Giovanni P. Galdi

On the steady translational self-propelled motion of a body in a Navier-Stokes fluid

In: Ravi P. Agarwal and František Neuman and Jaromír Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, [Part 1] Survey papers. Masaryk University, Brno, 1998. CD-ROM. pp. 63--80.

Persistent URL: http://dml.cz/dmlcz/700272

EQUADIFF 9 CD ROM, Brno 1997 PROCEEDINGS
Masaryk University pp. 63–80

On the Steady Translational Self-Propelled Motion of a Body in a Navier-Stokes Fluid

Giovanni P. Galdi

Department of Mathematics & Statistics,
301 Thackeray Hall, University of Pittsburgh,
Pittsburgh 15260, PA U.S.A.
Permanent address: Istituto di Ingegneria, Università di Ferrara,
Ferrara 44100 Italy
Email: galdi@math.pitt.edu

Abstract. There is a large interest, from both mathematical [4], [5], [14], [15] and physical [20], [3] point of view, in the motion of self-propelling bodies in a viscous fluid. This latter means that the body $\mathcal{B}$, say, moves without the action of an external force (like gravity, for instance), but just because of the interaction between its boundary Σ, say, and the fluid. Therefore, Σ serves as the "driver" of $\mathcal{B}$ and the distribution of velocity on Σ as its "thrust". In this paper we shall consider steady, translational self-propelled motion of a body in a Navier-Stokes fluid. In particular, we show the existence of a space $\mathcal{T}(\mathcal{B})$ of velocity distributions on $\mathcal{S}$ with the property that for any given translational velocity V of $\mathcal{B}$ there is one and only one element in $\mathcal{T}(\mathcal{B})$ which can move the body with velocity V. $\mathcal{T}(\mathcal{B})$ depends only on the geometric properties of $\mathcal{B}$ such as size or shape. In particular, it is independent of the orientation of $\mathcal{B}$ and on the fluid property.

AMS Subject Classification. 35Q, 76C

Keywords. Steady-state Navier-Stokes equations, self-propelled body, existence and uniqueness

1 Introduction

A body $\mathcal{B}$ moving in an infinite viscous fluid $\mathcal{F}$ undergoes a *self-propelled motion* if the net total force and torque, external to $\mathcal{B}$ and $\mathcal{F}$, acting on $\mathcal{B}$ are identically zero. Examples of self-propelled motions can be those performed by rockets, submarines, fishes, microorganisms, etc. This type of motion is possible because of the interaction between the boundary of the body Σ, say, and the fluid. Therefore, Σ serves as the "driver" of $\mathcal{B}$ and and the distribution of velocity on Σ as its "thrust".

Since the pioneering work of G. I. Taylor [20] on the propulsion of microscopic organisms, these problems have attracted the attention of many scientists, particularly with the objective of giving a fluid mechanical interpretation of the self-motion of ciliated and flagellated organisms, see, *e.g.* [3], [19], [2] and the bibliography cited therein. It should also be noticed that most of these results are derived under the assumption of zero Reynolds number, that is, Stokes approximation.

In this paper we shall consider steady, translational self-propelled motion of a body $\mathcal{B}$ in a Navier-Stokes fluid $\mathcal{F}$. By this we mean that $\mathcal{B}$ moves in $\mathcal{F}$ by purely translational motion, with constant velocity $-\xi \neq 0$, and that the motion of $\mathcal{F}$, as seen by an observer attached to $\mathcal{B}$, is independent of time. The shape of $\mathcal{B}$ is, of course, independent of time as well. Our goal is to investigate the class of velocity distributions on Σ (the "thrust") which makes $\mathcal{B}$ move with the given velocity $-\xi$. It is simple to see that this problem admits an *infinite* number of solutions corresponding to the same ξ; see Section 2. The objective of this work is to characterize a class of boundary velocities for which the problem admits *one and only one* solution. We shall show, among other things, that there exists a six-dimensional subspace $\mathcal{T}(\mathcal{B})$ of the space $L^2(\Sigma)$ with the following properties. $\mathcal{T}(\mathcal{B})$ depends only on the geometric properties of $\mathcal{B}$ such as size or shape and for any given ξ, there exists one and only one element of $\mathcal{T}(\mathcal{B})$ moving $\mathcal{B}$ with prescribed velocity $-\xi$. We thus give a general answer to a question which was addressed and/or partially solved by several authors. In this regard, we recall the paper of Lugovtsov & Lugovtsov [12] where particular examples are given of flow past a self-propelled body and to the contributions of Sennitskii [16], [17] who, by the method of matched asymptotic expansion, has constructed, for sufficiently small values of the Reynolds number an approximate solution in the case when $\mathcal{B}$ is a cylinder or a sphere, under different prescriptions of boundary velocity.[1] A similar type of question (momentumless flow) for $\mathcal{B}$ of arbitrary shape has been investigated and solved by Pukhnachev [14], [15] within the Stokes approximation. Recently, I have given a general existence and uniqueness theory for the full nonlinear problem, in the particular case when $\mathcal{B}$ has rotational symmetry [6].[2]

The paper is organized as follows. In Section 2 we formulate the problem and introduce some notation. In Section 3, we study the linearized version of the self-motion of $\mathcal{B}$ within the Stokes approximation and furnish, in particular, necessary and sufficient conditions on the distribution of velocity on Σ in order that $\mathcal{B}$ performs a steady, translational flow. These results contain, as a particular case, those of [14], [15] and are of fundamental importance in the investigation of the nonlinear problem which is the object of Section 4. There, we shall show that for any translational motion of $\mathcal{B}$ with velocity -ξ, there exists a uniquely determined velocity distribution on Σ, which lies in a six-dimensional "control" space $\mathcal{T}(\mathcal{B})$, provided $|\xi|$ is not "too large". $\mathcal{T}(\mathcal{B})$ depends only on $\mathcal{B}$. Furthermore, the set

[1] See also [18] for a dynamical counterpart of these problems.

[2] We would like to mention also a series of works aimed at investigating the asymptotic behaviour of the velocity field of the fluid within the wake behind $\mathcal{B}$. We refer, in particular, to the work of Birkhoff and Zarantonello [4], Finn [5] and, more recently, Pukhnachev [13], Kozono and Sohr [10], and Kozono, Sohr and Yamazaki [11].

of all translational motions of $\mathcal{B}$ is set in a one-to-one correspondence with a subspace $\mathcal{T}'(\mathcal{B})$ of $\mathcal{T}(\mathcal{B})$.

An interesting question we leave open is that of the uniqueness of the space $\mathcal{T}'(\mathcal{B})$. In other words, assume there is another space $\tilde{\mathcal{T}}(\mathcal{B})$ with the property that any translational motion of $\mathcal{B}$ determines and is determined by a unique element of $\tilde{\mathcal{T}}(\mathcal{B})$. The question is if and how $\mathcal{T}'(\mathcal{B})$ and $\tilde{\mathcal{T}}(\mathcal{B})$ are related to each other.

Acknowledgment. This paper is part of a keynote lecture I gave at the Conference Equadiff 9, held in Brno in August 1997. I am particularly grateful to Professors F. Neuman and J. Vosmanský for their kind invitation and warm hospitality.

