Jacques-Louis Lions
Vectors of Gevrey classes and applications


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VECTORS OF GEVREY CLASSES AND APPLICATIONS

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Introduction.

In several problems in partial differential equations one is led to study the space of functions $u$ defined in a domain $\Omega$ of $\mathbb{R}^n$ with smooth boundary $\Gamma$ and which satisfy conditions of the following type (we take here the simplest possible case):

1. $\left( \int |\Delta^k u|^2 \, dx \right)^{1/2} \leq c L^k M_k \quad \forall \, k,$
2. $\Delta^k u = 0 \quad \text{on} \quad \Gamma \quad \forall \, k,$

where $c$ and $L$ are suitable constants (which depend on $u$) and $M_k$ is a given sequence. For example, if

3. $M_k = (2k)!$

then (1) (2) imply that $u$ is analytic in $\overline{\Omega} = \Omega \cup \Gamma$ (assuming $\Gamma$ to be real-analytic). A much more general result of this type will be reported in Section 4 below.

Once one is led to study classes of functions satisfying conditions of type (1) (2), it is natural to put this question in a more general framework and to replace in (1) (2) $\Delta$ by an unbounded operator $A$ in a Banach space $E$, condition (2) being then replaced by

$\tilde{2} \quad u \in \text{domain of } A, \quad Au \in \text{domain of } A,$

and so on, and condition (1) being replaced by

$\tilde{1} \quad \|A^k u\| \leq c L^k M_k \quad \forall \, k,$

(where $\|\|\|$ denotes the norm in).

In Sections 1,2 we give some (simple) remarks on the spaces defined by

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(1) Expository lecture. All details and other results are contained in the book [4] by E. Magenes and the A. .
(1) (2) (the so-called “vectors of Gevrey class” when \( \{M_k\} \) is a Gevrey sequence) when \((-A)\) is the infinitesimal generator of a semi-group. [This contains (1) (2) by taking \( E = L^2(\Omega), A = -\Delta \), the domain of \( A \) consisting of those functions \( u \) which are zero on \( \Gamma \)].

The plan is as follows:
1. Domains \( D(A^\infty; M_k) \).
2. A criterion of non triviality.
3. The semi group on \( D(A^\infty; M_k) \).
4. The case when \( A \) is an elliptic operator.
5. Transposition.
7. Some examples.

Bibliography

1. Domains \( D(A^\infty; M_k) \).

Let \( E \) be a reflexive Banach space, norm \( || \cdot || \); let \( A \) be an unbounded operator given in \( E \); we assume (for semi-group theory we refer to [2], [10]):

(1.1) \((-A)\) is the infinitesimal generator of a continuous semi-group \( G(t) \) in \( E \). Let \( D(A) \) be the domain of \( A \). We set

\[
D(A^\infty) = \{ u \mid A^k u \in D(A) \quad \forall \ k \};
\]

it is well known [2], [10] that \( D(A^\infty) \) is dense in \( E \).

Let now \( \{M_k\} \) be a given sequence of positive numbers.

We define

\[
D(A^\infty; M_k) = \{ u \mid u \in D(A^\infty); \text{there exist constants } c \text{ and } L \text{ (depending on } u \text{) such that } ||A^k u|| \leq cL^k M_k \quad \forall \ k \}.
\]

Example 1.1.
If \( M_k = (k!)^2, \alpha > 1 \), the corresponding \( D(A^\infty; M_k) \) space is called: the space of vectors of Gevrey class \( \alpha \).

Example 1.2.
If \( M_k = k! \), the corresponding \( D(A^\infty; M_k) \) is the space of analytic vectors. (See [8])

Remark 1.1

Definition 1.2 is purely algebraic. There is a “natural” locally convex topology on \( D(A^\infty; M_k) \): firstly, fix \( L \) in (1.2) (but not \( C \)) and call \( D^L(A^\infty; M_k) \)
the corresponding space; provided with the norm $\sup_{k \geq 0} \frac{1}{L_k^k M_k} ||A^k u||$, it is a Banach space; then $D(A^\infty; M_k) = \text{inductive limit of } D^{L_n}(A^\infty; M_k)$, $L_n \to +\infty$. For details see [4].

Remark 1.2.
Hypothesis (1.1) is perfectly useless in Definition (1.2). But it will be useful in the proofs below.

