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Periodic solutions of nonlinear partial differential equations of evolution

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PERIODIC SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF EVOLUTION.

The first paper on the subject is perhaps [23] A. VITT in 1934. As it is seen from the list of papers known to the author and quoted below, there are now about sixty published papers on the theory considered. Therefore, it is impossible to cover the topic in the whole in this survey. Thus, only papers on partial differential equations of hyperbolic type, which appeared (or are in press) after my expository talk at Equadiff I (see [20]) in 1962, will be mentioned.

Let us start with the author's paper [21]. Here, the existence of ω-periodic solutions of a perturbed wave equation

$$(1)\qquad u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)$$

with boundary conditions

$$(2)\qquad u(t, 0) = u(t, \pi) = 0$$

where f is ω-periodic in t, is investigated by means of the Poincaré method. It is neccessary to distinguish three cases: (α) $\omega = 2\pi n$, n a natural number, (β), $\omega = 2\pi \frac{p}{q}$, p, q natural numbers, (γ) $\omega = 2\pi\alpha$, α an irrational number.

In the case $\omega = 2\pi n$ it is shown that the bifurcation equation of the problem reads either

$$(3)\qquad \int_0^{2\pi n} \int_0^{\pi} f(t, x, u(t, x), u_t(t, x), u_x(t, x))\, v(t, x)\, \mathrm{d}x\, \mathrm{d}t = 0,$$

$v(t, x) = \varphi(x + t) - \varphi(-x + t)$ for any 2π-periodic function φ of class C^2,

or

$$(3')\qquad f(t, x, u(t, x - t), u_t(t, x - t), u_x(t, x - t)\, \mathrm{d}t \equiv 0.$$

Using the latter form of it, it is proved that there exists a classical $2\pi n$-periodic solution for sufficiently small ε, if (i) f is sufficiently smooth and

$$f(t, 0, 0, 0, w, \varepsilon) = f(t, \pi, 0, 0, w, \varepsilon) = 0,$$

(ii) the equation

$$(4)\qquad G(s)(x) \equiv$$
$$\equiv \int_0^{2\pi n} f(\vartheta, x-\vartheta, s(x)-s(-x+2\vartheta), s'(x)-s'(-x+2\vartheta), s'(x)+s'(-x+2\vartheta))\,\mathrm{d}\vartheta = 0$$

has a solution $s^*(x)$ in a subspace $\tilde{C}_2$ of the space C^2, (iii) there exists the inverse operator

$$[G'_s(s^*)]^{-1} \in L[D \to \tilde{C}_2], \qquad \text{where } D = G(\tilde{C}_2)$$

A similar result is obtained for $\omega = 2\pi \frac{p}{q}$, p, q natural numbers. (A paper on a similar problem with nonhomogeneous boundary conditions has just been finished.)

J. KURZWEIL [10] applying his theory of integral manifolds of ordinary differential equations in the Banach space to the problem (1), (2) (with $\omega = 2\pi$) gets an analogous result assuming that besides (i), (ii) quoted above, the condition

(iii') $s^*(x)$ is an exponentially stable stationary solution of a certain ordinary differential equation in a Banach space holds. Then the found 2π-periodic solution of (1), (2) is also asymptotically stable.

In both the papers some particular cases are discussed, for which all assumptions take place. (E.g. in [21] $f = h(t, x) + \alpha u + \beta u^3$, or $f = h(t, x) + (1 - \alpha u^2) u_t$; in [10] an autonomous case is treated successfully, too.)

Usually, the verification of conditions (ii), (iii) is rather difficult. Therefore, the results assuring the two conditions to be fulfilled under some assumptions which may be verified easier, are desirable. Besides some older results [22], [24]—[28], [30] for $\omega \doteq 2\pi \frac{2k+1}{2l}$, k, l natural numbers, P. RABINOWITZ ([17]) assured the existence of a 2π-periodic solution of (1), (2) under the conditions that $f = f(t, x, u)$, f is sufficiently smooth and $\frac{\partial f}{\partial u}(t, x, u) < \beta < 0$ (β being a constant). His method is based on the fact that the bifurcation equation in the form (3) is an Euler equation of an apropriate variational problem, namely

$$\underset{u \in N}{\text{minimize}} \int_0^{2\pi} \int_0^{\pi} F(t, x, u)\,\mathrm{d}x\,\mathrm{d}t,$$

where $F(t, x, u) = \int^u f(t, x, v)\,\mathrm{d}v$ and N is the subspace in L_2 of functions of the form $\varphi(x + t) - \varphi(-x + t)$; φ being 2π-periodic.

In general, the case $\omega = 2\pi\alpha$, α an irrational number, seems to be rather difficult. Recently, G. T. SOKOLOV in [29] has shown the existence of an

ω-periodic solution of the problem (1), (2) (he writes it in a somewhat different way) for $\omega = 2\pi\sqrt{n}$, n a natural number, and $f = f(t, x, u)$.