2 Formulation of the Problem

Assume a body $\mathcal{B}$ moves of translational motion, with constant velocity $-\xi$ in a Navier-Stokes fluid. We denote by $\mathcal{D}$ the region occupied by the fluid (the exterior of $\mathcal{B}$) and by Σ the boundary of $\mathcal{B}$. If the fluid performs a time-independent flow, the relevant equations, written in a frame $\mathcal{S}$ attached to $\mathcal{B}$ and in dimensionless form, become[3]

$$\left.\begin{aligned} \Delta v &= \lambda v \cdot \nabla v + \nabla p \\ \operatorname{div} v &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$v = v_* \quad \text{on } \Sigma \tag{2.1}$$
$$\lim_{|x|\to\infty} v(x) = \xi.$$

Here v, p are velocity and pressure field, respectively, associated to the particles of the fluid. Moreover, λ is the dimensionless Reynolds number which has the form LU/ν, where L is a characteristic length (the diameter of $\mathcal{B}$, for example), U is a characteristic speed (the speed of $\mathcal{B}$, in which case ξ is of modulus one) and ν is the kinematical viscosity coefficient of the fluid. We shall now append to these equations the conditions describing that $\mathcal{B}$ is self-propelling. Since $\mathcal{B}$ moves at steady pace, these latter are expressed by the requirement that, relative to an inertial frame, the total momentum flux and moment of momentum flux through Σ balance the total force and total moment of force exerted by the fluid on $\mathcal{B}$, respectively; see [20] pp. 448–449. Reformulating these conditions in the moving frame $\mathcal{S}$, we then obtain

$$\int_\Sigma [-T(v,p)\cdot n + \lambda(v_* - \xi)v_* \cdot n] = 0 \tag{2.2}$$

and

$$\int_\Sigma x \times [-T(v,p)\cdot n + \lambda(v_* - \xi)v_* \cdot n] = 0, \tag{2.3}$$

[3] We suppose that there is no body force acting on the fluid.

where n is the unit inward normal to $\mathcal{B}$ and $T = T(v, p)$ is the stress tensor whose components are given by

$$T_{ij}(v,p) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - p\,\delta_{ij}, \quad i,j = 1,2,3.$$

The main goal of this paper is to investigate the following Problem $\mathcal{P}$: *Given* $\xi \neq 0$*, find a solution to problem* (2.1)–(2.3).

Notice that, unlike the "classical" formulation of the exterior problem, the value of the velocity field at Σ is *not* prescribed.

As stated, it becomes clear that Problem $\mathcal{P}$ always admits the trivial solution $v = \xi, p = \text{const}$ and that, in general, it admits an infinite number of solutions corresponding to the same ξ. Actually, assume $\mathcal{B}$ is a body of revolution around the x_1-axis (say) and let Φ be *any* harmonic function in $\mathcal{D}$ approaching $\xi \cdot x$ at large distances and satisfying the following parity condition

$$\Phi(x_1, x_2, x_3) = \Phi(x_1, -x_2, x_3) = \Phi(x_1, x_2, -x_3).$$

Set $v = \nabla\Phi, p = \frac{1}{2}\left(\nabla\Phi\right)^2$. By a straightforward calculation, one shows that $v,\ p$ is a solution to (2.1), with $v_* = \nabla\Phi\,|_\Sigma$. Moreover, using the parity requirements it is obvious that also condition (2.3) is satisfied. Also, integrating $(2.1)_1$ on the subdomain of $\mathcal{D}$ delimited by the surface Σ and by the surface Σ_R of a ball of radius R centered in $\mathcal{B}$, we find

$$\int_\Sigma (T(v,p)\cdot n - \lambda(v_* - \xi)v_* \cdot n) = \int_{\Sigma_R} (T(v,p)\cdot n - \lambda(v - \xi)v\cdot n). \qquad (2.4)$$

Thus, taking into account that $D^\sigma\Phi(x) = O(|x|^{-1-|\sigma|})$, $|\sigma| = 1, 2$, we let $R \to \infty$ in (2.4), to deduce that also condition (2.2) holds.

This example shows that, in order to preserve uniqueness for Problem $\mathcal{P}$, we must impose some other restrictions on the class of solutions. We shall therefore require that the trace v_* of v at Σ belongs to a suitable "control" space $\mathcal{T}$. Our objective is to determine $\mathcal{T}$ in such a way that Problem $\mathcal{P}$ admits (one and) only one solution. We shall show that this is always the case, provided $\lambda|\xi|$ is not too large.

We shall briefly recall the main notation used in this paper.

$\mathbb{R}^3$ is the three-dimensional Euclidean space and (e_1, e_2, e_3) the canonical orthonormal basis.

Unless otherwise explicitly stated, we shall assume $\mathcal{D}$ sufficiently regular, for instance, of class C^2.

For $\beta = (\beta_1, \beta_2, \beta_3)$, $\beta_i \geq 0$, we set

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}}, \quad |\beta| = \beta_1 + \beta_2 + \beta_3.$$

If $u = \{u_i\}$ is a vector function, by $D(u) = \{D_{ij}(u)\}$ we denote the symmetric part of $\nabla u = \left\{\dfrac{\partial u_i}{\partial x_j}\right\}$, that is

$$D_{ij}(u) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

We shall use standard notations for function spaces, see [1]. So, for instance, $L^q(\mathcal{A})$, $W^{m,q}(\mathcal{A})$, etc., will denote the usual Lebesgue and Sobolev spaces on the domain $\mathcal{A}$, with norms $\|\cdot\|_{q,\mathcal{A}}$ and $\|\cdot\|_{m,q,\mathcal{A}}$, respectively. Whenever confusion will not arise, we shall omit the subscript $\mathcal{A}$. The trace space on $\partial\mathcal{A}$ for functions from $W^{m,q}(\mathcal{A})$ will be denoted by $W^{m-1/q,q}(\partial\mathcal{A})$ and its norm by $\|\cdot\|_{m-1/q,q,\partial\mathcal{A}}$. For other notation we follow [1].