The "natural questions" are now:
(i) when is $D(A^\infty; M_k) \neq \{0\}$?
(ii) what is the "abstract" interest of $D(A^\infty; M_k)$?
(iii) how can one characterize, in "concrete" situations, the spaces $D(A^\infty; M_k)$ in "concrete" terms?

Partial answers to these questions are respectively given in Sections 2, 3, 4 below — some applications being given in Sections 5, 6, 7.

2. A criterion of non triviality.

Theorem 2.1. Let $\{M_k\}$ be a non quasi-analytic sequence(1) [1] [7]. Then $D(A^\infty; M_k)$ is dense in $E$.

Proof. 1) If $\{M_k\}$ is non quasi-analytic, one can find a sequence $\varphi_n$ of functions with the following properties [7] [9]

\begin{equation}
\varphi_n \in D(M_k), \varphi_n(t) = 0 \text{ if } t \leq 0 \text{ or } t \geq \varepsilon_n, \varepsilon_n \to 0 \text{ if } n \to \infty,
\end{equation}

\begin{equation}
\varphi_n \geq 0, \int_0^\infty \varphi_n(t) \, dt = 1.
\end{equation}

2) Define next $G(\varphi_n) \in L(E; E)$ by

\begin{equation}
G(\varphi_n) e = \int_0^\infty G(t) e \cdot \varphi_n(t) \, dt, e \in E
\end{equation}

One checks easily that $G(\varphi_n) e \in D(A^\infty)$ and that

\begin{equation}
A^k G(\varphi_n) e = G(\varepsilon_n^k) e \not\forall k.
\end{equation}

Thanks to the fact that $\varphi_n \in D(M_k)$, it follows that $G(\varphi_n) e \in D(A^\infty; M_k)$.

3) Let now $e$ be arbitrarily given in $E$; by (2.1) $G(\varphi_n) e \to e$ in $E$, and by 2), $G(\varphi_n) e \in D(A^\infty; M_k)$, hence the result follows.

Remark 2.1. It can happen that $D(A^\infty; M_k)$ is dense in $E$ even with $M_k = 1 \not\forall k$ example: assume that $A$ has a complete set in $E$ of eigenvectors $\omega_n$ then $A\omega_n = \lambda_n \omega_n$ hence $||A^k \omega_n|| \leq ||\omega_n|| \lambda_n^k$, i.e. belongs to $D(A^\infty; 1)$.

(1) This means: let $D(M_k)$ be the space of $C^\infty$ scalar functions $\varphi$ on $R$ with compact support and satisfying $|\ldots| \leq \varphi^{(k)}(0) \leq c L^k M_k \not\forall k$ then $D(M_k) \neq \{0\}$.
But in can happen that $D(A^{\infty}; M_k) = \{0\}$ if $M_k$ is quasi-analytic; example: $E = L^p(0, \infty), A = \frac{d}{dx}, D(A) = \left\{ f | f, \frac{df}{dx} \in L^p(0, \infty), f(0) = 0 \right\}$.

3. The semi-group on $D(A^{\infty}; M_k)$.

**Theorem 3.1.** The necessary and sufficient condition for $u \in E$ to be in $D(A^{\infty}; M_k)$ is that the function

$(3.1) \ G(.) u = "t \to G(t) u"$

is of class $M_k$ with values in $E$, i.e.:

$(3.2) \ \left\{ \begin{array}{c}
\text{for every finite } T \text{ there exist constants } C_1 \text{ and } L_1 \text{ (depending on } T \text{ and } u \text{) such that }
\| \frac{d^k}{dt^k} G(t) u \| \leq C_1 L_1^k M_k \quad \forall \ k, \ t \in [0, T].
\end{array} \right.$

**Remark 3.1.** This property justifies the terminology introduced in Examples 1.1 and 1.2.

**Proof of Theorem 3.1.**

1) $(3.2)$ implies $(1.2)$ (with $C = C_1, L = L_1$). Obvious, take $t = 0$ in $(3.2)$ and use $\frac{d^k G(t)}{dt^k} \cdot u \bigg|_0 = (-1)^k A^k u$.

2) $(1.2)$ implies $(3.2)$. Obvious too. Indeed $\frac{d^k}{dt^k} G(t) u = (-1)^k G(t) A^k u$

hence, for $t \in [0, T]$

$$\left\| \frac{d^k G(t)}{dt^k} u \right\| \leq \sup_{t \in [0, T]} \| G(t) \|_{L(E; E)} \| A^k u \|,$$

hence $(3.2)$ follows.

