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ON OSCILLATION OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Section A

Consider the linear differential equation

(1) $u^{(n)} + p(t)u = 0$

where $n \geq 3$ and the function $p : R_+ \to R$ is locally integrable. In addition, we assume that p satisfies one of the following inequalities

(2) $p(t) \geq 0$ for $t \in R_+$

or

(3) $p(t) \leq 0$ for $t \in R_+$.

A nontrivial solution of the equation (1) is said to be *oscillatory* if it has infinitely many zeros and *nonoscillatory* - otherwise.

The equation (1) is said to be *oscillatory* if it has an oscillatory solution and *nonoscillatory* if all its nontrivial solutions are nonoscillatory.

The equation (1) *has property* A if for even n each nontrivial solution of this equation is oscillatory and for odd n either oscillatory or satisfying the condition

(4) $|u^{(i)}(t)| \downarrow 0$ for $t \uparrow +\infty$ $(i = 0,\dots,n - 1)$.

The equation (1) *has property* B if for even n each nontrivial solution of this equation is either oscillatory, or satisfying condition (4) or the condition

(5) $|u^{(i)}(t)| \uparrow +\infty$ for $t \uparrow +\infty$ $(i = 0,\dots,n - 1)$

and for odd n either oscillatory, or satisfying the condition (5).

Let $l \in \{1,\dots,n - 1\}$. The equation (1) is said to be $(l,n - l)$ *conjugate* (at a neighborhood of $+\infty$) if for any $t_0 \geq 0$ there exist $t_2 > t_1 \geq t_0$ and a nontrivial solution of this equation such that

$$u^{(i)}(t_1) = 0 \quad (i = 0,\dots,l - 1),$$
$$u^{(i)}(t_2) = 0 \quad (i = 0,\dots,n - l - 1).$$

Otherwise the equation (1) is said to be $(l,n - l)$ *disconjugate* (at a neighborhood of $+\infty$).

With regard to the presence of properties A and B, oscillation of equations and $(n/2, n/2)$ conjugacy of equations of even order, see [1 - 5].

In what follows we study the connection of $(1,n-1)$ conjugacy with oscillation as well as with the presence of property A or B.

THEOREM 1. *Let the inequality (2) (the inequality (3)) hold. Then the following statements are equivalent:*

a) *the equation (1) has property A (property B);*

b) *for any* $\ell \in \{1,\dots,n-1\}$ *such that* $\ell + n$ *is* odd (even), *the equation (1) is* $(\ell, n-\ell)$ *conjugate;*

c) *the equation (1) is* $(n-1,1)$ *conjugate* $((1/2(3+(-1)^n), n - 1/2(3+(-1)^n))$ *conjugate).*

Define the numbers ℓ_n^* *and* ℓ_{*n} *by the following equalities.*

$$(6) \qquad \ell_n^* = \begin{cases} n/2 - 1 & \text{if } n \equiv 0 \pmod 4, \\ n/2 & \text{if } n \equiv 2 \pmod 4, \\ n - 1/2 & \text{if } n \equiv 1 \pmod 4, \\ n + 1/2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

$$(7) \qquad \ell_{*n} = \begin{cases} n - 1 - \ell_n^* & \text{if } n \equiv 0 \pmod 2, \\ n - \ell_n^* & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

THEOREM 2. *Let the inequality (2) (the inequality (3)) hold. Then the equation (1) is oscillatory if and only if it is* $(\ell_n^*, n - \ell_n^*)$ *conjugate* $((\ell_{*n}, n - \ell_{*n})$ *conjugate).*

To prove this assertion the following statements are used:

LEMMA 1. *Let the inequality (2) (the inequality (3)) hold, p be not trivial in any neighborhood of* $+\infty$, $\ell \in \{1,\dots,n-1\}$ *and* $\ell + n$ *is* odd (even). *Then the equation (1) is* $(\ell, n-\ell)$ *disconjugate if and only if there exists a solution of this equation such that*

$$(8_\ell) \qquad \begin{aligned} &u^{(i)}(t)u(t) > 0 \quad \text{for } t \ge t_0 \ (i = 0,\dots,\ell-1), \\ &(-1)^{i+\ell}u^{(i)}(t)u(t) > 0 \ \text{for } t \ge t_0 \ (i = \ell,\dots,n-1). \end{aligned}$$

COROLLARY. *Let the inequality (2) (the inequality (3)) hold,* $\ell \in \{1,\dots,n-1\}$ *and* $\ell + n$ *is* odd (even). *Then the equation (1) has the solution satisfying the condition* (8_ℓ) *if and only if the equation*

$$u^{(n)} + (-1)^n \cdot p(t)u = 0$$

has the solution satisfying $(8_{n-\ell})$.

LEMMA 2. *Let the inequality (2) (the inequality (3)) hold,* $\ell \in \{2,\dots,[n/2]\}$, $\ell + n$ *is* odd (even) *and there exists a solution of the equation (1) satisfying the condition* (8_ℓ). *Then the equation*

$$u^{(n)} - p(t)u = 0$$

has a solution satisfying the condition $(8_{\ell-1})$.