3 The Stokes Approximation

We shall begin to consider the limiting situation of $\lambda \to 0$ of Problem $\mathcal{P}$. If we formally take $\lambda = 0$ into equations (2.1), (2.2), (2.3) we get the following problem

$$\left.\begin{aligned} \Delta v_0 &= \nabla p_0 \\ \operatorname{div} v_0 &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$v_0 = v_{0*} \quad \text{on } \Sigma$$
$$\lim_{|x|\to\infty} v_0(x) = \xi \tag{3.1}$$
$$\int_\Sigma T(v_0,p_0)\cdot n = 0$$
$$\int_\Sigma x \times T(v_0,p_0)\cdot n = 0.$$

We shall show that for any $\xi \neq 0$ there is a unique solution to (3.1) with v_{0*} in a suitable “control” space, see (3.17)

To prove this, we introduce some auxiliary fields, see [9] Chapter 5, [6]. Let $\{h_i,p_i\}$, $\{H_i,P_i\}$, $i = 1,2,3$, be the solutions to the following Stokes problems

$$\left.\begin{aligned} \Delta h^{(i)} &= \nabla p^{(i)} \\ \operatorname{div} h^{(i)} &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$h^{(i)} = e_i \quad \text{on } \Sigma \tag{3.2}$$
$$\lim_{|x|\to\infty} h^{(i)}(x) = 0,$$

and

$$\left.\begin{aligned} \Delta H^{(i)} &= \nabla P^{(i)} \\ \operatorname{div} H^{(i)} &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$H^{(i)} = e_i \times x \quad \text{on } \Sigma \tag{3.3}$$
$$\lim_{|x|\to\infty} H^{(i)}(x) = 0.$$

We also set

$$\begin{aligned} g^{(i)} &:= T(h^{(i)}, p^{(i)}) \cdot n\,|_{\Sigma}\ , \quad i = 1,2,3, \\ G^{(i)} &:= T(H^{(i)}, P^{(i)}) \cdot n\,|_{\Sigma}\ , \quad i = 1,2,3. \end{aligned} \tag{3.4}$$

The vector functions $g^{(i)} = g^{(i)}(x)$ and $G^{(i)} = G^{(i)}(x)$ depend only on the geometric properties of $\mathcal{B}$ such as size or shape. In particular, they do not depend on the orientation of $\mathcal{B}$ and on the fluid property. These functions $g^{(i)} = g^{(i)}(x)$ and $G^{(i)} = G^{(i)}(x)$ will play an important role and, in particular, we are interested in their linear independence properties. In this regard, we have.

Lemma 3.1. *The system of vector functions*

$$\mathsf{S} = \{g^{(i)}, G^{(i)}\}$$

is linearly independent.

Proof. Assume there are scalars γ_i, δ_i, $i = 1,2,3$ such that

$$\gamma_i g^{(i)}(x) + \delta_i G^{(i)}(x) = 0, \quad \text{for all } x \in \Sigma. \tag{3.5}$$

Setting

$$H = \gamma_i h^{(i)} + \delta_i H^{(i)}, \quad P = \gamma_i p^{(i)} + \delta_i P^{(i)}, \tag{3.6}$$

from (3.2), (3.3) we immediately deduce that H, P satisfy the following problem

$$\left.\begin{aligned} \Delta H &= \nabla P \\ \operatorname{div} H &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$\lim_{|x|\to\infty} H(x) = 0 \tag{3.7}$$
$$H(x) = \gamma + \delta \times x, \quad x \in \Sigma.$$

Moreover, condition (3.5) gives

$$T(H,P) \cdot n = 0 \ \text{ at } \Sigma. \tag{3.8}$$

Multiplying (3.7)$_1$ by H, integrating by parts over $\mathcal{D}$, and and using well-known asymptotic properties of solutions to the Stokes problem, [7] Chapter V, we obtain

$$\int_{\mathcal{D}} |D(H)|^2 = \int_{\Sigma} (\gamma + \delta \times x) \cdot T(H, P) \cdot n.$$

In view of (3.8) we then deduce $H(x) = \gamma + \delta \times x$, for all $x \in \mathcal{D}$, and since $H(x)$ vanishes for $|x| \to \infty$ we conclude $\gamma = \delta = 0$, proving the assertion. □

We shall next furnish necessary and sufficient conditions in order that $\mathcal{B}$ performs a steady, translational self-propelled motion within the Stokes approximation with prescribed translational velocity; see (3.15). To this end, we multiply (3.1)$_1$ by $h^{(i)}$ and integrate by parts over $\mathcal{D}$ to find

$$e_i \cdot \int_{\Sigma} T(v_0, p_0) \cdot n = 2 \int_{\mathcal{D}} D(h^{(i)}) : D(v_0), \quad i = 1, 2, 3.$$

Likewise, multiplying (3.2)$_1$ by $v_0 + V_0$ and integrating by parts over $\mathcal{D}$, we obtain

$$\int_{\Sigma} (v_{0*} - \xi) \cdot g^{(i)} = 2 \int_{\mathcal{D}} D(h^{(i)}) : D(v_0), \quad i = 1, 2, 3. \tag{3.9}$$

These two displayed relations then imply

$$\int_{\Sigma} (v_{0*} - \xi) \cdot g^{(i)} = e_i \cdot \int_{\Sigma} T(v_0, p_0) \cdot n. \tag{3.10}$$

In a similar fashion, multiplying (3.1)$_1$ by $H^{(i)}$ and (3.3) by $v_0 - \xi$, respectively, and integrating by parts over $\mathcal{D}$ we find

$$e_i \cdot \int_{\Sigma} x \times T(v_0, p_0) \cdot n = 2 \int_{\mathcal{D}} D(H^{(i)}) : D(v_0), \quad i = 1, 2, 3.$$

and

$$\int_{\Sigma} (v_{0*} - \xi) \cdot G^{(i)} = 2 \int_{\mathcal{D}} D(H^{(i)}) : D(v_0), \quad i = 1, 2, 3 \tag{3.11}$$

which in turn give

$$\int_{\Sigma} (v_{0*} - \xi) \cdot G^{(i)} = e_i \cdot \int_{\Sigma} x \times T(v_0, p_0) \cdot n. \tag{3.12}$$

Let us consider the matrices $(i, j = 1, 2, 3)$, see [9] Chapter 5[4]

$$K_{ij} = -\int_{\Sigma} g_j^{(i)}, \quad C_{ij} = -\int_{\Sigma} \left(x \times g^{(i)}\right)_j, \tag{3.13}$$

[4] Notice that, in general, the matrix C depends on the origin of the axis.

and the vectors

$$\mathcal{V}_i = \int_\Sigma v_{0*} \cdot g^{(i)}, \quad \mathcal{W}_i = \int_\Sigma v_{0*} \cdot G^{(i)}, \quad i = 1, 2, 3, \tag{3.14}$$

from (3.10), (3.12) we find that v_{0*} generates a steady, translational self-propelled motion if and only if the following condition holds

$$\begin{aligned} \mathcal{V} &= K \cdot \xi_0 \\ \mathcal{W} &= C^\dagger \cdot \xi_0 . \end{aligned} \tag{3.15}$$

In view of the linear independence of the system $\mathsf{S} = \{g^{(i)}, G^{(i)}\}$, see Lemma 3.1, we have that, setting

$$M_{ij} = \int_\Sigma g^{(i)} \cdot g^{(j)}, \quad N_{ij} = \int_\Sigma g^{(i)} \cdot G^{(j)}, \quad O_{ij} = \int_\Sigma G^{(i)} \cdot G^{(j)},$$

the 6×6 matrix

$$\begin{pmatrix} M & N \\ N^\dagger & O \end{pmatrix} \tag{3.16}$$

is invertible. Therefore, for any $\xi \in \mathbb{R}^3$ there exists a vector field $v_{0*} = \alpha_i g^{(i)} + \beta_i G^{(i)}$ with uniquely determined α, β satisfying (3.15), (3.14). However, it is also clear from (3.15) that in general it is *not* true that all vectors of the form (3.14) will generate a steady translational flow. This will happen *only* if certain compatibility conditions are satisfied. For example, if $\mathcal{B}$ has spherical symmetry, then one proves [9] Chapter 5, that $C^\dagger \equiv 0$. Consequently, from $(3.15)_2$ it follows that one must prescribe v_* in such a way that $\mathcal{W} = 0$. More generally, (3.15) will admit a solution ξ_0, if the data $\mathcal{V}$ and $\mathcal{W}$ satisfy the compatibility condition $\mathcal{W} = C^\dagger K^{-1} \mathcal{V}$.[5] An *arbitrary* prescription of $\mathcal{V}$ and $\mathcal{W}$ will, in general, produce also a rotation for $\mathcal{B}$, a possibility which is not considered in this paper, and this will be the object of future research.

