Jan Franců

Homogenization of scalar hysteresis operators

In: Zuzana Došlá and Jaromír Kuben and Jaromír Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, [Part 3] Papers. Masaryk University, Brno, 1998. CD-ROM. pp. 111--122.

Persistent URL: http://dml.cz/dmlcz/700280

EQUADIFF 9 CD ROM, Brno 1997 PAPERS
Masaryk University pp. 111–122

Homogenization of Scalar Hysteresis Operators

Jan Franců*

Department of Mathematics. Technical University Brno
Technická 2,
616 96 Brno, Czech Republic
Email: francu@fme.vutbr.cz

Abstract. Scalar hysteresis operators of Ishlinskii stop type are studied. Homogenization problem for hyperbolic equation with hysteresis operator is formulated. Formula for the homogenized operator is derived.

AMS Subject Classification. 35B27, 73B27, 73E05

Keywords. Scalar hysteresis operators, Ishlinskii operator, homogenization

Introduction

The title of the contribution contains words homogenization and hysteresis. Homogenization is a mathematical method used in modelling composite materials with periodic structure. It consists in replacing the heterogeneous material modelled by equations with periodic coefficients with an equivalent homogeneous material modelled by constant coefficients, see e. g. [1], [2], [3], [4] and many others.

Hysteresis is one of nonlinear phenomena that appears in evolutionary nonlinear problems of continuum mechanics, see e. g. [5], [6], [7], [8], [9], [10]. The basic feature of hysteresis behavior is a memory effect and irreversibility of the process, the response of the material by loading differs from the response by unloading. The behavior of the material is well characterized by the *hysteresis loop*. In mechanics hysteresis operators model plastic deformation.

We shall deal with homogenization of a one-dimensional boundary value problem, that can be interpreted as a vibration of a plastic rod. We assume that all quantities depend on length variable $x \in \mathbb{R}$ only. The plasticity of the rod is modelled by Ishlinskii hysteresis operator based on the stop operators. Existence of the solutions of these problems was proved e. g. in [10]. The proof needs properties of stop and play operators and Ishlinskii operators of stop and play type studied in Section 2 and 3.

* This research has been supported by grant No. 201/97/0153 of Grant Agency of Czech Republic.

This is the preliminary version of the paper.

The plan of the contribution is the following. In Section 1 we start with basic elements used in modelling of materials. Section 2 deals with the stop $\mathcal{S}_h$ and the play $\mathcal{P}_h$ operators and their properties. Combination in parallel of the stop or the play operators leads to Ishlinskii operator studied in Section 3. One-dimensional homogenization problem is introduced in Section 4. The last Section 5 deals with form of the homogenized operator. The complete proof and the convergence of the solutions is subject of future research.

1 Basic elements

In this section we briefly recall basic one-dimensional models of deformation of solid materials. We consider a one-dimensional homogeneous solid (a homogeneous rod of unit length and of uniform small cross-section). We assume a uniform (independent of place) strain e caused by loading. The strain corresponds to a displacement $u(x)$ satisfying $e = u_x (\equiv \mathrm{d}u/\mathrm{d}x)$ or equivalently $u(x) = e \cdot x$. Thus fixing one end $u(0) = 0$ the displacement in the second end $x = 1$ corresponds to the strain: $e = u(1)$. The response of the material is the stress σ. It is also supposed to be uniform in the rod. The material is described by the constitutive relation between $\sigma(t)$ and $e(t)$ represented by the stress-strain σ-e (or strain-stress e-σ) diagram. The behavior is often modelled by a mechanical device (string, pipe, friction element etc.). Piston-in-cylinder element represents a geometric model.

Elastic element $\mathcal{E}$

The elastic element means linear stress-strain relation

$$\sigma = A \cdot e \,, \tag{1}$$

where A is the elasticity constant (Young modulus of elasticity). In this model the stress depends on instant value of strain independently on preceding course of deformation. The elastic element is modelled by a string and its graph in the stress-strain diagram forms a line crossing the origin with slope A.

