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In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 155--159.

Persistent URL: http://dml.cz/dmlcz/700223

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

ARZELA — LIKE THEOREM WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS AND CONTROL THEORY

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§ 1. In numereous problems concerning given differential equations or control systems the sequences of functions

(1.1) $$y = g_i(x), \qquad (i = 1, 2, \ldots)$$

approximating solutions of respective equations for instance one equation

(1.2) $$y' = f(x, y)$$

play an important role.

Usually one starts with equation (1.2) and then determine suitable sequences (1.1).

Our paper deals with the following opposite:

Problem Z. *Given the sequence* (1.1), *such an equation* (1.2) *called "asymptotically inducted by sequence* (1.1)" *is to be found, that the limits of all convergent subsequences* $y = k_i(x)$ *of* (1.1) *satisfy* (1.2). It is evident that the required function $f(x, y)$ can be determined exclusively on the accumulation set L for sequence of curves (1.1). It can happen that L reduces to a single arc.

Problem Z is considered globally.

We shall give conditions for subsequences k_i to converge towards an "*extensive solution*" of (1.1) i.e. solution tending to the boundary of open set W (containing L) at both ends.

Such conditions are given in Theorem A which proved to be very convenient for didactic purposes because of many applications (for instance global existence theorems, continuous dependence on and differentiability in respect to initial values, constructing of approximate solutions).

The curves (1.1) occuring in Theorem A are arcs.

In some applications of the classical Arzela's theorem on equicontinuous sequences (1.1) is of no use because one is obliged to use functions (1.1) with graphs consisting of finite number of points. In this case we replace the notion of equicontinuity by notion of asymptotic smoothness and extensivity (Theorem B).

Theorem C gives a construction of a control system "asymptotically induced" by the sequence (1.1).

§ 2. **Theorem A.** *Let $W \in R^2$ be an open non-empty set. Consider a sequence of real functions $g_i(x)$ defined, continuous and having the right-hand (finite) derivative D_+g_i on open intervals J_i, $(i = 1,2, \dots)$.*

Denote by g_i the graph of $g_i(x)$ and put

$$(2.1) \qquad L = \{(x, y) : (x, y) \in W, \ \liminf_{i\to\infty} r((x, y), g_i) = 0\}$$

where $r((x, y), B)$ denotes the distance of point (x, y) from set B.

Suppose the following implication:
if for any subsequence $g_{k(i)}$

$$(2.2) \qquad (x_i, g_{k(i)}(x_i)) \to (x, y) \in W,$$

then two following conditions (2.3), (2.4) hold:

(2.3) *there exist a neighbourhood of x contained for large i in $J_{k(i)}$,*

(2.4) $\lim D_+ g_{k(i)}(x_i)$ *exists and is finite.*

Under above assumptions:
the limit (2.4) depends on (x, y) only and is independent of the particular choice of the subsequence $g_{k(i)}$.

This limit denoted by $F(x, y)$ is defined and continuous on L [see (2.1)].

(2.5) *For any point $(x_0, y_0) \in L$ there exists such a subsequence $g_{m(i)}$ that* $\lim g_{m(i)}(x) = h(x)$,

where $h(x)$ is an "extensive" solution of the equation

$$(2.6) \qquad y' = F(x, y),$$

i.e. *$h(x)$ is an open arc "tending" at both ends to the boundary of W.*

Moreover if through each point of L passes a unique solution of (2.6) and

$$\lim r((x_0, y_0), g_i) = 0$$

then the original sequence g_i is convergent to the extensive solution passing through (x_0, y_0).

§ **3.** Remark 1. The sequence g_i satisfying the implication (2.2) $\Rightarrow$ (2.3) will be called expansive on W.

The differential equation (2.6) will be called to be "induced by sequence g_i".

This notions will be generalized in the following.

Remark 2. The proof of this theorem can be based on the classical Arzela's Lemma and on the theorem on differentiability of the limit.

Theorem A can serve as the starting point for generalizations. For this purpose we introduce some definitions.

§ 4. Limits restreint, complete and exact in Hausdorff-like sense.

Let $A = \{A_i\}$ be a sequence of sets $A_i \subset P = R^2$. We define

(4.1) $\zeta(A) = \lim \operatorname{restr} A_i = \{z : z \in P, \lim r(z, A_i) = 0\}$,

(4.2) $\eta(A) = \lim \operatorname{compl} A_i = \{z : z \in P, \liminf r(z, A_i) = 0\}$.

If $\zeta(A) = \eta(A)$ we say that A is *H-convergent* and we define

(4.3) $\lambda(A) = \lim \operatorname{exact} A_i = \zeta(A) = \eta(A)$.

(4.4) Proposition. For $A_i \neq 0$ there exists H-convergent subsequence of $\{A_i\}$.

§ 5. Univalent sets and smooth functions.