To state the main results obtained above, it is convenient to introduce the following 6-dimensional subspace of $L^2(\Sigma)$

$$\mathcal{T}(\mathcal{B}) = \left\{ u \in L^2(\Sigma) : \ u = \alpha_i g^{(i)} + \beta_i G^{(i)}, \text{ for some } \alpha, \beta \in \mathbb{R}^3 \right\}. \tag{3.17}$$

As we noticed, $\mathcal{T}(\mathcal{B})$ depends only on the geometric properties of $\mathcal{B}$ such as size or shape. In particular, it is independent of the orientation of $\mathcal{B}$ and on the fluid property.

Taking into account classical existence and uniqueness theorems for the exterior Stokes problem, see [7] Chapter V, we may then summarize the results obtained thus far in the following.

Theorem 3.1. *Let $\mathcal{B}$ have a locally lipschitzian boundary Σ. Then, for any $\xi \in \mathbb{R}^3$ there exists a unique solution v_0, p_0 to problem $(3.1)_{1,2,4,5,6}$ such that the restriction v_{0*} of v_0 to Σ belongs to $\mathcal{T}(\mathcal{B})$.*

[5] The matrix K is always invertible [9] Chapter 5.

4 Existence and Uniqueness for Problem $\mathcal{P}$

To solve Problem $\mathcal{P}$, we shall put it into an equivalent form. For a given $\xi \neq 0$, we set

$$v = u + \xi,$$

and find that $(2.1)_{1,2,4}$ is equivalent to

$$\left.\begin{array}{r} \Delta u + \lambda\xi\cdot\nabla u = \lambda\,\mathrm{div}\,F(u) + \nabla p \\ \mathrm{div}\,u = 0 \end{array}\right\} \text{ in } \mathcal{D}$$
$$\lim_{|x|\to\infty} u(x) = 0 \tag{4.1}$$

while the self-propelling conditions (2.2), (2.3) become

$$\begin{aligned} \int_\Sigma u\cdot g^{(i)} &= \lambda\int_{\mathcal{D}} F : \nabla h^{(i)} + \lambda\xi\cdot\int_{\mathcal{D}} \nabla h^{(i)}\cdot u, \quad i = 1,2,3 \\ \int_\Sigma u\cdot G^{(i)} &= \lambda\int_{\mathcal{D}} F : \nabla H^{(i)} + \lambda\xi\cdot\int_{\mathcal{D}} \nabla H^{(i)}\cdot u, \quad i = 1,2,3. \end{aligned} \tag{4.2}$$

In (4.1), (4.2) we set

$$F(u) := u\otimes u, \tag{4.3}$$

and the vectors $g^{(i)}, G^{(i)}$ defined in (3.4). The identities (4.2) are obtained by multiplying first $(4.1)_1$ by $h^{(i)}$ and $H^{(i)}$, then (3.2), (3.3) by u, integrating by parts and proceeding as in the proof given for the case of the Stokes approximation (see (3.10), (3.12)). If the value of u at the boundary Σ is requested to be in the class $\mathcal{T}(\mathcal{B})$, (4.2) becomes

$$\begin{aligned} \alpha_j M_{ij} + \beta_j N_{ij} &= \lambda\int_{\mathcal{D}} F(u) : \nabla h^{(i)} + \lambda\xi\cdot\int_{\mathcal{D}} \nabla h^{(i)}\cdot u, \quad i = 1,2,3 \\ \alpha_j N_{ji} + \beta_j O_{ij} &= \lambda\int_{\mathcal{D}} F(u) : \nabla H^{(i)} + \lambda\xi\cdot\int_{\mathcal{D}} \nabla H^{(i)}\cdot u, \quad i = 1,2,3. \end{aligned} \tag{4.4}$$

for some $\alpha, \beta \in \mathbb{R}^3$.

A solution to (4.1), (4.3) and (4.4) will be obtained as a fixed point in a suitable Banach space. To this end, for $q \in (1, 3/2)$ we put

$$\langle\!\langle u \rangle\!\rangle_{\lambda,q} := (\lambda|\xi|)^{1/2}\|u\|_{2q/(2-q)} + (\lambda|\xi|)^{1/4}\|u\|_4 + \|u\|_{3q/(3-2q)} + \|D^2 u\|_{2,q} + \|\nabla u\|_2, \tag{4.5}$$

and set

$$\mathcal{X}^q = \left\{\varphi \in L^1_{loc} :\ \langle\!\langle u \rangle\!\rangle_{\lambda,q} < \infty\right\}.$$

Clearly, $\mathcal{X}^q$ is a Banach space. We shall denote by $\mathcal{X}^q_\delta$ the ball of radius δ in $\mathcal{X}^q$.

We next consider the following map

$$\mathcal{N}: \ \varphi \in \mathcal{X}^q_\delta \to (\alpha, \beta) \to u$$

where α, β satisfy the following conditions

$$\begin{aligned} \alpha_j M_{ij} + \beta_j N_{ij} &= \lambda \int_{\mathcal{D}} F(\varphi) : \nabla h^{(i)} + \lambda \xi \cdot \int_{\mathcal{D}} \nabla h^{(i)} \cdot \varphi, \quad i = 1,2,3 \\ \alpha_j N_{ji} + \beta_j O_{ij} &= \lambda \int_{\mathcal{D}} F(\varphi) : \nabla H^{(i)} + \lambda \xi \cdot \int_{\mathcal{D}} \nabla H^{(i)} \cdot \varphi, \quad i = 1,2,3. \end{aligned} \tag{4.6}$$

while u satisfies

$$\left.\begin{aligned} \Delta u + \lambda \xi \cdot \nabla u &= \lambda \operatorname{div} F(\varphi) + \nabla p \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$u = \alpha_j g^{(j)} + \beta_j G^{(j)} \ \text{ at } \Sigma \tag{4.7}$$
$$\lim_{|x|\to\infty} u(x) = 0.$$

We have the following.