By means of theorems 1 and 2 and by the results of [4,5] we can derive the sufficient conditions under which the equation (1) is (1,n - 1) conjugate.

THEOREM 3. *Let* $\ell \in \{1,\dots,n-1\}$ *and either the inequality* (2) *hold and* $n+\ell$ *is* odd, *or the inequality* (3) *hold and* $n+\ell$ *is* even. *Then the equation* (1) *is* $(\ell, n-\ell)$ *conjugate if one of the following conditions is fulfilled:*

1) $\limsup_{t\to+\infty} t \int_t^{+\infty} s^{n-2}|p(s)|ds > (n-1)!$;

2) $n+\ell$ *is* odd *and*
$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2}p(s)ds > M_n^*$;

3) n *and* ℓ *are* even *and*
$\liminf_{t\to+\infty} t^2 \int_t^{+\infty} s^{n-3}|p(s)|ds > 1/2M_n^*$;

4) n *and* ℓ *are* odd *and*
$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2}|p(s)|ds > M_{*n}$,

where M_n^* *and* M_{*n} *are the largest local maximum of the polynomials*

(9) $P_n^*(x) = -x(x-1)\dots(x-n+1)$

and

(10) $P_{*n}(x) = x(x-1)\dots(x-n+1)$,

respectively.

THEOREM 4. *Let the inequality* (2) *hold. The equation* (1) *is* $(\ell_n^*, n-\ell_n^*)$ *conjugate if one of the following conditions is fulfilled:*

1) $\limsup_{t\to+\infty} t \int_t^{+\infty} s^{n-2}p(s)ds > \ell_n^*!(n-\ell_n^*)!$;

2) $\limsup_{t\to+\infty} \ln t \int_t^{+\infty} s^{n-1}[p(s) - m_n^* s^{-n}]_+ ds = +\infty$,
$\int_1^{+\infty} t^{n-1}[p(t) - m_n^* t^{-n}]_- \ln^2 t\, dt < +\infty$;

3) $n \not\equiv 3 \pmod 4$ *and*
$\liminf_{t\to+\infty} t^{\ell_n^*} \int_t^{+\infty} s^{n-\ell_n^*-1} p(s)ds > m_n^*/\ell_n^*$;

4) $n \equiv 3 \pmod 4$ *and*
$\liminf_{t\to+\infty} t^{\ell_n^*-1} \int_t^{+\infty} s^{n-\ell_n^*} p(s)ds > m_n^*/\ell_n^* - 1$

where ℓ_n^* *is defined by the equality* (6) *and* m_n^* *is the least local maximum of the polynomial* (9).

THEOREM 5. *Let the inequality* (3) *hold. The equation* (1) *is* $(\ell_{*n}, n-\ell_{*n})$ *conjugate if one of the following conditions is fulfilled:*

1) $\limsup_{t\to+\infty} t \int_t^{+\infty} s^{n-2} |p(s)| ds > \ell_{*n}!(n - \ell_{*n})!$;

2) $\limsup_{t\to+\infty} \ln t \int_t^{+\infty} s^{n-1} [p(s) + m_{*n} s^{-n}]_- ds = +\infty$;

$\int_1^{+\infty} t^{n-1} [p(t) + m_{*n} t^{-n}]_+ \ln^2 t dt < +\infty$;

3) $n \not\equiv 1 \pmod 4$ and

$\liminf_{t\to+\infty} t^{\ell_{*n}} \int_t^{+\infty} s^{n-\ell_{*n}-1} |p(s)| ds > m_{*n}/\ell_{*n}$;

4) $n \equiv 1 \pmod 4$ and

$\liminf_{t\to+\infty} t^{\ell_{*n}-1} \int_t^{+\infty} s^{n-\ell_{*n}} |p(s)| ds > m_{*n}/\ell_{*n} - 1$

where ℓ_{*n} *is defined by the equality (7) and* m_{*n} *is the least local maximum of the polynomial (10).*

In the case when $n \equiv 2 \pmod 4$ and p is nonnegative or $n \equiv 0 \pmod 4$ and p is nonpositive Theorems 3 - 5 precise certain results of I.Glazman ([1],Theorems 9,11 and 12). In order to verify this fact it suffice to take into consideration that for any positive integer m

$$\frac{[(m-1)!]^2}{2m-1} \left(\sum_{k=1}^{m} \frac{(-1)^{k-1}}{2m-k} C_{m-1}^{k-1} \right)^{-2} = \frac{[(2m-1)!]^2}{(2m-1)[(m-1)!]^2} \geq (2m-1)! \geq (m!)^2$$

and, in addition, if $n \equiv 2 \pmod 4$, then $\ell_n^* = n/2$, $m_n^* = [(n-1)!!]^2 2^{-n}$ and if $n \equiv 0 \pmod 4$, then $\ell_{*n} = n/2$, $m_{*n} = [(n-1)!!]^2 2^{-n}$.

R e f e r e n c e s

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