It follows easily from Theorem 3.1 that (see [4] for details).

**Theorem 3.2.** For every $t$, $G(t)$ is a continuous linear mapping from $D(A^{\infty}; M_k)$ into itself; the semi group $G(t)$ in $D(A^{\infty}; M_k)$ is $C^\infty$ (and of infinitesimal generator $-A$).

One can also show [4] that if for a suitable constant $d$

$(3.3) \ M_k^{+f} \leq d^{k+j} M_f M_j \quad \forall \ k, j$

then for every $u \in D(A^{\infty}; M_k)$ the function $t \to G(t) u$ is of class $M_k$ in $t \geq 0$

with values in $D(A^{\infty}; M_k)$ (i.e., for every finite $T$, there exists a bounded set $B$ in $D(A^{\infty}; M_k)$ and a constant $L$ such that $\frac{1}{L^k M_k} \frac{d^k}{dt^k} G(t) u \in B \quad \forall \ k, t \in [0, T]$).
4. The case when $A$ is an elliptic operator.

Let us recall first a classical definition: a complex-valued function $\varphi$ defined on a compact set of $\mathbb{R}^n$ is said of Gevrey order $\beta > 1$ (resp. real analytic) if for suitable constants $c$ and $L$ one has

$$|D^p \varphi(x)| \leq cL^{p_1 + \ldots + p_n}(p_1! p_2! \ldots p_n)!$$

(resp. $\beta = 1$)

$\forall p = \{p_1, \ldots, p_n\}, \forall x \in$ compact set of definition of $\varphi$.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, of boundary $\Gamma$; we assume

(4.1) $\{\Gamma$ is a $(n-1)$ dimensional variety, of Gevrey order $\beta$ (resp. real analytic)$\}$

Let $A$ be a differential operator in $\Omega$; we assume that

(4.2) $A$ is an elliptic operator of order $2m$ (and properly elliptic if $n = 2$) and that

(4.3) the coefficients of $A$ are of Gevrey order $\beta$ (resp. real analytic) in $\overline{\Omega}$.

We are going to characterize $D(A^\infty; M_k)$, taking

(4.4) $E = L^2(\Omega)$.

(4.5) $D(A) = \{u \mid u \in H^{2m}_{\text{loc}}(\Omega) \cap H^m_\text{rad}(\Omega)\}$ (that is: $D^p u \in L^2(\Omega) \forall p, |p| \leq 2m, D^p u = 0$ on $\Gamma \forall, |p| \leq m - 1$),

and when we choose

(4.6) $M_k = [(2km)!]^2$.

One can prove (see [5], [6], [4]):

**Theorem 4.1.** We assume the hypotheses (4.1), (4.2), (4.3) to hold choosing $D(A)$ and $M_k$ by (4.5) (4.6) one has

(4.7) $\{D(A^\infty; M_k) \equiv$ functions of Gevrey order $\beta$ in $\overline{\Omega}$ (resp. real analytic) which satisfy the boundary conditions $"A^k u \in H^m_\text{rad}(\Omega) \forall k"$.

**Remark 4.1.** Under the hypothesis (4.2), $-A$ is the infinitesimal generator of a semi-group in $E$ and even of an analytical semi-group. [2], [10].

One can replace $E = L^2(\Omega)$ by $L^p(\Omega)$, $1 < p < \infty, p \neq 2$, without changing $D(A^\infty; M_k)$.

**Remark 4.2.** The same result holds true for other boundary conditions than the Dirichlet boundary conditions considered above. — See [4].

**Remark 4.3.** If $u$ satisfies $||A^k u|| \leq cL^k(2km)! \forall k$ and no boundary conditions, then one can conclude that $u$ is real analytic on every compact subset of $\Omega$; see [3]; this result is contained in Theorem 4.1.

**Remark 4.4.** A more general result is proved in [4] when we also consider "non-zero boundary conditions".
5. Transposition

Since $E$ is assumed to be a reflexive Banach Space (actually "reflexive" is used here for the first time — and in a non essential manner!) all what we said in Sections 1, 2, 3 is valid after replacing $E$ by $E'$ = dual of $E$
$G(t)$ by $G^*(t)$ = adjoint of $G(t)$
$A$ by $A^*$, $A^*$ being the adjoint of $A$ in the sense of unbounded operators in $E$ or the (opposite to the) infinitesimal generator of the adjoint semi-group $G^*(t)$.