Rigid-plastic element $\mathcal{R}$

The element is characterized by a parameter r ($r > 0$). Three cases occur: either the stress σ is inside the interval $(-r, r)$, then the strain e does not change, or the stress σ reach r, then e can grow, or $\sigma = -r$ then e may decrease. With cyclic loading and deloading we obtain a rectangular hysteresis loop.

The element can be modelled by a mechanical system with friction, the parameter r of the model represents the friction coefficient of the Coulomb friction law. The behavior can be also described by a variational inequality

$$\sigma(t) \in [-h, h] \,, \quad \text{such that} \quad \frac{\mathrm{d}e(t)}{\mathrm{d}t}(\sigma(t) - \widetilde{\sigma}) \geq 0 \qquad \forall\, \widetilde{\sigma} \in [-h, h] \,. \tag{2}$$

Other basic elements

In material modelling also other elements are used, e. g. viscous and brittle element, but they do not correspond to our framework of behaviour of plastic materials.

Combination of elements

The elements can be combined either in series or in parallel or in more complicated systems. Let us consider n elements and denote their strains by e_i, their stresses by σ_i, $i = 1, 2, \ldots, n$, the total strain by e and the total stress of the system by σ. Then in case of combination in parallel the deformation is common and the stresses are added:

$$e = e_1 = e_2 = \cdots = e_n\,, \qquad \sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n \tag{3}$$

while in case of combination in series the stress is common and the deformations are added:

$$e = e_1 + e_2 + \cdots + e_n\,, \qquad \sigma = \sigma_1 = \sigma_2 = \cdots = \sigma_n\,. \tag{4}$$

Many other combinations are used, we shall deal with the following serial elasto-plastic combination.

Elasto-plastic element.

Combining elastic and rigid-plastic elements in series $\mathcal{E} - \mathcal{R}$ we obtain an elasto-plastic element. Its hysteresis loop is similar to that of plastic element, only its vertical segments are slanted, the slope reflects the elasticity constant.

Geometric model: Piston-in-cylinder

Piston-in-cylinder model represents a geometric equivalent of elasto-plastic element. Let us consider a cylinder of length $2h$ with a piston moving inside the cylinder. Denoting the input — the absolute (with respect to the coordinate system) position of the piston by u, the relative position of the piston with respect to the cylinder by $\mathcal{S}_h$ and the absolute position of the cylinder by $\mathcal{P}_h$. Clearly, $u = \mathcal{P}_h + \mathcal{S}_h$. The relative position of the piston is proportional to the stress $\sigma = \eta \cdot \mathcal{S}_h$.

Let us consider an increasing deformation $e(t)$. The piston is moving in the cylinder, $\mathcal{S}_h(t)$ which is proportional to the stress $\sigma(t)$ is increasing (elastic deformation). If the piston reaches the end of the cylinder and $e(t)$ is still increasing, then $\mathcal{S}_h(t)$ and the stress $\sigma(t)$ remain constant but the cylinder $\mathcal{P}_h(t)$ starts to move (plastic deformation). Now, if the $e(t)$ starts decreasing, the piston starts moving backwards. In the beginning the piston is moving in the cylinder, $\mathcal{S}_h(t)$ (and the stress $\sigma(t)$) is decreasing (elastic deformation) down to the opposite end of the cylinder when again $\mathcal{S}_h(t)$ stops and $\mathcal{P}_h(t)$ starts to decrease (plastic deformation). The position of the cylinder $\mathcal{P}_h(t)$ describes an internal state yielding memory effects.

Let us remark the degenerated cases. For $h = \infty$ the stop operator converts to identity: $\mathcal{S}_\infty = \mathrm{I}$ and for $h = 0$ the play operator becomes identity: $\mathcal{P}_0 = \mathrm{I}$; in both cases the operators degenerate to the elastic element with $A = 1$.