Lemma 4.1. *There is a positive constant $C = C(\mathcal{B}, q)$ such that if $\lambda|\xi| < C$, the map $\mathcal{N}$ is a contraction on $\mathcal{X}^q_\delta$, for $\delta = |\xi|$.*

Proof. Applying Theorems VII.7.1 and VII.7.2 of [7] to (4.7), we obtain

$$\langle\!\langle u \rangle\!\rangle_{\lambda,q} + \|\nabla p\|_q \le c\lambda \left[\|\operatorname{div} F(\varphi)\|_q + \|F(\varphi)\|_2 + (|\alpha| + |\beta|)/\lambda\right]. \tag{4.8}$$

Using the Hölder and Sobolev inequalities, we readily deduce

$$\begin{aligned} \|\operatorname{div} F(\varphi)\|_q &\le c\|\varphi\|_{2q/(2-q)}\|\nabla\varphi\|_2 \le c(\lambda|\xi|)^{-1/2}\langle\!\langle \varphi \rangle\!\rangle^2_{\lambda,q} \\ \|F(\varphi)\|_2 &\le c\|\varphi\|_4^2 \le c(\lambda|\xi|)^{-1/2}\langle\!\langle \varphi \rangle\!\rangle^2_{\lambda,q}. \end{aligned} \tag{4.9}$$

The constant c in (4.9) depends only on $\mathcal{B}$, q and C_0 whenever $\lambda|\xi| \le C_0$. Replacing (4.9) into (4.8), we obtain

$$\langle\!\langle u \rangle\!\rangle_{\lambda,q} + \|\nabla p\|_q \le c\left(\lambda^{1/2}|\xi|^{-1/2}\langle\!\langle \varphi \rangle\!\rangle^2_{\lambda,q} + |\alpha| + |\beta|\right). \tag{4.10}$$

Furthermore, recalling that (see [7] Chapter V)

$$\nabla h^{(i)}, \nabla H^{(i)} \in L^{s'}(\mathcal{D}), \quad \text{all } s' > 3/2,$$

by the Hölder inequality we deduce, with $Z^{(i)}$ denoting either $h^{(i)}$ or $H^{(i)}$, $i = 1,2,3$,

$$\left|\int_D F(\varphi) : \nabla Z^{(i)}\right| + \left|\xi \cdot \int_D \nabla Z^{(i)} \cdot \varphi\right| \le c\left((\lambda|\xi|)^{-1/2}\langle\!\langle \varphi \rangle\!\rangle^2_{\lambda,q} + |\xi|\|\nabla Z^{(i)}\|_{s'}\|\varphi\|_s\right),$$

Assume $1 < q < 6/5$, and choose $s' = 2q/(3q-2)$. We then find

$$\left|\int_D F(\varphi) : \nabla Z^{(i)}\right| + \left|\xi \cdot \int_D \nabla Z^{(i)} \cdot \varphi\right| \leq c\left((\lambda|\xi|)^{-1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}^2 + \lambda^{-1/2}|\xi|^{1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}\right). \quad (4.11)$$

Using this latter inequality into (4.6), and recalling that the matrix (3.16) is nonsingular (see Lemma 3.1), we infer for $\lambda|\xi| \leq c$

$$|\alpha| + |\beta| \leq C\left(\lambda^{1/2}|\xi|^{-1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}^2 + \lambda^{1/2}|\xi|^{1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}\right), \quad (4.12)$$

where $C = C(\mathcal{B}, q)$. Collecting (4.8) and (4.12), we arrive at the following inequality

$$\langle\!\langle u \rangle\!\rangle_{\lambda,q} + \|\nabla p\|_q \leq c\left(\lambda^{1/2}|\xi|^{-1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}^2 + \lambda^{1/2}|\xi|^{1/2} \langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}\right). \quad (4.13)$$

If $\varphi \in X_\delta^q$, from (4.13) we obtain

$$\langle\!\langle u \rangle\!\rangle_{\lambda,q} + \|\nabla p\|_q \leq \delta c\left(\lambda^{1/2}|\xi|^{-1/2}\delta + \lambda^{1/2}|\xi|^{1/2}\right),$$

and so, if

$$\lambda|\xi| < (1/2c)^2, \quad (4.14)$$

we may choose

$$\delta = |\xi|, \quad (4.15)$$

and we prove that $\mathcal{N}$ transforms X_δ^q into itself. Once this has been established, it is easy to show that $\mathcal{N}$ is, in fact, a contraction. Actually, setting $\phi = \varphi_2 - \varphi_1$, $\varphi_1, \varphi_2 \in X_\delta^q$, from (4.3) and $(4.13)_2$ we obtain

$$\begin{aligned}\|\operatorname{div}(F(\varphi_2) - F(\varphi_1))\|_q &\leq c\left(\|\varphi_1\|_{2q/(2-q)}\|\nabla\phi\|_2 + \|\phi\|_{2q/(2-q)}\|\nabla\varphi_2\|_2\right) \\ &\leq c(\lambda|\xi|)^{-1/2}\delta\langle\!\langle \phi \rangle\!\rangle_{\lambda,q}\end{aligned}$$

$$\|F(\varphi_2) - F(\varphi_1)\|_2 \leq c\left((\|\varphi_2\|_4 + \|\varphi_1\|_4\right)\|\phi\|_4 \leq c(\lambda|\xi|)^{-1/2}\delta\langle\!\langle \phi \rangle\!\rangle_{\lambda,q}. \quad (4.16)$$

Thus, setting $w = u_2 - u_1 \equiv \mathcal{N}(\varphi_2) - \mathcal{N}(\varphi_2)$, $\pi = p_2 - p_1$, $A = \alpha_2 - \alpha_1$, $B = \beta_2 - \beta_1$ (with the obvious meaning for the symbols), from (4.7), (4.16) and Theorems VII.7.1 and VII.7.2 of [7], we find

$$\langle\!\langle w \rangle\!\rangle_{\lambda,q} + \|\nabla\pi\|_q \leq c\left(\lambda^{1/2}|\xi|^{-1/2}\delta\langle\!\langle \phi \rangle\!\rangle_{\lambda,q} + |A| + |B|\right). \quad (4.17)$$

Moreover, from (4.6), by an argument similar to that leading to (4.12), we obtain

$$|A| + |B| \leq c\lambda^{1/2}|\xi|^{-1/2}\delta\langle\!\langle \varphi \rangle\!\rangle_{\lambda,q}^2 \quad (4.18)$$

which, once replaced into (4.17), furnishes

$$\langle\!\langle w\rangle\!\rangle_{\lambda,q} \le c\lambda^{1/2}|\xi|^{-1/2}\delta\langle\!\langle w\rangle\!\rangle_{\lambda,q}.$$

Therefore, with the choice (4.15), if λ and ξ satisfy conditions of the type (4.14), this latter inequality implies that $\mathcal{N}$ is a contraction on $\mathcal{X}^q_\delta$, and the lemma is proved. □

From the previous lemma, we can obtain an existence result for Problem $\mathcal{P}$. To this end, for $w \in \mathcal{T}(\mathcal{B})$, we set

$$\|w\|_{\mathcal{T}} =: \sum_{i=1}^{3}\left(|\alpha_i| + |\beta_i|\right),$$

where

$$\alpha_i = \int_\Sigma w\cdot g^{(i)},\ \ \beta_i = \int_\Sigma w\cdot G^{(i)},\quad i=1,2,3.$$

We have.

