EQUADIFF 2

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In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 89--92.

Persistent URL: `http://dml.cz/dmlcz/700201`

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

INVARIANT MANIFOLDS FOR FLOWS

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The purpose of this paper is to present a geometric approach to the theory of invariant manifolds of differential systems. Let the concept of an invariant manifold of a differential system be illustrated by the following simple and rather typical example (which is frequently met in applications to electrical systems).

$$\dot{x} = Ax, \qquad \dot{\varphi} = 0, \tag{1}$$

$$\dot{x} = Ax + f_1(x, \varphi, t), \qquad \dot{\varphi} = f_2(x, \varphi, t). \tag{2}$$

Here x, f_1 are n-vectors, A is an $n \times n$-matrix, φ is a coordinate vector on an m-dimensional torus Φ. Assume that the real parts of the characteristic numbers of A are different from zero. The subset of $E_n \times \Phi \times E_1$, which consists of all points $(0, \varphi, \check{t})$, $\varphi \in \Phi$, $\check{t} \in E_1$, is obviously invariant with respect to (1), i.e. if $x(\check{t}) = 0$, then $x(t) \equiv 0$ for a solution (x, φ) of (1). If f_1, f_2 are sufficiently small, then a similar situation holds for (2), more precisely there exists a map p from $\Phi \times E_1$ to E_n such that if $\tilde{x} = p(\tilde{\varphi}, \check{t})$, then there exists a solution (x, φ) of (2) defined on E_1, $x(\check{t}) = \tilde{x}$, $\varphi(\check{t}) = \tilde{\varphi}$ and $x(t) = p(\varphi(t), t)$ for $t \in E_1$. The map p is unique and the set P of all $(\tilde{x}, \tilde{\varphi}, \check{t})$, $\tilde{x} = p(\tilde{\varphi}, \check{t})$ is the invariant manifold of (2). The behaviour of the solutions of (2) near P is similar to the behaviour of solutions of (1) near the plane $\tilde{x} = 0$.

Usually it is assumed that the perturbation f_i, $i = 1, 2$ is small in that sense that it fulfils one of the following conditions

(I) $f_i(0, \varphi, t) = 0, \qquad ||f_i(x_1, \varphi, t) - f_i(x_2, \varphi, t)|| \leqq L \, ||x_1 - x_2||, \qquad i = 1, 2$

L being small (which is usually fulfilled in the way that f_i contains higher powers in x only),

(II) $\qquad f_i(x, \varphi, t, \varepsilon) = \varepsilon g_i(x, \varphi, t), \qquad i = 1, 2,$

g_i fulfilling some boundedness conditions, ε being a parameter, which is at our disposal and which may be chosen sufficiently small,

(III) $\qquad f_i(x, \varphi, t, \varepsilon) = h_i(x, \varphi, t/\varepsilon), \qquad i = 1, 2,$

$h_i(x, \varphi, \tau)$ being periodic or almost periodic in τ, the average of h_i with respect to τ being zero and ε being again the small parameter.

Or it may be assumed that f_i is a sum of three terms, each of which fulfils one of conditions (I), (II), (III). Theorems of the above type were proved for a large number of various situations, for example the matrix A need not be constant, there may appear a small parameter ε on the right hand side of some rows of (2) or on the left hand sides of some rows (i.e. at derivatives of some components of x or φ) and recently similar theorems were proved for equations with time-lags or for functional differential equations.

The unifying theory may be obtained by a geometric approach. The basic concept is the one of a flow, which is more general then the concept of a dynamical system. It differs from the concept of a dynamical system in the following way: the solution $y(t, \check{y}, \check{t})$ which passes through $\check{y}$ in the moment $\check{t}$ need not exist on the whole real axis but on some interval $\langle \check{t}, t_1 \rangle$, $t_1 > \check{t}$ and uniqueness of solutions is required with t increasing only. The values of the solutions y are from a metric space or from a Banach space. By a flow Y we shall mean a set of functions fulfilling some axioms and in special cases Y may be the set of all solutions of a differential equation or of a functional differential equation. The elements y of a flow Y will be called solutions.

The conditions which guarrantee the existence and uniqueness of an invariant manifold for a flow, cannot be stated in detail here. They may be described roughly as follows: the space Y, where the solutions y of the flow Y take their values from, may be represented as a product of two spaces X and Φ and the x- and φ-components of the solutions y satisfy some inequalities.

General Theorem: *If the above conditions are satisfied, then there exists a unique map p from $\Phi \times E_1$ to X such that if $\tilde{x} = p(\tilde{\varphi}, \check{t})$, then there exists a solution $y = (x, \varphi)$ from the flow, which is defined on the whole real axis $x(\check{t}) = \tilde{x}$, $\varphi(\check{t}) = \tilde{\varphi}$ and $x(t) = p(\varphi(t), t)$ for $t \in E_1$. Again the set P of all $(\tilde{x}, \tilde{\varphi}, \check{t})$, $\tilde{x} = p(\tilde{\varphi}, \check{t})$ is an invariant subset of the flow Y and it is possible to describe the behaviour of solutions from Y near P.*

Let several features of the General Theorem be emphasized.