2 Stop and Play operator

The introduced piston-in-cylinder model yields two operators $\mathcal{S}_h$ called the stop and $\mathcal{P}_h$ the play. They form the base for more complicated models. The operators are functional, they map a function to a function. They have a parameter h — it corresponds to the half length of the cylinder. Besides the input function $u(t)$ we need to set the initial states s_h, p_h — the value of the operators in the initial time. The operators are complementary, i. e. they are connected by the relation

$$(\mathcal{S}_h u)(t) + (\mathcal{P}_h u)(t) = u(t) \quad \forall\, u, \quad \forall\, t \tag{5}$$

and satisfy

$$|(\mathcal{S}_h u)(t)| \le h \quad \text{and} \quad |(\mathcal{P}_h u)(t) - u(t)| \le h. \tag{6}$$

The initial states $s_h, p_h \in \mathbb{R}$ are supposed to satisfy the compatibility conditions in the initial time $t = a$ analogous to (5), (6):

$$s_h + p_h = u(a) \qquad \text{and} \qquad |s_h| \le h\,, \qquad |p_h - u(a)| \le h\,,\,. \tag{7}$$

If no initial values are stated we can assigns to the input $u(t)$ the natural initial values s_h for the stop operator $\mathcal{S}_h$

$$s_h = \begin{cases} h & \text{if } u(a) \ge h \\ u(a) & \text{if } |u(a)| < h \\ -h & \text{if } u(a) \le -h \end{cases} \tag{8}$$

and similarly the natural initial value p_h for the play operator $\mathcal{P}_h$

$$p_h = \begin{cases} u(a) - h & \text{if } u(a) \ge h \\ 0 & \text{if } |u(a)| < h \\ u(a) + h & \text{if } u(a) \le -h \end{cases}\,. \tag{9}$$

The operators are introduced by preceding geometric model, nevertheless let us give their exact definition.

Max-min definition

The definition, see e. g. [5], [8] starts with defining the operator for piecewise monotone functions, then it is extended to all continuous functions:

Let $I = [a, b]$ be a time interval and u a continuous piecewise monotone function on I. Let us denote by t_i the turning points of u i. e. $a = t_0 < t_1 < \cdots < t_k = b$ and the continuous input u is monotone (non-decreasing or non-increasing) on each subinterval $I_1, \ldots, I_n$ ($I_i = (t_{i-1}, t_i]$). Moreover we may assume that in each t_i $(0 < i < n)$ the function u changes from non-decreasing to non-increasing or the other way round.

Then the value of the stop operator $\mathcal{S}_h$ is function $\mathcal{S}_h u : I \to \mathbb{R}$ defined by

$$(\mathcal{S}_h u)(t) = \begin{cases} s_h & \text{for } t = a\,, \\ \min\{(S_h u)(t_{i-1}) + u(t) - u(t_{i-1}), h\} & \text{for } t \in I_i \text{ if } u \text{ is non-decreasing on } I_i\,, \\ \max\{(S_h u)(t_{i-1}) + u(t) - u(t_{i-1}), -h\} & \text{for } t \in I_i \text{ if } u \text{ is non-increasing on } I_i\,. \end{cases} \tag{10}$$

Since the *play operator* is complementary it can be defined either by a similar max-min definition or simply by

$$(\mathcal{P}_h u)(t) = u(t) - (\mathcal{S}_h u)(t)\,. \tag{11}$$

Since both operators are Lipschitz continuous, see Lemma 2, the definition can be extended to continuous functions by continuity.

Definition by variational inequality

The stop and play operator can be also introduced by a variational inequality, similar to (4), see [10], [9].

Let $u \in W^{1,1}(I)$ be an input function on the interval $I = [a, b]$ and s_h the initial state satisfying $|s_h| < h$. Let $x(t)$ be a solution of the following problem:

$$\textit{Find } x(t) \in W^{1,1}(I) \textit{ such that:} \quad x(t) \in [-h, h]\,, \qquad x(a) = s_h(a)\,,$$
$$\textit{and for almost all } \; t \in I \; \textit{ the following inequality holds}$$
$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t) - \frac{\mathrm{d}x}{\mathrm{d}t}(t)\right)(x(t) - \widetilde{x}) \geq 0 \quad \forall \widetilde{x} \in [-h, h]\,. \tag{12}$$

Since the problem admits unique solution $x(t)$ we use it to definition of the stop operator $\mathcal{S}_h$ and play operator $\mathcal{P}_h$:

$$(\mathcal{S}_h u)(t) = x(t)\,, \qquad (\mathcal{P}_h u)(t) = u(t) - x(t)\,. \tag{13}$$

It can be proved that both ways of introducing the play and the stop operators define the same operators.