Theorem 4.1. *Let $\xi \ne 0$, be given and let $1 < q < 6/5$. Then, there exists $C = C(\mathcal{B}, q) > 0$, such that if $\lambda|\xi| \le C$, Problem $\mathcal{P}$ admits at least one solution v, p, with $v_* \in \mathcal{T}(\mathcal{B})$ and such that $v, p \in C^\infty(\mathcal{D})$,*

$$(v-\xi) \in L^{2q/(2-q)}(\mathcal{D}),\quad \nabla v \in L^{4q/(4-q)}(\mathcal{D}) \cap L^2(\mathcal{D}),\quad D^2 v \in L^q(\mathcal{D})$$

$$p \in L^{3q/(3-q)}(\mathcal{D}), \nabla p \in L^q(\mathcal{D}).$$

Moreover, the following estimates hold

$$\begin{aligned}\langle\!\langle v-\xi\rangle\!\rangle_{\lambda,q} + \|\nabla p\|_q &\le c_1|\xi| \\ |\xi| \le c_2\|v_*\|_{\mathcal{T}} &\le c_3|\xi|\end{aligned} \tag{4.19}$$

where $\langle\!\langle\ \rangle\!\rangle_{\lambda,q}$ is defined in (4.5), and $c_i = c_i(\mathcal{B}, q)$, $i = 1,2,3$.

Proof. The smoothness of v, p comes from known results on the regularity of solutions to the Navier-Stokes equations in exterior domains, [8] Theorem IX.1.1. Thus, in view of Lemma 4.1, to prove the result completely, it remains to show the second inequality in (4.19). Multiplying $(3.2)_1$ by $v-\xi$ and integrating by parts over $\mathcal{D}$, we find

$$\int_\Sigma (v_*-\xi)\cdot g^{(i)} = 2\int_{\mathcal{D}} D(h^{(i)}) : D(v),\quad i=1,2,3. \tag{4.20}$$

The matrix (3.16) is not singular and so, from (4.20) we find

$$\|v_*\|_{\mathcal{T}} \le c\left(|\xi| + \|\nabla v\|_2\right),$$

which in conjunction with $(4.19)_1$ allows us to conclude

$$\|v_*\|_{\mathcal{T}} \le c|\xi|.$$

Let us now prove the reverse inequality. Since the matrix K defined in (3.13) is nonsingular, [9] Chapter 5, recalling that $u = v - \xi$, from (4.2) we find

$$|\xi| \le c\left(\|v_*\|_{\mathcal{T}} + \lambda \max_i \left| \int_D F(v-\xi) : \nabla h^{(i)} + \xi \cdot \int_D \nabla h^{(i)} \cdot (v-\xi)\right|\right).$$

Using (4.11) in this inequality we deduce

$$|\xi| \le c\left(\|v_*\|_{\mathcal{T}} + \lambda^{1/2}|\xi|^{-1/2}\langle\!\langle v-\xi\rangle\!\rangle^2_{\lambda,q} + \lambda^{1/2}|\xi|^{1/2}\langle\!\langle v-\xi\rangle\!\rangle_{\lambda,q}\right)$$

and so, with the help of $(4.19)_1$, from this latter relation we conclude

$$|\xi| \le c\left(\|v_*\|_{\mathcal{T}} + \lambda^{1/2}|\xi|^{3/2}\right).$$

Choosing $\lambda^{1/2}|\xi|^{1/2}c < 1$ we obtain

$$|\xi| \le c\|v_*\|_{\mathcal{T}}$$

and the proof of the theorem is completed. □

Our next objective is to investigate uniqueness for Problem $\mathcal{P}$. In this regard, we propose the following result whose proof is similar to Theorem 4.2 of [6] and therefore it will be omitted.

Lemma 4.2. *Let v, p be a solution to (2.1), with $\nabla v \in L^2(\mathcal{D})$, corresponding to $\xi \neq 0$ and $v_* \in W^{2-1/q_0,q_0}(\Sigma)$, $q_0 > 3$. Furthermore, let $q \in (1, 3/2)$. Then, there exists a positive constant $\lambda_1 = \lambda_1(\mathcal{B}, q, q_0)$ such that, if*

$$\lambda(\|v_*\|_{2-1/q_0,q_0(\Sigma)} + |\xi|) \le \lambda_1,$$

we have

$$(v-\xi) \in L^{2q/(2-q)}(\mathcal{D}), \quad (v-\xi)(1+|x|) \in L^\infty(\mathcal{D}), \quad \nabla v \in L^{4q/(4-q)}(\mathcal{D})$$

and the following estimate holds

$$(\lambda|\xi|)^{1/2}\|v-\xi\|_{2q/(2-q)} + (\lambda|\xi|)^{1/4}\|\nabla v\|_{4q/(4-q)} + \|\nabla v\|_2 + \|(v-\xi)(1+|x|)\|_\infty \le c(\|v_*\|_{2-1/q_0,q_0,\Sigma} + |\xi|).$$

with $c = c(\mathcal{B}, q, q_0)$.

We are now in a position to prove the following uniqueness result.

Theorem 4.2. *Let $\mathcal{S}_\xi$ be the class of solutions v, p to (2.1)–(2.3) corresponding to a given $\xi \neq 0$, such that*

(i) $\nabla v \in L^2(\mathcal{D})$;

(ii) $v_* \in \mathcal{T}(\mathcal{B})$;

(iii) $\|v_*\|_{\mathcal{T}} \le C_0|\xi|$ *for some* $C_0 > 0$.

Then, there exists $C = C(\mathcal{B}, C_0) > 0$ *such that if* $\lambda|\xi| < C$, $\mathcal{S}_\xi$ *is constituted by at most one element.*

Proof. Let v, p and u, p_1 be two elements of $\mathcal{S}_\xi$ and let

$$U = u - v, \quad \pi = p_1 - p.$$

From (2.1) and (4.2) we obtain

$$\left.\begin{aligned} \Delta U - \lambda\xi\cdot\nabla U &= \lambda\left[U\cdot\nabla u + (v-\xi)\cdot\nabla U\right] + \nabla\pi \\ \operatorname{div} U &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$\lim_{|x|\to\infty} U(x) = 0 \tag{4.21}$$

and $(i = 1,2,3)$

$$\int_\Sigma U_*\cdot g^{(i)} = \lambda\int_D \left[(v-\xi)\cdot\nabla h^{(i)}\cdot U + U\cdot\nabla h^{(i)}\cdot(u-\xi)\right] + \lambda\xi\cdot\int_D \nabla h^{(i)}\cdot U$$
$$\int_\Sigma U_*\cdot G^{(i)} = \lambda\int_D \left[(v-\xi)\cdot\nabla H^{(i)}\cdot U + U\cdot\nabla H^{(i)}\cdot(u-\xi)\right] + \lambda\xi\cdot\int_D \nabla H^{(i)}\cdot U. \tag{4.22}$$

where U_* is the restriction of U at Σ. Set

$$\langle U\rangle_{\lambda,q} \equiv (\lambda|\xi|)^{1/2}\|U\|_{2q/(2-q)} + \|U\|_{3q/(3-2q)} + \|\nabla U\|_2.$$