Consequently:

(5.1) $G^*(t)$ is a semi-group in $D(A^*; M)'.$

If we make the hypothesis (see Theorem 1.1):

(5.2) $D(A^*; M)$ is dense in $E'$

then we can identify $E$ to a sub-space of the dual $D(A^*; M)'$ of $D(A^*; M)$; summing up, we have

(5.3) $D(A; M) < E < D(A^*; M)$!

Taking the adjoint of (5.1) we obtain:

(5.4) $[G^*(t)]'$ is a semi-group in $D(A^*; M)'$.

But one easily checks that $(G^*(t))'$ is an extension of $G(t)$, that we can still denote by $G(t)$. Therefore:

(5.5) $G(t)$ is a semi-group in $D(A^*; M)'$, which is $C^0$ and whose infinitesimal generators is $-A$.

For more details, see [4].

Remark 5.1. In the applications, $D(A^*; M)'$ is not a space of distributions but a space of functionals (analytic functionals of Gerver's functionals). Structure theorems for the elements of $D(A^*; M)'$ are given in [4].


If $-A$ is the infinitesimal generator of a semi-group $G(t)$, then the unique solution of the Cauchy problem

(6.1) $Au + u' = 0 \quad \left( u' = \frac{du}{dt} \right)$,

(6.2) $\begin{cases} u(t) \in D(A), \\ u(0) = u_0 \end{cases}$

is given by

(6.3) $u(t) = G(t) u_0$. 

See [2], [10].
Thanks to Theorem 3.2 and its "transposed" version (5.5) we obtain:

**Theorem 6.1.** We assume that (5.2) holds true — For \( u_0 \) given in \( D(A^\infty; M_k) \) (resp. in \( D(A^{*\infty}; M_k)' \)) the Cauchy problem (6.1), (6.2) admits a unique solution, given by (6.3), which is \( C^\infty \) from \( t \geq 0 \to D(A^\infty; M_k) \) (resp. \( D(A^{*\infty}; M_k)' \)). Moreover, in case (3.3) holds true, the solution \( u(t) \) is of class \( M_k \).

Remark 6.1. In case \( G(t) \) is analytic (see Remark 4.1) then, even starting with \( u_0 \in D(A^{*\infty}; M_k)' \) (i.e. with an extremely general Cauchy data), one has \( u(t) \in D(A^\infty; M_k) \forall t > 0 \).

See [4].

7. Some examples.

We take the two as simple as possible cases.

7.1. Heat equation.

Combining results of Sections 4 and 6 we obtain the following result: let \( u_0 \) be given in \( \Omega \), satisfying

\[
\begin{align*}
\text{(7.1)} \quad &u_0 \text{ is of Gevrey order } \beta \text{ (resp. real analytic) in } \Omega, \text{ and } \Delta^k u_0 = 0 \text{ on } \Omega, \\
\text{(7.2)} \quad &-\Delta u + \frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times ]0, \infty[, \\
\text{(7.3)} \quad &u(x, t) = 0 \text{ if } x \in \Gamma, \ t > 0, \\
\text{(7.4)} \quad &u(x, 0) = u_0(x), \ x \in \Omega
\end{align*}
\]

is of Gevrey order \( \beta \) in \( x \) (resp. real analytic if \( \beta = 1 \)) and of Gevrey order \( 2\beta \) in \( t \).

We have just to take: \( M_k = [(2k)!] \) in the general theory.

Moreover in this case Remark 6.1 applies —

7.2. Wave equation.

We consider now

\[
\begin{align*}
\text{(7.5)} \quad &-\Delta u + \frac{\partial^2 u}{\partial t^2} = 0 \text{ in } \Omega \times ]0, \infty[, \\
\text{(7.6)} \quad &u(x, t) = 0 \text{ if } x \in \Gamma, \ t > 0, \\
\text{(7.7)} \quad &\frac{\partial u}{\partial t} \ (x, 0) = u_{01}(x), \ x \in \Omega
\end{align*}
\]
Writing (7.5) as a first order system in $t$ one can apply semi-group theory. One obtains:

$$\text{if } u_0 \text{ and } u_0, \text{ satisfy conditions analogous to (7.1) for } u_0, \text{ then } u(x, t)$$

is of Gevrey order $\beta$ in $x$ and in $t$.


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