Inverse operators

Both operators $\mathcal{S}$ and $\mathcal{P}$ are not invertible, since they are not injective. Adding a multiple of identity denoted by I the operators become injective and invertible: inverse of the stop type operator is the play type operator and vice versa.

Lemma 1. *Let a, b, c, d, h, k be positive constants. Then we have the following relations*

$$(a\,\mathrm{I} + b\,\mathcal{S}_h)^{-1} = c\,\mathrm{I} + d\,\mathcal{P}_k\,, \tag{14}$$

where

$$c = \frac{1}{a+b}\,, \qquad d = \frac{1}{a} - \frac{1}{a+b}\,, \qquad k = (a+b)h\,,$$

and conversely

$$(c\,\mathrm{I} + d\,\mathcal{P}_k)^{-1} = a\,\mathrm{I} + b\,\mathcal{S}_h\,, \tag{15}$$

where

$$a = \frac{1}{c+d}\,, \qquad b = \frac{1}{c} - \frac{1}{c+d}\,, \qquad h = ck\,.$$

Scaling, continuity and monotony properties

Due to definition we have the following dependence on the parameter h:

$$\mathcal{S}_h u = h\,\mathcal{S}_1\left(\frac{u}{h}\right)\,, \qquad \mathcal{P}_h u = h\,\mathcal{P}_1\left(\frac{u}{h}\right)\,. \tag{16}$$

The stop and the play operators are Lipschitz continuous, see e. g. [5], [8]:

Lemma 2. *Let $u_1(t), u_2(t)$ be two inputs on $I = [a, b]$ with the same initial values s_h, p_h. Then for all $t \in (a, b]$ we have*

$$\|(\mathcal{S}_h u_1)(t) - (\mathcal{S}_h u_2)(t)| \le 2 \max_{s \le t} |u_1(s) - u_2(s)| \tag{17}$$

$$\|(\mathcal{P}_h u_1)(t) - (\mathcal{P}_h u_2)(t)| \le \max_{s \le t} |u_1(s) - u_2(s)| \tag{18}$$

The operators conserve monotony properties, non-decreasing inputs yields non-decreasing output and non-increasing input yields non-increasing output. The properties are formulated in [5], [8], [9].

3 Ishlinskii operators

Simple stop $\mathcal{S}_h$ and play $\mathcal{P}_h$ operators yield hysteresis loops consisting of segments with two slopes. By weighted parallel finite or infinite or continuum combination of stop (or play) operators (including a multiple of identity I) we obtain more general hysteresis loops.

Stop type Ishlinskii operators

Let us consider a parallel weighted combination of n different stop operators, in physical setting elasto-plastic elements. Each element has two parameters: h_i half-length of the cylinder and η_i "stiffness" of the element, it describes stress contribution of the element $\sigma_i = \eta_i \cdot f_h$. Thus the combination is determined by a family of n pairs (h_i, η_i) $(i = 1, 2, \ldots, n)$, ordered $0 < h_1 < h_2 < \cdots < h_n \leq \infty$, and $\eta_i \geq 0$. Admitting the case $h_n = \infty$ we include a multiple of identity $\eta_n \, \mathrm{I}$ — simple elastic element combined in parallel. Let us consider an input $e(t)$. Then the output stress is

$$\sigma(t) = (\mathcal{F}(e))\,(t) \equiv \sum_{i=1}^{n} \eta_i \left(\mathcal{S}_{h_i} e\right)(t) \,. \tag{19}$$

The operator $\mathcal{F}$ is called *discrete Ishlinskii operator*. It has to be completed by the initial values of stop operators s_{h_i}. They can be given or set by (8) in case of intact initial state.