(i) If the flow Y fulfils the conditions from the General Theorem, then every flow Z, which is sufficiently close to Y, fulfils conditions of the same type and therefore there exists an invariant subset of Z. The fact that flows Y and Z are close is described by two numbers $\zeta > 0$, $T > 0$, ζ being small and T being large and it is required that the following inequalities hold

$$||y(t, \tilde{u}, \check{t}) - z(t, \tilde{u}, \check{t})|| \leqq \zeta \quad \text{for} \quad \check{t} \leqq t \leqq \check{t} + T, \tag{3}$$

$$||y(t, \tilde{u}, \check{t}) - y(t, \tilde{v}, \check{t}) - z(t, \tilde{u}, \check{t}) + z(t, \tilde{v}, \check{t})|| \leqq \zeta \, ||\tilde{u} - \tilde{v}|| \quad \text{for} \quad \check{t} \leqq t \leqq \check{t} + T. \tag{4}$$

Assume that flows Y and Z are the sets of all solutions of

(5) $$\dot{y} = f(y, t),$$

(6) $$\dot{z} = g(z, t),$$

the space Y where the solutions y and z take their values from being a Banach space.

Theorem CDP (Continuous Dependence on a Parameter): *If f and g fulfil some boundedness conditions, then flows Y and Z are close in the above sense if* $\| \int_t^{t+\Delta} [f(y, \sigma) - g(y, \sigma)] \, d\sigma$ *is sufficiently small for all y, t and $0 < \Delta \leqq 1$.*

Theorem CDP may be applied in the special case that

(7) $$\dot{y} = h(y, t/\varepsilon),$$

(8) $$\dot{z} = h_0(z),$$

$h_0(z) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} h(z, \sigma) \, d\sigma$ the limit being uniform with respect to z and t. This way the averaging principle is included into the above theory without any transformation of coordinates.

(ii) The theory of invariant manifolds (or subsets) may be developed for metric spaces. It is clear that the norm of the difference of y and z in (3) is to be replaced by the distance; (4) in the case of metric spaces is formulated in a more complicated way. Usually the invariant subset is a product of a torus with an Euclidian space, but in the above theory the invariant subset may be a general complete metric space Φ. The theory simplifies considerably, if Φ has the following property:

(A) if Ψ is a continuous map from Φ to Φ, if Ψ^{-1} exists and fulfils a Lipschitz condition, then $\Psi(\Phi) = \Phi$. It is very easy to prove that every finite-dimensional manifold has the property (A) and there exist spaces having property (A), which are not manifolds.

(iii) There are no periodicity or almostperiodicity conditions in General Theorem. If it happens that the flow Y is periodic [i.e. if f is periodic in t in the case that Y is the set of solutions of (5)], it is verified easily that the invariant subset remains an invariant subset, if it is shifted in the time by the period of the flow; as the invariant manifold is unique, it is necessarily periodic. In a similar way almostperiodicity may be treated.

(iv) General Theorem may be applied if the behaviour of solutions near the invariant subset is like the behaviour of solutions of a differential system near a saddle point; the case that the invariant subset is exponentially stable is the most simple one.

(v) Systems with discrete time — i.e. transformations — are included in General Theorem.

(vi) General Theorem may be applied in case of singular perturbations.

As one of the applications of the above theory the following result may be mentioned: It is well known that solutions of differential equations with time lags or of functional differential equations cannot be prolonged with t decreasing in general. It may be deduced from the above theory that the solutions of a functionally perturbed ordinary differential equation, which are defined on E_1, fill up an $(n + 1)$-dimensional manifold, if the unperturbed equation is a (nonlinear) ordinary differential equation in E_n or in an n-dimensional manifold the right hand side of which fulfils some boundedness conditions. The reason is in the very simple structure of the flow which corresponds to the unperturbed equation considered as a functional equation: the x-component of any solution from this flow tends to zero extremely rapilly. Of course the necessary boundedness conditions are not fulfilled by equation $\dot{x}(t) = Ax(t) + \varepsilon Bx(t - 1)$ — it is well known that there exist solutions $x_j = e^{\lambda_j t}, j = 1, 2, 3, \ldots$ — but the above result always applies, if a functional perturbation term is added to the right hand side of $\dot{\varphi} = g(\varphi)$, φ being a co-ordinate vector on a compact-manifold (and some smoothness conditions being fulfilled).

Finally let some results on the Van der Pol Perturbation of a Vibrating String be described. Consider the problem

(9) $$u_{tt} - u_{xx} = \varepsilon h(u)\, u_t, \qquad 0 \leqq x \leqq 1, \qquad u(t, 0) = u(t, 1) = 0,$$

h having similar properties as $1 - u^2$. This problem may be transformed to an ordinary differential equation in a function space of the type (7). There are no time-independent solutions of the averaged equation (8), which are continuous (in the space variable), but there exists an infinity of discontinuous ones. Some of them are exponentially stable, other ones are unstable so that the picture rendered by the averaged equation is rather complicated. For the unperturbed equation it may be proved that there exist smooth solutions, tending with $t \to \infty$ to periodic ones, which are discontinuous. Thus it is shown that there exist discontinuous periodic solutions of (9) and that discontinuous solutions appear in a natural way, if (9) is examined.

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