Applying Theorem VII.7.1 of [7] to (4.21), and using the Hölder inequality, Lemma 4.2 and the assumptions (i)–(iii), we find

$$\begin{aligned} \langle U\rangle_{\lambda,q} &\le c(\lambda\|U\cdot\nabla u + (v-\xi)\cdot\nabla U\|_q + \|U_*\|_{\mathcal{T}}) \\ &\le c\left(\lambda\|U\|_{2q/(2-q)}\|\nabla u\|_2 + \lambda\|v-\xi\|_{2q/(2-q)}\|\nabla U\|_2 + \|U_*\|_{\mathcal{T}}\right) \\ &\le c\left(\lambda^{1/2}|\xi|^{1/2}\langle U\rangle_{\lambda,q} + \|U_*\|_{\mathcal{T}}\right). \end{aligned} \tag{4.23}$$

Furthermore, from (4.22) we obtain, with $Z^{(i)} = h^{(i)}, H^{(i)}$,

$$\begin{aligned} \left|\int_\Sigma U_*\cdot Z^{(i)}\right| &\le \lambda\|\nabla Z^{(i)}\|_{6q/(13q-12)}\|U\|_{3q/(3-2q)}(\|u-\xi\|_{2q/(2-q)} + \\ &\qquad \|v-\xi\|_{2q/(2-q)}) + \lambda|\xi|\,\|\nabla Z^{(i)}\|_{2q/(3q-2)}\|U\|_{2q/(2-q)}. \end{aligned} \tag{4.24}$$

Recalling that $\nabla Z^{(i)} \in L^r(\mathcal{D})$ for all $r > 3/2$, [7], we choose $q \in (1, 12/9)$ and obtain from (4.24) the following inequality

$$\|U_*\|_{\mathcal{T}} \le c\lambda^{1/2}|\xi|^{1/2}\langle U\rangle_{\lambda,q}. \tag{4.25}$$

Replacing (4.25) into (4.23), and taking $\lambda|\xi|$ less than a suitable constant depending only on $\mathcal{B}$ and C_0, we prove uniqueness. □

Remark 4.1. Theorem 4.1 ensures that for any $\xi \neq 0$, the class $\mathcal{S}_\xi$ defined in Theorem 4.2 is not empty.

Theorems 4.1 and 4.2 prove the existence of a map $\mathcal{M}$ from the space T of translational motions of $\mathcal{B}$ onto a subspace $\mathcal{T}'(\mathcal{B})$ of $\mathcal{T}(\mathcal{B})$. We know from the linear theory of Section 3 that $\mathcal{T}'(\mathcal{B})$ is expected to be *strictly* contained into $\mathcal{T}(\mathcal{B})$, due to the fact that a velocity distribution in $\mathcal{T}(\mathcal{B})/\mathcal{T}'(\mathcal{B})$ will produce, in general, also a rotation for $\mathcal{B}$. However, we shall show in the next theorem that the map $\mathcal{M}$ is in fact one-to-one on $\mathcal{T}'(\mathcal{B})$.

Theorem 4.3. *Let v, p and v_1, p_1 be two solutions to Problem $\mathcal{P}$, as given in Theorem 4.1, corresponding to ξ and ξ_1, respectively, with $\xi \neq \xi_1$. Let v_* and v_{1*} be their restrictions at Σ. Then, there exists a positive constant $C = C(\mathcal{B})$ such that if*

$$\lambda|\xi| < C, \tag{4.26}$$

necessarily $v_* \not\equiv v_{1*}$

Proof. Assume, by contradiction, $v_* \equiv v_{1*}$, and let

$$u = v - \xi, \quad u_1 = v_1 - \xi_1, \quad \mu = \xi - \xi_1$$
$$U = u - u_1, \quad \pi = p - p_1$$

We then obtain

$$\left.\begin{aligned} \Delta U - \lambda\xi \cdot \nabla U &= \lambda\left(\mu \cdot \nabla u_1 + U \cdot \nabla u + u_1 \cdot \nabla U\right) + \nabla\pi \\ \operatorname{div} U &= 0 \end{aligned}\right\} \text{ in } \mathcal{D}$$
$$U = -\mu \quad \text{on } \Sigma \tag{4.27}$$
$$\lim_{|x|\to\infty} U(x) = 0.$$

Moreover, from $(4.1)_1$, we find that

$$\mu \cdot \int_\Sigma g^{(i)} = \lambda \int_{\mathcal{D}} U \cdot \nabla h^{(i)} \cdot u + \lambda \int_{\mathcal{D}} u_1 \cdot \nabla h^{(i)} \cdot U + \lambda\xi \cdot \int_{\mathcal{D}} \nabla U \cdot h^{(i)} + \lambda\mu \cdot \int_{\mathcal{D}} \nabla h^{(i)} \cdot u_1, \quad i = 1, 2, 3. \tag{4.28}$$

Applying Theorem VII.7.1 of [7] to (4.27) we find for $q \in (1, 3/2)$

$$\begin{aligned} &\lambda\|\xi \cdot \nabla U\|_q + (\lambda|\xi|)^{1/2}\|U\|_{2q/(2-q)} + (\lambda|\xi|)^{1/4}\|\nabla U\|_{4q/(4-q)} + \|\nabla U\|_{3q/(3-q)} \\ &\leq c\lambda\left(|\mu|\|\nabla u_1\|_q + \|U\|_{2q/(2-q)}\|\nabla u\|_2 + \|u_1\|_4\|\nabla U\|_{4q/(4-q)} + \frac{|\mu|}{\lambda}\right). \end{aligned} \tag{4.29}$$

From Lemma 4.2 and $(4.19)_2$, we have

$$\|u_1\|_4 + \|\nabla u\|_2 \le c\|v_*\|_{\mathcal{T}},$$

and, for $q \in (4/3, 3/2)$,

$$\|\nabla u_1\|_q \le c(\lambda\|v_*\|_{\mathcal{T}})^{-1/4}\|v_*\|_{\mathcal{T}}.$$

Thus, from (4.29) and $(4.19)_2$ we recover the following inequality

$$\begin{aligned}\lambda\|\xi\cdot\nabla U\|_q + (\lambda\|v_*\|_{\mathcal{T}})^{1/2}\|U\|_{2q/(2-q)} + (\lambda\|v_*\|_{\mathcal{T}})^{1/4}\|\nabla U\|_{4q/(4-q)} +\\ \|\nabla U\|_{3q/(3-q)} \le c\lambda\Big[|\mu|(\lambda\|v_*\|_{\mathcal{T}})^{-1/4}\|v_*\|_{\mathcal{T}} +\\ \|v_*\|_{\mathcal{T}}\left(\|U\|_{2q/(2-q)} + \|\nabla U\|_{4q/(4-q)}\right) + |\mu|/\lambda\Big]\,. \end{aligned}\tag{4.30}$$