The stress-strain diagram to the increasing loading $e(t)$ from the intact state of this material forms a concave non-decreasing function Φ — another characterization of the hysteresis behavior called *virgin curve*.

Using (19) we can compute the function Φ. Put the deformation $e(r) = r$ for $r \in [0, \infty)$ with $s_{h_i} = 0$. Then we have

$$\Phi(r) = \sum_{i} \eta_i \mathcal{S}_{h_i}(r) = \sum_{h_i \leq r} \eta_i \mathcal{S}_{h_i}(r) + \sum_{h_i > r} \eta_i \mathcal{S}_{h_i}(r) \,.$$

Since in course of this loading $\mathcal{S}_h(r) = h$ for $h \leq r$ and $\mathcal{S}_h(r) = r$ for $h > r$ we obtain

$$\Phi(r) = \sum_{h_i \leq r} \eta_i h_i + r \sum_{h_i > r} \eta_i \,. \tag{20}$$

The function $\Phi(r)$ is clearly concave on $[0, \infty)$. Considering the negative loading from the intact state the function $\Phi(r)$ can be extended to negative values to an odd function by $\Phi(r) = -\Phi(-r)$.

The Ishlinskii operator represents a generalization of parallel combination of elastic-plastic elements to the case of infinite elements with continuously increasing parameter h. The family of pairs (h_i, η_i) is replaced by a non-negative "stiffness" function $\eta(h)$. Further we replace the sum in (21) by an integral (possible η_∞ remains) and we obtain the *Ishlinskii* operator $\mathcal{F}$

$$\sigma(t) = (\mathcal{F}(e))\,(t) \equiv \int_0^\infty \eta(h)\, \left(\mathcal{S}_h e\right)(t) \mathrm{d}h + \eta_\infty e(t) \,. \tag{21}$$

The operator $\mathcal{F}(e)$ should be completed by the initial values s_h that can be given or defined by (8).

In the stress-strain diagram the cyclic loading has concave increasing arches and convex decreasing arches.

Introducing a function $\mu(h)$ by

$$\mu(h) = \int_0^h \eta(r)\mathrm{d}r \qquad \text{or} \qquad \mu'(h) = \eta(h)\,, \quad \mu(0) = 0 \tag{22}$$

we can replace the Riemann integral by Stieltjes integral

$$\sigma(t) = (\mathcal{F}(e))\,(t) \equiv \int_0^\infty (\mathcal{S}_h e)\,(t)\mathrm{d}\mu(h) + \eta_\infty e(t)\,. \tag{23}$$

The advantage of the formula is the fact that it includes even the discrete case of the Ishlinskii operator (19) taking piecewise constant function

$$\mu(r) = \sum_{h_i<r} \eta_i\,. \tag{24}$$

Characteristics of the Ishlinskii operator

Besides the function $\eta(h)$ or $\mu(h)$ the Ishlinskii operator can be characterized by the function $\varPhi$ of the stress-strain diagram of monotone loading from intact state. Using $\mathcal{S}_h r = h$ for $h \le r$ and $\mathcal{S}_h(r) = r$ for $h > r$ from (21) or (23) we can compute the function $\varPhi$ by means of η_∞ and $\eta(h)$ or $\mu(h)$:

$$\varPhi(r) = \int_0^r h\eta(h)\mathrm{d}h + r\int_r^\infty \eta(h)\mathrm{d}h + \eta_\infty r\,, \tag{25}$$

$$\varPhi(r) = \int_0^r h\mathrm{d}\mu(h) + r\left[\mu(\infty) - \mu(r) + \eta_\infty\right]\,. \tag{26}$$

On the other hand since $\varPhi'(r) = \mu(\infty) - \mu(r) + \eta_\infty$ we can express $\mu(r)$ by means of function $\varPhi(r)$:

$$\eta_\infty = \varPhi'(\infty)\,, \qquad \mu(r) = \varPhi'(0) - \varPhi'(r)\,. \tag{27}$$

The function $\varPhi(r)$ is continuous and concave on $[0,\infty)$ with $\varPhi(0) = 0$. If $\eta_\infty > 0$ or $\varPhi(\infty) = \infty$, then the function $\varPhi(r)$ is increasing and it has an inverse function $\varPsi(s)$ on $[0,\infty)$ defined by

$$\varPsi(s) = r \qquad \text{iff} \qquad \varPhi(r) = s\,.$$

If $\varPhi(\infty) \equiv \varPhi_\infty < \infty$, then its inverse function $\varPsi$ is defined on $[0,\varPhi_\infty]$ only. Moreover, it may be multivalued for $s = \varPhi_\infty$. In both cases $\varPsi$ is a convex function.

