^\alpha, n - \ell_n^\alpha)\) conjugate if one of the following conditions is fulfilled:

1) \(\limsup_{t \to +\infty} t \int_0^\infty \frac{n-2}{n!} |p(s)| ds > \ell_n^\alpha(n - \ell_n^\alpha)! ;\)

2) \(\limsup_{t \to +\infty} \ln t \int_0^\infty \frac{n-1}{(n-1)!} |p(s)| ds = +\infty ,

\(\int_0^\infty t^{n-1}[p(t) - m_n^{\alpha-1} n^\alpha] \ln^2 t dt < +\infty ;\)

3) \(n \not\equiv 3 \pmod{4}\) and

\(\liminf_{t \to +\infty} t^{\ell_n^\alpha} \int_0^\infty \frac{n-\ell_n^\alpha-1}{n!} |p(s)| ds > \frac{m_n^\alpha}{\ell_n^\alpha} ;\)

4) \(n \equiv 3 \pmod{4}\) and

\(\liminf_{t \to +\infty} t^{\ell_n^\alpha-1} \int_0^\infty \frac{n-\ell_n^\alpha}{n!} |p(s)| ds > \frac{m_n^\alpha}{\ell_n^\alpha} - 1\)

where \(\ell_n^\alpha\) is defined by the equality (6) and \(m_n^\alpha\) is the least local maximum of the polynomial (9).

THEOREM 5. Let the inequality (3) hold. The equation (1) is \((\ell_n^{\alpha+1}, n - \ell_n^{\alpha+1})\) conjugate if one of the following conditions is fulfilled:
1) \( \lim_{t \to +\infty} \sup_t \int_{0}^{+\infty} x \delta^{-2} |p(x)| \, dx > \ell_{\delta n}(n - \ell_{\delta n})!; \)

2) \( \lim_{t \to +\infty} \inf_t \int_{0}^{+\infty} x \delta^{-n-1} [p(x) + m_{\delta n} \delta^{-n}] \, dx = +\infty; \)

3) \( n \not\equiv 1 \pmod{4} \) and

4) \( n \equiv 1 \pmod{4} \) and

where \( \ell_{\delta n} \) is defined by the equality (7) and \( m_{\delta n} \) is the least local maximum of the polynomial \( 10). \)

In the case when \( n \equiv 2 \pmod{4} \) and \( p \) is nonnegative or \( n \equiv 0 \pmod{4} \) and \( p \) is nonpositive Theorems 3–5 precise certain results of I. Glazman ([1], Theorems 9, 11 and 12). In order to verify this fact it suffice to take into consideration that for any positive integer \( m \)

\[
\frac{[(m-1)!!]^2}{2^{m-1} \binom{2m-1}{m-1}} \left( \sum_{k=1}^{m} \frac{(-1)^{k-1}}{2^{m-k} \binom{2m-1}{m-1}} C_{m-1}^{k-1} \right)^{-2} \geq \frac{[(2m-1)!!]^2}{(2m-1)!!(2m-1)!!} \geq (2m-1)! \geq (m!)^2
\]

and, in addition, if \( n \equiv 2 \pmod{4}, \) then \( \ell_{\delta n} = n/2, \) \( m_{\delta n} = [(n-1)!!] \cdot 2^{-\frac{n}{2}} \)

and if \( n \equiv 0 \pmod{4}, \) then \( \ell_{\delta n} = n/2, \) \( m_{\delta n} = [(n-1)!!] \cdot 2^{-\frac{n}{2}}. \)

References