L. CESARI investigates the problem given by

$$u_{tx} = f(t, x, u, u_t, u_x) \tag{5}$$

$$u(t, 0) = u_0(t) \tag{6}$$

f and u_0 being ω-periodic in t and he asks when it is possible to choose the function $u(0, x) \equiv u_0(0) + v(x)$, $v(0) = 0$, on a sufficiently narrow strip $-a \leq x \leq a$ so that the solution of the problem (5), (6) be ω-periodic in t. He makes use of the fact that the modified problem given by

$$u_{tx} = f(t, x, u, u_t, u_x) - \tag{5'}$$

$$-\frac{1}{\omega}\int_0^{\omega} f(\vartheta, x, u(\vartheta, x), u_t(\vartheta, x), u_x(\vartheta, x))\,\mathrm{d}\vartheta$$

and by the condition (6) has always an ω-periodic solution if a is sufficiently small and f and u_0 are sufficiently smooth. After some anticipatory results in [2], [3] he proves in [4] that there exists an ω-periodic solution of (5), (6) for a sufficiently small a if the following assumptions are fulfilled: (i) f is sufficiently smooth, (ii) the equation

$$\int_0^{\omega} f(\vartheta, 0, u_0(\vartheta), \dot{u}_0(\vartheta), q(\vartheta)\,(\mu))\,\mathrm{d}\vartheta = 0, \tag{7}$$

where $q(t)\,(\mu)$ is the solution of the problem

$$\frac{\mathrm{d}q}{\mathrm{d}t} = f(t, 0, u_0(t), \dot{u}_0(t), q(t)), \qquad q(0) = \mu,$$

has at least one solution $\mu = \mu^*$, (iii) the Jacobian of the equation (7) at the point $\mu = \mu^*$ is nonvanishing. (In CESARI's papers the quantities μ, f, u_0 etc. are supposed to be vectors.)

Besides this CESARI ([5], [6]) studies the problem (5), (6) for

$$f = \varepsilon[\psi(t, x) + Cu + \alpha(x)\,u_t + \beta(t)\,u_x + \varepsilon g(t, x, u, u_t, u_x)],$$

where ψ, α, β and g and u_0 and v are ω-periodic in t and x and he seeks a solution ω-periodic in both variables. Making use of the successive approximation method and the Fourier method he proves that under the condition $C \neq 0$ and some other less fundamental conditions an ω-periodic solution exists.

(Let us note right now that for a perturbed telegraph equation of a similar type i.e.

$$u_{tx} = \psi(t, x) + Cu + \alpha(x)\,u_t + \beta(t)\,u_x + \varepsilon g(t, x, u, u_t, u_x)$$

he also derives an existence theorem for an ω-periodic solution adding to the conditions above the requirement that the limit equation ($\varepsilon = 0$) have an ω-periodic solution of the form $u_0(t) + v_0(x)$.)

In [1] A. K. AZIZ investigates the existence of an ω-periodic solution of the modified problem (5′) under more general assumptions than CESARI does.

In [7] F. A. FICKEN and B. A. FLEISHMAN investigated the problem

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x) + \varepsilon f(t, x, u, u_t, u_x) \tag{8}$$

either for $-\infty < x < +\infty$ or for $0 \leq x \leq \pi$ with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{9}$$

They suppose $a > 0$, $b = 0$, $c > 0$, $f = -u^3$ and h ω-periodic in t and sufficiently smooth. Then they prove the existence of an ω-periodic solution for sufficiently small ε by examining the transition operator $U(t, x)$ at points $t = \tau + n\omega$, for $n \to \infty$, n a natural number.

Making use of the same method J. HAVLOVÁ in [9] generalized their result to the case $a \neq 0$, b arbitrary, $\frac{b^2}{4} + c > 0$, f sufficiently smooth. (V. VÍTEK is preparing a paper on a similar equation in two spatial variables.)

A more general equation

$$u_{tt} + 2\gamma u_t + Au + F(t, u) = f(t) \quad , \tag{10}$$

where A is a positive selfadjoint operator, $F(t, u)$ is a nonlinear sufficiently smooth operator with $F(t, 0) = 0$ is treated also by the same method in [11] by K. MASUDA and the existence of a generalized ω-periodic solution for sufficiently small f is assured.

Recently, the equation (10) was attacked by J. HAVLOVÁ under somewhat different assumptions by means of the Fourier method and the existence of a classical ω-periodic solution was shown.

Lately, a special case of the problem (8), (9), namely $a = b = 0$, $c \neq 0$ was studied by the author and for $\left| k^2 + c - m^2 \left(\frac{2\pi}{\omega}\right)^2 \right| > \delta > 0$ ($k \neq 0$, m natural number), and $f = f(t, x, u)$ the existence of a classical ω-periodic solution was proved.

As early as in 1956 G. PRODI ([14]) treated successfully the strongly nonlinear equation

$$u_{tt} - \Delta u + g(t, x, u_t) = f(t, x, \operatorname{grad} u) \tag{11}$$

with $u = 0$ on the boundary of a domain G. In the last two years, another strongly nonlinear differential equation

$$u_{tt} - \Delta u + g(u_t) = f(t, x) \tag{12}$$

was studied by G. PROUSE ([16]) in the case $g(v) = v + |v|\, v$ and by G. PRODI

([15]) in the case $g(v) \approx |v|^{\varrho-1}v$ ($\varrho \geq 1$) for $v \to \pm\infty$, g being continuous and monotone. They derive some apriori estimates for periodic solutions and make use of the Galerkin method.

In all these cases the existence of a generalized ω-periodic solutions, only, is assured; it would be difficult, however, to describe the Banach spaces in which the solutions lie.

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