Play type Ishlinskii operators

Replacing the stop operator $\mathcal{S}_h$ by the play operator $\mathcal{P}_h$ we obtain Ishlinskii operators of play type. Let us state the formula with input $u(t)$, weight function $\zeta(h)$ and ζ_0 multiple of identity:

$$(\mathcal{G}(u))\,(t) = \int_0^\infty \zeta(h)\,(\mathcal{P}_h u)\,(t)\mathrm{d}h + \zeta_0 u \tag{28}$$

or with function $\nu(r) = \int_0^r \zeta(s)\mathrm{d}s$ and Stieltjes integral

$$(\mathcal{G}(u))(t) = \int_0^\infty (\mathcal{P}_h u)(t)\mathrm{d}\nu(h) + \zeta_0 u\,. \qquad (29)$$

The operator $\mathcal{G}$ can be characterized by a function $\Psi(s)$ describing the response of the operator to monotone increasing load from the intact state. Using $\mathcal{P}_h(s) = s - h$ for $h < s$ and $\mathcal{P}_h(s) = 0$ for $h \geq s$ from (28) or (29) we obtain formula for the function $\Psi(s)$:

$$\Psi(s) = \int_0^s (s-h)\zeta(h)\mathrm{d}h + \zeta_0 s = s\nu(s) - \int_0^s h\mathrm{d}\nu(h) + \zeta_0 s\,. \qquad (30)$$

On the other hand since $\Psi'(s) = \nu(s) + \zeta_0$ the function $\Psi(s)$ yields $\nu(s)$ and ζ_0:

$$\zeta_0 = \Psi'(0)\,, \qquad \nu(s) = \Psi'(s) - \Psi'(0)\,. \qquad (31)$$

The hysteresis loops have convex increasing arches and concave decreasing arches. Again if the function Ψ is defined on $[0,\infty)$ it is invertible, its inverse is the Ishlinskii operator of stop type and vice versa. Thus the operators of the play type are used in modelling the inverse hysteresis constitutive relations i. e. dependence of strain on stress.

Ishlinskii operators satisfy similar continuity and monotony properties like the stop and play operators.

4 Homogenization problem

Let us consider a two component periodically ordered layered material. We assume that both materials denoted as material A and B are characterized by Ishlinskii operators F_A and F_B with corresponding characteristics η_A, μ_A, Φ_A and η_B, μ_B, Φ_B.

Material with a periodic layered structure

The homogenization approach, see [1], [2], [3], [4] consists in considering a sequence of periodically ordered materials with a diminishing period ε.

Let $d_A : d_B$ $(d_A + d_B = 1)$ be the proportion of the thicknesses of the components A and B. We define a space dependent Ishlinskii operator periodic in space

$$\mathcal{F}(y)(e)(t) = \int_0^\infty (\mathcal{S}_h e)(t)\,\mathrm{d}\mu(y,h) + \eta_\infty(y)e(t)\,, \qquad (32)$$

where $\mu(y,h)$ and $\eta_\infty(y)$ are functions periodic in y with period one defined by

$$\mu(y,h) = \begin{cases} \mu_A(h) & \text{for} \quad y \in [k, k+d_A) \quad k \in \mathbb{Z}\,, \\ \mu_B(h) & \text{for} \quad y \in [k-d_B, k) \quad k \in \mathbb{Z}\,. \end{cases} \qquad (33)$$

The function $\eta_\infty(y)$ is defined similarly.