For $B \subset P$ we define by

B^{I} = projection of B on x-axis,

B^{II} = projection of B on y-axis.

We say that set B is *univalent* if

$$p \in B, \qquad q \in B, \qquad p^{\mathrm{I}} = q^{\mathrm{I}} \Rightarrow p = q.$$

The univalent sets will be considered as functions of variable x. The whole of such functions (or sets) will be denoted by Unival.

Let $f \in$ Unival and put $g = f \cap W$.

f is called *extensive* if g is continuous, g^{I} is open and $(x, g(x))$ tends to the boundary of W, as x tends to the boundary of g^{I}.

f is called smooth if g is continuous and closed in W.

§ 6. Suppose that

1) $g = \{g_i\}$, $g_i \in$ Unival,

2) (x_0, y_0) is an arbitrary point of W,

3) $k = \{k_i\}$ is an arbitrary H-convergent subsequence of g for which $(x_0, y_0) \in \lambda(k)$.

(6.1) g is called *asymptotically smooth in* W if $[x_i \in k_i^{\mathrm{I}},\ x_i \to x_0] \Rightarrow [k_i(x_i) \to y_0]$.

(6.2) g is called *asymptotically extensive in* W if x_0 is an interior point of $[\eta(k)]^{\mathrm{I}}$.

(6.3) g is called *asymptotically lipschitzian in W* if
$[x_i \in k_i^{\mathrm{I}},\ s_i \in k_i^{\mathrm{I}},\ x_i \to x_0,\ s_i \to x_0,\ k_i(x_i) \to y_0] \Rightarrow$
$\Rightarrow [\limsup |(k_i(s_i) - k_i(x_i))/(s_i - x_i)| < +\infty]$.

(6.4) Obviously if g is asymptotically lipschitzian it is asymptotically smooth.

Theorem B. *If $g_i \in$ **Unival**, $g = \{g_i\}$ is asymptotically smooth and asymptotically extensive (in W), $(x_0, y_0) \in \eta(g)$ then there exists such an H-convergent subsequence k of g that $(x_0, y_0) \in \lambda(k)$ and $\lambda(k)$ is univalent, extensive and smooth in W.*

Remark 3. If moreover g is asymptotically lipschitzian then $\lambda(k)$ is locally lipschitzian.

Remark 4. The theorem B generalizes the classical lemma of Arzela on equicontinuous sequences of functions. It is used in the following.

§ 7. For $G \subset P$ we define

Convex $G =$ closed convex hull of G.

For $a \in P$, $b \in P$, $a^{\mathrm{I}} \neq b^{\mathrm{I}}$ we put

$$\text{slope}\,(a, b) = (b^{\mathrm{II}} - a^{\mathrm{II}})/(b^{\mathrm{I}} - a^{\mathrm{I}}).$$

If $f \in$ **Unival**, Q is open, $Q \subset P$, we define

$$\text{slope}\,(f, Q) = \bigcup_{a,b} \text{slope}\,(a, b), \text{ for } a \in f \cap Q,\ b \in f \cap Q.$$

For $g = \{g_i\}$, where $g_i \in$ Unival, g asymptotically lipschitzian in W, $\varnothing \neq Q$ open, we put

$$B(i, Q) = \bigcup_{j/i}^{\infty} \text{slope}\,(g_j, Q).$$

For $(x, y) \in \zeta(g) \cap W$ we define

(7.1) $$C(x, y) = \textbf{Convex}\,[\lim_{i,Q \to (\infty,x,y)} \text{exact}\, B(i, Q)].$$

The relation (7.1) means that for any such sequence of open non-empty sets Q_i that $(x, y) \in Q_i$, diameter $Q_i \to 0$, we have:

$$C(x, y) = \text{Convex}\,[\lim_{i \to \infty} \text{exact}\, B(i, Q_i)].$$

(7.2) Definition. *The contingent condition*

(7.3) $$D^* y(x) \in C(x, y(x)),$$

where D^* denotes the contingent derivative is called contingent equation asymptotically induced by sequence g.

Theorem C. *Suppose that g is asymptotically lipschitzian and asymptotically extensive in W.*

Then $C(x, y)$ is upper semicontinuous (in respect to inclusion) in $\eta(g) \cap W$.

If $(x_0, y_0) \in \eta(g) \cap W$ then there exists such a H-convergent subsequence k of g, that $\lambda(k) \cap W$ is an extensive solution of contingent equation (7.3) *passing through* (x_0, y_0).

(7.4) Remark 5. In our case the classical assumption of Zaremba—Marchaud theory on contingent and paratingent equations are satisfied.

(7.5) Remark 6. The condition (7.3) can be considered as a control system with eliminated control variables.

§ 8. Remark 7. Theorem B of § 6 can be easily reformulated for the case $P = R^m \times R^n$, where m, n are arbitrary positive integers. It can be even generalized for the case $P = H \times V$, H and V being suitable topological spaces.