Moreover, from the Hölder inequality, we also obtain

$$\begin{aligned}\left|\int_{\mathcal{D}} U\cdot\nabla h^{(i)}\cdot u\right| + \left|\int_{\mathcal{D}} u_1\cdot\nabla h^{(i)}\cdot U\right| \le\\ \|U\|_{2q/(2-q)}\|\nabla h^{(i)}\|_2\left(\|u_1\|_{q/(q-1)} + \|u\|_{q/(q-1)}\right)\end{aligned}$$

and, since by Lemma 4.2 and $(4.19)_2$, for $q < 3/2$ it is

$$\|u_1\|_{q/(q-1)} + \|u\|_{q/(q-1)} \le c\left(\|u_1(1+|x|)\|_\infty + \|u(1+|x|)\|_\infty\right) \le c\|v_*\|_{\mathcal{T}},$$

we obtain

$$\left|\int_{\mathcal{D}} U\cdot\nabla h^{(i)}\cdot u\right| + \left|\int_{\mathcal{D}} u_1\cdot\nabla h^{(i)}\cdot U\right| \le c\|U\|_{2q/(2-q)}\|v_*\|_{\mathcal{T}} \tag{4.31}$$

Also, again from Lemma 4.2 and $(4.19)_2$, for $s \in (2,3)$ we find

$$\lambda\left|\mu\cdot\int_{\mathcal{D}}\nabla h^{(i)}\cdot u_1\right| \le c\lambda|\mu|\|u\|_s\|\nabla h^{(i)}\|_{s/(s-1)} \le c|\mu|(\lambda\|v_*\|_{\mathcal{T}})^{1/2}. \tag{4.32}$$

Finally, for $q < q_1 < 3/2$ we have

$$\left|\int_{\mathcal{D}} \xi\cdot\nabla U\cdot h^{(i)}\right| \le \|\xi\cdot\nabla U\|_{q_1}\|h^{(i)}\|_{q_1/(q_1-1)} \le C\|\xi\cdot\nabla U\|_{q_1},$$

and, by the convexity inequality,

$$\|\xi\cdot\nabla U\|_{q_1} \le \|\xi\cdot\nabla U\|_q^\theta\|\xi\cdot\nabla U\|_{3q/(3-q)}^{1-\theta}, \quad \theta\in(0,1).$$

Thus, by $(4.19)_2$ and Young's inequality we deduce

$$\begin{aligned}\lambda\left|\int_{\mathcal{D}}\xi\cdot\nabla U\cdot h^{(i)}\right| &\le c(\lambda|\xi|)^{1-\theta}\left(\lambda^\theta\|\xi\cdot\nabla U\|_q^\theta\|\nabla U\|_{3q/(3-q)}^{1-\theta}\right)\\ &\le c(\lambda\|v_*\|_{\mathcal{T}})^{1-\theta}\left(\lambda\|\xi\cdot\nabla U\|_q + \|\nabla U\|_{3q/(3-q)}\right).\end{aligned}\tag{4.33}$$

Collecting (4.31)–(4.33) and using (4.28), we find that there exists a constant $C = C(\mathcal{B}) > 0$ such that if (4.26) holds then

$$|\mu| \le c \left[\lambda \|v_*\|_{\mathcal{T}} \|U\|_{2q/(2-q)} + (\lambda \|v_*\|_{\mathcal{T}})^{1-\theta} \left(\lambda \|\xi \cdot \nabla U\|_q + \|\nabla U\|_{3q/(3-q)}\right)\right].$$

Replacing this inequality into (4.30), it is immediate to show that there exists a positive constant C depending only on $\mathcal{B}$ such that if (4.26) is satisfied, we then get

$$(\lambda \|v_*\|_{\mathcal{T}})^{1/2} \|U\|_{2q/(2-q)} + (\lambda \|v_*\|_{\mathcal{T}})^{1/4} \|\nabla U\|_{4q/(4-q)} \le 0,$$

which implies, in particular, $\xi = \xi_1$, which contradicts the assumption. The theorem is completely proved. □

References

1. Adams, R. A. 1975, *Sobolev Spaces*, Academic Press, New York
2. Blake, J. R., and Otto, S. R., 1996, Ciliary Propulsion, Chaotic Filtration and a 'Blinking' Stokeslet, *J. Engineering Math.*, **30**, 151–168
3. Brennen, C., and Winet, H., 1977, Fluid Mechanics of Propulsion by Cilia and Flagella, *Ann. Rev. Fluid Mech.*, **9**, 339–398
4. Birkhoff, G., and Zarantonello, E. H., 1957, *Jets, Wakes, and Cavities,* Academic Press.
5. Finn, R., 1965, On the Exterior Stationary Problem for the Navier-Stokes Equations, and Associated Perturbation Problems, *Arch. Rational Mech. Anal.,* **19**, 363–406
6. Galdi, G. P., 1998, On the Steady, Translational Self-Propelled Motion of a Symmetric Body in a Navier-Stokes Fluid, *Quaderni di Matematica della II Università di Napoli*, Vol I, to be published
7. Galdi, G. P., 1994, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 38, Springer-Verlag
8. Galdi, G. P., 1994, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Nonlinear Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 39, Springer-Verlag
9. Happel, V., and Brenner, H., 1965, *Low Reynolds Number Hydrodynamics*, Prentice Hall.
10. Kozono, H., and Sohr, H., 1993, On Stationary Navier-Stokes Equations in Unbounded Domains, *Ricerche Mat.*, **42**, 69–86.
11. Kozono, H., Sohr, H., and Yamazaki, M., 1997, Representation Formula, Net Force and Energy Relation to the Stationary Navier-Stokes Equations in 3-Dimensional Exterior Domains, *Kyushu J. Math.*, **51**, 239–260.
12. Lugovtsov, A. A., and Lugovtsov, B. A., 1971, Example of a Viscous Incompressible Flow Past a Body with Moving Boundary, *Dynamics of Continuous Media*, Novosibirsk, **8**, 49–55 (in Russian)
13. Pukhnachev, V. V., 1989, Asymptotics of a Velocity Field at Considerable Distances From a Self-Propelled Body, *J. Appl. Mech. Tech. Phys.*, **30**, 52–60.
14. Pukhnachev, V. V., 1990, Stokes Approximation in a Problem of the Flow Around a Self-Propelled Body, *Boundary Value Problems in Mathematical Physics*, Naukova Dumka, Kiev, 65–73 (in Russian)

15. Pukhnachev, V. V., 1990, The Problem of Momentumless Flow for the Navier-Stokes Equations, *Springer Lecture Notes in Mathematics*, **1431**, Springer-Verlag, 87–94
16. Sennitskii, V. L., 1978, Liquid Flow Around a Self-Propelled Body, *J. Appl. Mech. Tech. Phys.*, **3**, 15–27
17. Sennitskii, V. L., 1984, An Example of Axisymmetric Fluid Flow Around a Self-Propelled Body, *J. Appl. Mech. Tech. Phys.*, **25**, 526–530
18. Sennitskii, V. L., 1990, Self-Propulsion of a Body in a Fluid, *J. Appl. Mech. Tech. Phys.*, **31**, 266–272
19. Shapere, A. and Wilczek, F., 1989, Geometry of Self-Propulsion at Low Reynolds Number, *J. Fluid Mech.*, **198**, 557–585
20. Taylor, G.I., 1951, Analysis of the Swimming of Microscopic Organisms, *Proc. Royal Soc. London A,* **209**, 447–461.