Now we define a sequence of operators $\mathcal{F}^\varepsilon$ with diminishing period $\varepsilon \to 0$

$$\mathcal{F}^\varepsilon(x)(e) = \mathcal{F}\left(\frac{x}{\varepsilon}\right)(e)\,. \qquad (34)$$

Boundary value problems

We shall consider a problem modelling longitudinal vibration of the rod. Let us consider a thin rod with space variable $x \in [0,l]$ made of a periodically layered material. Let us denote the longitudinal displacement by $u(x,t)$ and the stress by $\sigma(x,t)$. We combine the equation of motion $\sigma_x + f = \rho\, u_{tt}$, where ρ is the density and $f(x,t)$ the applied force, with the stress-strain relation $\sigma = \mathcal{F}(e)$, where $\mathcal{F}(x)(e)$ is the space dependent Ishlinskii operator and $e(x,t) = u_x(x,t)$. Since the material has periodic structure, the Ishlinskii operator $\mathcal{F}(x)$ is periodic with the period ε. We denote it by a superscript ε, see (34). We also denote the corresponding solution by u^ε.

Thus for any $\varepsilon > 0$ we obtain the following equation

$$[\mathcal{F}^\varepsilon(x)(u^\varepsilon_x)]_x + f = \rho\, u^\varepsilon_{tt} \quad \text{for} \quad x \in (0,l)\,, \quad t > 0\,. \tag{35}$$

We complete the equation (35) with suitable boundary conditions, e. g. fixed ends of the rod

$$u^\varepsilon(0,t) = 0\,, \qquad u^\varepsilon(l,t) = 0 \quad \text{for} \quad t > 0 \tag{36}$$

and initial conditions with a given initial displacement u_0 and a given initial velocity u_1

$$u^\varepsilon(x,0) = u_0(x)\,, \qquad u^\varepsilon_t(x,0) = u_1(x) \quad \text{for} \quad x \in (0,l)\,. \tag{37}$$

Finally we have to add the initial state of the stop operators $\mathcal{S}_h$ in the Ishlinskii operator

$$s_h(x) \quad \text{for} \quad x \in (0,l) \quad h > 0\,, \tag{38}$$

satisfying $|s_h(x)| \le h$. In case of the intact initial state we use (8).

Homogenization problem

Taking a sequence ε_i converging to zero (we shall write $\varepsilon \to 0$ only) we obtain a sequence of boundary value problems (35) with a sequence of Ishlinskii operators (34) completed by boundary and initial conditions (36), (37) and the Ishlinskii operator initial state condition (38). Assuming the solvability of the problems, see [10], we have a sequence of the corresponding solutions u^ε.

The following steps of homogenization are to be carried out. We have to show that the sequence $\{u^\varepsilon\}$ converge to a function u^0 which is a solution to the so called homogenized problem of the similar form. The homogenized problem consists of the equation

$$\left[\mathcal{F}^0 u^0_x\right]_x + f = \rho\, u^0_{tt} \quad \text{for} \quad x \in (0,l) \quad t > 0\,. \tag{39}$$

with the homogenized operator $\mathcal{F}^0$ independent of the space variable x. The problem is completed by analogous boundary and initial conditions

$$u^0(0,t) = 0\,, \qquad u^0(l,t) = 0 \quad \text{for} \quad t > 0\,, \tag{40}$$

$$u^0(x,0) = u_0(x)\,, \qquad u_t^0(x,0) = u_1(x) \quad \text{for} \quad x \in (0,l)\,. \tag{41}$$

Assuming that $\mathcal{F}^0$ is also an Ishlinskii operator we add the same condition giving the initial state of its stop operators $s_h(x)$.

5 Homogenized operator

We try to derive the form of the homogenized operator $\mathcal{F}^0$. Let us consider the uniformly loaded layered rod. Both components described by Ishlinskii operators $\mathcal{F}_A, \mathcal{F}_B$ can be considered as combination in series of the elements, namely the layers satisfy the rule (4).

Denoting the deformation and stress in both materials A, B by e_A, e_B and σ_A, σ_B, the serial combination rule yields

$$\sigma = \sigma_A = \sigma_B\,. \tag{42}$$

for the total stress. Taking into account the ratio $d_A : d_B$ of the thickness of the layers A, B $(d_A + d_B = 1)$ we can write

$$e = d_A\, e_A + d_B\, e_B\,. \tag{43}$$

Inserting inverses $e_A = \mathcal{G}_A(\sigma_A)$, $e_B = \mathcal{G}_B(\sigma_B)$ of the constitutive relations $\sigma_A = \mathcal{F}_A(e_A)$, $\sigma_B = \mathcal{F}_B(e_B)$ into (43) we obtain

$$e = d_A\, \mathcal{G}_A(\sigma) + d_B\, \mathcal{G}_B(\sigma) = (d_A\, \mathcal{G}_A + d_B\, \mathcal{G}_B)\,(\sigma)\,,$$

which implies the limit constitutive relation

$$\sigma = \mathcal{F}^0(e) \equiv (d_A\, \mathcal{G}_A + d_B\, \mathcal{G}_B)^{-1}\,(e)\,. \tag{44}$$

The last equality gives little information on the homogenized operator $\mathcal{F}^0$. If we consider an increasing uniform loading from the intact state, we can replace the Ishlinskii operator by the corresponding functions Φ_A, Φ_B and rewrite the relation (44) to

$$\Phi^0(r) = \left(d_A\, \Phi_A^{-1} + d_B\, \Phi_B^{-1}\right)^{-1}(r) = (d_A\, \Psi_A + d_B\, \Psi_B)^{-1}\,(r)\,. \tag{45}$$

We have arrived to the following result:

Theorem 3. *Under assumptions that the homogenized operator $\mathcal{F}^0$ of the homogenized problem (39)–(41) is the Ishlinskii operator of stop type, it is defined by*

$$F^0(e)(t) = \int_0^\infty (\mathcal{S}_h e)\,(t)\mathrm{d}\mu^0(h)\,. \tag{46}$$

The weight function $\mu^0(r)$ is given by

$$\mu^0(r) = \left(\Phi^0\right)'(0) - \left(\Phi^0\right)'(r)\,, \tag{47}$$

where $\Phi^0(r)$ is given by (45).

Justification of the assumption and proof of the convergence $u^\varepsilon \to u^0$ is subject of the future research.

References

[1] *I. Babuška:* Homogenization approach in engineering. In Computing methods in applied sciences and engineering, Lecture notes in Econ. and Math. Systems 134, Springer, Berlin 1976, 137–153.

[2] *A. Bensoussan, J. L. Lions, G. Papanicolaou:* Asymptotic analysis for periodic structure, North Holland, 1978.

[3] *E. Sanchez Palencia:* Non-homogeneous media and vibration theory, Lecture Notes in Physics 127, Springer 1980.

[4] *N. S. Bakhvalov, G. P. Panasenko:* Averaging of processes in periodic media (Russian), Nauka, Moscow 1984.

[5] *P. Krejčí:* Methods of solving equations with hysteresis, Proceedings of 12 Seminar on P.D.E. (Czech), Lipovce 1987, 7–60, Union of Czechoslovak Mathematicians and Physicists Prague, Technical University Plzeň, 1987.

[6] *M. A. Krasnosel'skii, A. V. Pokrovskii:* Systems with hysteresis, Springer, Berlin 1989. Russian edition: Nauka, Moscow 1983.

[7] *P. Krejčí:* Modelling of singularities in elastoplastic materials with fatigue, Appl. Math. **39** (1994), 137–160.

[8] *M. Brokate:* Hysteresis Operators. In: Phase Transitions and Hysteresis, Lecture Notes of C.I.M.E., Montecatini Terme July 13–21, 1993, 1–38, Springer Berlin, 1994.

[9] *A. Visintin:* Differential models of hysteresis, Springer, Berlin Heidelberg 1994.

[10] *P. Krejčí:* Hysteresis, convexity and dissipation in hyperbolic equations, Gakuto Int. Series Math. Sci. & Appl., Vol. 8, Gakkōtosho, Tokyo 1996.